Groups with infinitely many ends acting analytically on the circle

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Résumé

Cet article est un retour sur deux travaux remarquables autour des actions non minimales de groupes sur le cercle : le théorème de Duminy sur les bouts des feuilles semi-exceptionnelles et le résultat de Ghys de liberté des groupes en régularité analytique. Notre premier résultat concerne les groupes de difféomorphismes analytiques avec une infinité de bouts : si l'action n'est pas expansive, alors le groupe est virtuellement libre. Le deuxième résultat est un théorème de Duminy dans le cadre des feuilletages minimaux de codimension un : soit les feuilles non expansibles ont une infinité de bouts, soit l'holonomie préserve une structure projective.

Abstract

This article takes the inspiration from two milestones in the study of non minimal actions of groups on the circle: Duminy's theorem about the number of ends of semi-exceptional leaves and Ghys' freeness result in analytic regularity. Our first result concerns groups of analytic diffeomorphisms with infinitely many ends: if the action is non expanding, then the group is virtually free. The second result is a Duminy's theorem for minimal codimension one foliations: either non expandable leaves have infinitely many ends, or the holonomy pseudogroup preserves a projective structure.

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1 Dynamics and structure of locally discrete groups of analytic circle diffeomorphisms

The description of minimal invariant compact sets is a classical way for easily distinguishing the dynamics of groups of circle homeomorphisms: for any group $G \leq \mathrm{Homeo}_+(\mathbf{S}^1)$, a minimal invariant compact set for the action of G can either be a finite orbit, the whole circle or a Cantor set (the latter is usually referred to as the *exceptional* case).

When the regularity of the action is at least C^2, we can hope this trichotomy to give further information about the dynamics and the algebraic structure of the group: one of the pioneering statements in this direction is that if the action of a finitely generated (f.g.) group G has an exceptional minimal set then there must be an element with hyperbolic fixed points. This result, known as Sacksteder's Theorem [27], has many important consequences: for example it implies the well-known Denjoy's Theorem [4], namely G cannot be cyclic.

Recent works of Deroin, Filimonov, Kleptsyn and Navas [5,7,11] suggest a shift in distinguishing the actions on the circle by means of the discreteness properties of the group, following the line of previous results by Ghys [15], Shcherbakov *et al.* [10], Nakai [22], Loray and Rebelo [19,25,26].

Definition 1. A group $G \leq \mathrm{Diff}^1_+(\mathbf{S}^1)$ is *locally discrete* (more precisely, C^1 locally discrete) if for any interval $I \subset \mathbf{S}^1$ intersecting its minimal set, the restriction of the identity to I is isolated in the C^1 topology among the set of restrictions to I of the diffeomorphisms in G.

It comes from [10,22] that groups acting analytically with an exceptional minimal set are locally discrete. However there are more locally discrete actions than that: any Fuchsian group is locally discrete, even if its action is minimal. At the "boundary" of non minimal actions there lay groups acting minimally but in a non expanding way, whose description is central in the study of locally discrete dynamics on the circle:

Definition 2. A point $x \in \mathbf{S}^1$ is *non expandable* for the action of a group $G \le \mathrm{Diff}^1_+(\mathbf{S}^1)$ if for any $g \in G$ the derivative of g at x is not greater than 1. We denote by $\mathrm{NE} = \mathrm{NE}(G)$ the set of non expandable points of G.

The action of a group $G \le \mathrm{Diff}^1_+(\mathbf{S}^1)$ is *expanding* if $\mathrm{NE} = \emptyset$.

It has been conjectured by Deroin, Kleptsyn and Navas that the presence of non expandable points in the minimal set for the action of a group $G \le \mathrm{Diff}^2_+(\mathbf{S}^1)$ entails that the group is locally discrete in restriction to the minimal set and has many other nice properties, as a Markov partition on the minimal set (see [5, 7]). The following picture is a slight variation on the *paradigm* presented in the survey [5] (see also the list of conjectures in [12]) and is the motivation for this work.

Main Conjecture (Locally discrete groups)**.** *Let $G \le \mathrm{Diff}^\omega_+(\mathbf{S}^1)$ be a finitely generated locally discrete group. Then*

- ***dynamical structure:*** *the action admits a Markov partition of the minimal set;*
- ***algebraic structure:*** *if the action of G is minimal and expanding, then G is analytically conjugate to a finite central extension of a cocompact Fuchsian group, whereas the group is virtually free in any other case.*

If non locally discrete, groups preserving a probability measure are either conjugated to a group of rotations or have a finite orbit. If no probability measure is preserved, the description can be resumed roughly as follows (see the works previously cited [10, 19, 22, 25, 26]):

Theorem 1 (Locally non discrete groups)**.** *Let $G \le \mathrm{Diff}^\omega_+(\mathbf{S}^1)$ be a finitely generated C^1 locally non discrete group which does not preserve any probability measure. Then*

- ***dynamical structure:*** *the action is minimal and expanding;*
- ***algebraic and topological structure:*** *the group has a local flow in a local C^1 closure and is topologically rigid.*

The existence of a Markov partition for locally discrete groups actually follows from the algebraic structure: in the case of Fuchsian groups it is known since the work of Bowen and Series [2], whereas when the group is virtually free it is a consequence of the so-called *properties* $(\star)$ and $(\Lambda\star)$ [8, 11].

Definition 3. A group $G \le \mathrm{Diff}^1_+(\mathbf{S}^1)$ acting minimally on the circle has *property* $(\star)$ if for every $x \in \mathrm{NE}$ there are g_+ and g_- in G such that x is an isolated fixed point from the right (resp. from the left) for g_+ (resp. g_-).

Definition 4. A group $G \le \mathrm{Diff}^1_+(\mathbf{S}^1)$ acting on the circle with an exceptional minimal set Λ has *property* $(\Lambda\star)$ if for every $x \in \mathrm{NE} \cap \Lambda$ there are g_+ and g_- in G such that x is an isolated fixed point in Λ from the right (resp. from the left) for $g_+|_\Lambda$ (resp. $g_-|_\Lambda$).

Properties $(\star)$ and $(\Lambda\star)$ imply that there are only finitely many non expandable points in the minimal set, whence Deroin, Filimonov, Kleptsyn and Navas are able to construct a Markov partition [7, 11] of the minimal set (see Theorem 5 below).

Plainly, groups whose action is expanding have property $(\star)$ or $(\Lambda\star)$. In presence of non expandable points, properties $(\star)$ and $(\Lambda\star)$ have been established for some classes of groups of analytic circle diffeomorphisms: virtually free groups [8], one-ended finitely presented groups of bounded torsion [12]. By a theorem of Ghys [13] (stated as Theorem 3 below) relying on the work of Duminy (see Theorem 8 below), only virtually free groups can have an exceptional minimal set (in analytic regularity), and thus every f.g. group $G \leq \mathrm{Diff}_+^\omega(\mathbf{S}^1)$ with an exceptional minimal set has property $(\Lambda\star)$. On the other hand, it is conjectured that every f.g. group acting minimally on the circle has property $(\star)$ (as a matter of fact, this was the "main conjecture" before the new paradigm was presented in [5]).

Our first main result enlarges the list of f.g. groups $G \leq \mathrm{Diff}_+^\omega(\mathbf{S}^1)$ for which property $(\star)$ holds and goes beyond that, providing information on the algebraic structure. Before stating it, let us recall the notion of *ends* of a group.

Definition 5. Let X be a connected topological space. The *number of ends* $e(X)$ of X is the least upper bound, possibly infinite, for the number of unbounded connected components of the complementary sets $X - K$, where K runs through the compact subsets of X.

If G is a group generated by the finite set $\mathcal{G}$, we define the number of ends $e(G)$ of G to be the number of ends of the *Cayley graph* of G relative to $\mathcal{G}$. This is the graph whose vertices are the elements of G and two elements $g, h \in G$ are joined by an edge if $g^{-1}h \in \mathcal{G}$. The graph metric induces the *length metric* in G given by the following expression

$$d_{\mathcal{G}}(g,h) = \min\{\ell \mid g^{-1}h = s_1 \cdots s_\ell,\, s_j \in \mathcal{G} \cup \mathcal{G}^{-1}\}.$$

It is a classical fact [3, §8.30] that the number of ends of a group does not depend on the choice of the finite generating set (since Cayley graphs associated to different finite generating systems are quasi-isometric). Moreover, f.g. groups can only have 0, 1, 2 or infinitely many ends and groups with 0 or 2 ends are not of particular interest, for they are respectively finite or virtually infinite cyclic, *i.e.* they contain $\mathbf{Z}$ as a finite index subgroup (we refer to [3, §8.32] for further details). Although they represent a broader class, groups with infinitely many ends may also be characterized algebraically after the celebrated Stallings' Theorem (Theorem 6 here).

Here we can state our first result that will be proved all along §3:

Theorem A. *Let G be a f.g. subgroup of $\mathrm{Diff}_+^\omega(\mathbf{S}^1)$ with infinitely many ends acting minimally on the circle. Then the group G has property $(\star)$.*

Moreover, if there are non expandable points, then G is virtually free.

Since free groups have infinitely many ends, Theorem A extends the main result of [8]:

Theorem 2 (Deroin, Kleptsyn, Navas)**.** *Let $G \leq \mathrm{Diff}_+^\omega(\mathbf{S}^1)$ be a virtually free group acting minimally on the circle. Then the group G has property $(\star)$.*

In fact, the proof of Theorem A relies on an interplay of Theorem 2 with Stallings' Theorem, following ideas of Hector and Ghys [13] that we sketch in §3.1.

Our second result, exposed in §4, not only puts particular emphasis on groups with infinitely many ends, but also shows that actions with non expandable points are very close to actions with an exceptional minimal set. The orbit of a non expandable point plays the role of the gaps associated to an exceptional minimal set. In this analogy the non expandable point is identified with a maximal gap which cannot be expanded. In this sense it is a close analogue of Duminy's and Ghys' theorems.

Theorem B. *Let G be a f.g. subgroup of $\mathrm{Diff}_+^3(\mathbf{S}^1)$ acting minimally, with property $(\star)$ and non expandable points. Then G has infinitely many ends.*

In particular, if furthermore $G \leq \mathrm{Diff}_+^\omega(\mathbf{S}^1)$, then it is virtually free.

The latter result will actually be proved keeping in mind the more general language of pseudogroups, for which it is more correct to state:

Theorem C. *Let G be a compactly generated pseudogroup of C^r diffeomorphisms, $r \geq 3$, of a one-dimensional manifold N such that the action of G is minimal, has property $(\star)$ and a non expandable point $x \in N$. Then the Schreier graph (relative to a system of compact generation) of the orbit of x has infinitely many ends or G preserves a C^r projective structure.*

Notice that, above, both cases can arise simultaneously. This happens, for example, for $\mathrm{PSL}(2, \mathrm{Z})$.

We refer to [1, 16] for the definitions related to pseudogroups. Here we just recall that the *Schreier graph* of an orbit X, denoted by $\mathrm{Sch}(X, \mathcal{G})$ is the graph whose vertices are the elements of X and two elements $x, y \in G$ are joined by an edge if there exists $s \in \mathcal{G}$ so that $s(x) = y$. The graph metric induces the *length* metric on X:

$$d_{\mathcal{G}}^X(x, y) = \min\left\{d_{\mathcal{G}}(id, g) \mid g(x) = y\right\}.$$

It is not the aim of this paper to give a deep discussion on pseudogroup actions, this setting should be treated in a forthcoming work. Theorem C translates easily to the framework of codimension one foliations on compact manifolds: if the foliation is minimal with non expanding C^r holonomy, $r \geq 3$, then either any leaf which contains a non expandable point has infinitely many ends or the foliation is transversely C^r projective.

It is important to stress that the assumption for C^3 regularity is essential for our proof: up to present, we are able to offer a proof only using control on the *projective distortion* of the elements of the pseudogroups, which classically requires three derivatives. However, we hope that Theorem C can be generalized to actions of pseudogroups of class C^2.

2 A review of known results in the direction of the Main Conjecture

At the light of Theorems A and B, the Main Conjecture is not far to be confirmed true:

"Missing Piece" Conjecture. *Let $G \leq \mathrm{Diff}_+^\omega(\mathbf{S}^1)$ be a f.g. one-ended group, not finitely presented or with elements of unbounded torsion. Then the action of G is expanding.*

For the reader's sake we shall provide a description of what has been done up to present, hence explaining how the "Missing Piece" Conjecture completes the picture. For further details, we refer again to the survey [5].

Finite orbits – If a group $G \leq \mathrm{Diff}_+^\omega(\mathbf{S}^1)$ has a finite orbit, then there is finite index subgroup having fixed points. From [21, Proposition 3.7] we have:

Proposition 1. *A group acting with a global fixed point must either be cyclic (if the action is locally discrete) or have a local vector field in its local closure (otherwise).*

Exceptional minimal set – If a group $G \leq \mathrm{Diff}_+^\omega(\mathbf{S}^1)$ acts with an exceptional minimal set, then it is locally discrete. Indeed, if it was not the case, one would be able to find an interval $I \subset \mathbf{S}^1$, intersecting the minimal set, such that the restriction of the identity $id|_I$ is not C^1 isolated in $G|_I = \{f|_I, f \in G\}$. Then, Loray–Rebelo–Nakai–Scherbakov arguments imply that on some subinterval of I there are local flows in the C^1-local closure of the action of G. These arguments go back to papers [10, 19, 22]; we will be using them in the form of [8, Proposition 2.8]:

Proposition 2. *Let I be an interval on which certain real analytic nontrivial diffeomorphisms $f_k \in G$ are defined. Suppose that f_k nontrivially converges to the identity in the C^1 topology on I, and let f*

be another C^1 diffeomorphism having a hyperbolic fixed point on I. Then there exists a (local) C^1 change of coordinates $\phi : I_0 \longrightarrow [-1,2]$ on some subinterval $I_0 \subset I$ after which the pseudogroup G generated by the f_k's and f contains in its $C^1([0,1],[-1,2])$-closure a (local) translation subgroup:

$$\overline{\{\phi g \phi^{-1}|_{[0,1]} \mid g \in G\}} \supset \{x \mapsto x + s \mid s \in [-1,1]\}.$$

Having this, one immediately obtains a contradiction with the local non minimality of the action of G.

In addition, groups of analytic diffeomorphisms acting with an exceptional minimal set are virtually free [13] and hence admit a Markov partition of the minimal set after the results of [7,8,11]:

Theorem 3 (Ghys)**.** *Let $G \leq \mathrm{Diff}_+^\omega(\mathbf{S}^1)$ be a f.g. group acting with an exceptional minimal set. Then G is virtually free.*

Minimal action – If a group $G \leq \mathrm{Diff}_+^\omega(\mathbf{S}^1)$ acts minimally and is non locally discrete, by arguments close to those of the paragraph above, it has local vector fields in its C^1 local closure and this implies that either it is conjugated to a group of rotations or it has no non expandable point (see [8, Remark 3.9]).

If a group $G \leq \mathrm{Diff}_+^\omega(\mathbf{S}^1)$ acts minimally and is locally discrete there are two possibilities: either it has non expandable points or it does not. In the first situation the "Missing Piece" Conjecture and Theorem B imply that the group is virtually free and after [7,11] it has a Markov partition. In the other case we have the following (unpublished) result:

Theorem 4 (Deroin)**.** *Let $G \leq \mathrm{Diff}_+^\omega(\mathbf{S}^1)$ be a locally discrete f.g. group whose action on the circle is minimal and expanding. Then G is analytically conjugated to a finite central extension of a cocompact Fuchsian group.*

It is perhaps worthwhile to extend this discussion to the description of the dynamical properties of actions on the circle, actually constituting the original motivation of this whole study. Going back to the 80s, it was observed, after the work of Shub and Sullivan [29], that expanding actions of groups $G \leq \mathrm{Diff}_+^{1+\alpha}(\mathbf{S}^1)$ have nice *ergodic* properties: if the action is minimal then it is also ergodic with respect to the Lebesgue measure, whereas if the action has an exceptional minimal set Λ, then the Lebesgue measure of Λ is zero and the complementary set $\mathbf{S}^1 - \Lambda$ splits into finitely many distinct orbits. This was close to what was known for $\mathbf{Z}$ actions of C^2 circle diffeomorphisms: it was proved (independently) by Katok and Herman that they are not only minimal (as Denjoy's theorem states), but also Lebesgue ergodic [17].

Remark 1. The notion of ergodicity is naturally extended to transformations with *quasi-invariant* measures, as for example the Lebesgue measure for any C^1 action. In this precise case, it means that any G-invariant subset of the circle has either full or zero Lebesgue measure.

One of the key ingredients behind these results is the technique of control of the affine distortion of the action (highly exploited throughout this paper as well).

By that time, this lead to conjecture that such a picture should hold in any circumstance in which the control of distortion can be assured.

Conjecture 1 (Ghys, Sullivan)**.** *Let $G \leq \mathrm{Diff}_+^2(\mathbf{S}^1)$ be a f.g. group whose action on the circle is minimal. Then the action is also Lebesgue ergodic.*

Conjecture 2 (Ghys, Sullivan; Hector)**.** *Let $G \leq \mathrm{Diff}_+^2(\mathbf{S}^1)$ be a f.g. group whose action on the circle has an exceptional minimal set Λ. Then the Lebesgue measure of Λ is zero and the complementary set $\mathbf{S}^1 - \Lambda$ splits into finitely many orbits of intervals.*

Properties ($\star$) and ($\Lambda\star$) were first introduced in [7] as finitary properties under which these conjectures could be established: as we have already said, from the set NE of non expandable points it is possible to define a Markov partition of the minimal set, with a non uniformly expanding map encoding the dynamics of G, allowing to extend the technique of Shub and Sullivan in order to prove the Conjectures 1 and 2 for groups with properties ($\star$) and ($\Lambda\star$) respectively. For further reference needed in §4, we recall the result of [11] in the case of minimal actions:

Theorem 5 (Filimonov, Kleptsyn). *Let $G \leq \mathrm{Diff}^2_+(\mathbf{S}^1)$ be a f.g. group whose action is minimal and with property ($\star$). Let k be the number of non expandable points of G, and write $\mathrm{NE} = \{x_1, \ldots, x_k\}$. Then there exists a partition of the circle $\mathbf{S}^1$ into finitely many open intervals*

$$\mathcal{I} = \left\{ I_1, \ldots, I_k, I_1^+, I_1^-, \ldots, I_\ell^+, I_\ell^- \right\},$$

an expansion constant $\lambda > 1$ and elements $g_I \in G$, $I \in \mathcal{I}$ such that:

- **i.** *for every $I \in \mathcal{I}$, the image $g_I(I)$ is a union of intervals in $\mathcal{I}$;*
- **ii.** *we have $g_I'|_I \geq \lambda$ for every $I = I_1, \ldots, I_k$;*
- **iii.** *the intervals I_i^+ and I_i^- are adjacent respectively on the right and on the left to the non expandable x_i, which is the unique fixed point, topologically repelling, for $g_{I_i^+}$ (resp. $g_{I_i^-}$) on the interval I_i^+ (resp. I_i^-); moreover x_i is the unique non expandable point in $g_{I_i^\pm}(I_i^\pm)$;*
- **iv.** *for every $I = I_1^\pm, \ldots, I_\ell^\pm$, set*
$$k_I : I \longrightarrow \mathbf{N}$$
to be the function $k_I(x) = \min\{k \in \mathbf{N} \mid g_I^k(x) \notin I\}$ and
$$j : I \longrightarrow \{1, \ldots, k\}$$
defined by the condition $g_I^{k_I(x)}(x) \in I_{j(x)}$. Then for every $x \in I$, $\left(g_{I_{j(x)}} \circ g_I^{k_I(x)} \right)'(x) \geq \lambda$.

It is worthwhile to observe that Theorem B was first conjectured in [12] as a moral consequence of Theorem 5: the (non uniformly) expanding maps g_I's give a way to decompose the Schreier graphs of all but finitely many orbits into a finite number of trees [11], thus suggesting freeness in the structure.

3 Theorem A: Property ($\star$) for groups with infinitely many ends

3.1 Stallings' theorem and virtually free groups

Groups with infinitely many ends are directly associated to Stallings' characterization as amalgamated products or HNN extensions over finite groups (see for example [28] for the definitions).

Theorem 6 (Stallings). *Let G be a f.g. group with infinitely many ends. Then G is either an amalgamated product $G_1 *_Z G_2$ over a finite group Z (different from G_1 and G_2) or an HNN extension $H*_Z$ over a finite group Z (different from H).*

Given a f.g. group G with infinitely many ends, we shall call a *Stallings' decomposition* any possible decomposition of G as an amalgamated product or an HNN extension over a finite group.

An idea which can be traced back to Hector (and Ghys) [13] is that we can use the knowledge on the action of G to restrict the possible Stallings' decompositions of G.

As an illustrative example, let us sketch the proof of Hector's result, stating that a f.g. torsion-free group G acting analytically on the circle with an exceptional minimal set is *free* [13, Proposition 4.1].

First of all, in the case of exceptional actions we have (see [13, Théorème 2.9] and [15, 23]):

Theorem 7 (Hector's lemma)**.** *Let $G \leq \mathrm{Diff}^{\omega}_{+}(\mathbf{S}^1)$ be a f.g. subgroup which acts with an exceptional minimal set Λ. Then the stabilizer in G of any point is either trivial or infinite cyclic.*

Duminy's Theorem (Theorem 8) implies that G has infinitely many ends, so there is a Stallings' decomposition. Since the group is torsion-free, the Stallings' decomposition must be a free product $G = G_1 * G_2$. Now, neither factor acts minimally (otherwise G does). If one of the factors acts with an exceptional minimal set, then we can expand the free product $G_1 * G_2$ until the moment we get $G = H_1 * \ldots * H_n$ with every H_i acting with some periodic orbit: this procedure has to stop in a finite number of steps, for the rank (least number of generators) of the factors is less than the rank of the group (Grushko's formula [20]). Hence, for every H_i there is a finite index subgroup K_i which fixes pointwise the periodic orbit. Since the action is analytic, Hector's lemma implies that every K_i is either infinite cyclic or trivial and actually $K_i = H_i$ since G is torsion-free. Thus, G is free, as claimed.

The main ingredient of the proof of Theorem A is to analyse the factors in Stallings' decomposition of such group, as we have just illustrated. Let $G \leq \mathrm{Diff}^{\omega}_{+}(\mathbf{S}^1)$ be a f.g. subgroup with infinitely many ends acting minimally on the circle. By Stallings' theorem, we know that either $G = G_1 *_Z G_2$ or $G = H*_Z$, with Z a finite group.

3.2 First (possible) case: No Stallings' factor acts minimally

In this case, using Proposition 1 (replacing Hector's lemma) and Ghys' Theorem 3, we deduce that every Stallings' factor is virtually free (possibly finite), so that G is either an amalgamated product of virtually free groups over a finite group or an HNN extension of a virtually free group over a finite group. This implies that G itself is virtually free after a classical theorem by Karrass, Pietrowski and Solitar [18].

3.3 Second (impossible) case: There is one factor acting minimally

Under this assumption, we shall prove that G is not locally discrete following some of the main arguments in [8]. This shows in particular that the action of G must be expanding.

Remark that it is enough to study the case where $G = G_1 *_Z G_2$ is an amalgamated product, since any HNN extension $H*_Z$ contains copies of $H *_Z H$ as subgroups: in $H*_Z$, if we denote by σ the *stable letter* (that is, the element conjugating the two embedded copies of Z), then H and $\sigma H \sigma^{-1}$ generate a subgroup which is isomorphic to $H *_Z H$.

Thus, from now on, we suppose that G is an amalgamated product $G_1 *_Z G_2$ over a finite group Z, and we assume that G_1 acts minimally. For simplicity, we suppose that G is generated by a finite set $\mathcal{G} = \mathcal{G}_1 \sqcup \mathcal{G}_2$, with $\mathcal{G}_i$ generating G_i and symmetric. We consider the length metric on the group G associated to this generating system and for every $n \in \mathbf{N}$ we define $B(n)$ to be the ball of radius n centred at the identity. Let us illustrate the main lines of the proof before getting involved in technicalities. This will be also the opportunity to introduce some notation.

We fix a non expandable point $x_0 \in \mathrm{NE}$, and for any finite set $E \subset G$, we let x_E denote the closest point on the right of x_0 among the points in the orbit $E.x_0$ distinct from x_0 (this exists for any E which is not contained in the stabilizer of x_0). This point corresponds to some $g_E \in E$ which is uniquely defined modulo $\mathrm{Stab}_G(x_0)$: $x_E = g_E(x_0)$. The length of the interval $J_E = [x_0, x_E]$ will be denoted by ℓ_E.

We also take count of the number of elements fixing x_0: we define

$$c_E = \max_{g \in G} \# \left(E \cap g \, \mathrm{Stab}_G(x_0)\right).$$

Notice that, under our assumption of analytic regularity, the stabilizer of x_0 is either trivial or infinite cyclic (this is a consequence of Proposition 1), hence c_E is respectively either *bounded* (by 1) or a *linear* function of $\rho(E)$, where $\rho(E)$ is the *outer radius* of E, that is, the minimal $n \in \mathbf{N}$ such that $E \subset B(n)$.

As in [8, 12], the proof is carried on in three different stages, which will be exposed separately in the next paragraphs.

Step 1. – The first and most important step (Proposition 4) is to describe a *sufficient condition* guaranteeing that for a prescribed sequence of finite subsets $E(n) \subset G$, setting $F(n) = E(n)^{-1}E(n)$, the elements $g_{F(n)}$ are "locally" converging C^1 to the identity. To have this, letting

$$S_E = \sum_{g \in E} g'(x_0),$$

it is enough that

$$\frac{\rho(E(n))^2}{S_{E(n)}} = o(1) \quad \text{as } n \text{ goes to infinity.} \tag{1}$$

This criterion does not provide a contradiction to the hypothesis of local discreteness of G, since we are only able to show that $g_{F(n)}$ is closer and closer to *id in restriction* to (a complex extension of) an interval containing $J_{F(n)}$, which is unfortunately shrinking to x_0.

Step 2. – We then show that it is very easy to find examples of sequences $(E(n))_{n \in \mathbf{N}}$ which satisfy the criterion above, even in a very strong way. For this, we use two key facts:

1. G_1 acts minimally, whence we find a non negligible sum $\sum_{g \in B_1(n)} g'(x)$, where $B_1(n)$ is the ball of radius n in G_1 with respect to the generating set $\mathcal{G}_1$ (Proposition 6).

2. The tree-like structure of the amalgamated product allows to move from one G_1 slice in G to another, so that we can increase the lower bound for $S_{E(n)}$ up to an exponential bound (Proposition 8): there exists $a > 1$ such that

$$S_{E(n)} \geq a^{\rho(E(n))}.$$

Step 3. – The last arguments are more of combinatorial nature. The key idea relies on a result of Ghys [15, Proposition 2.7] about groups of analytic local diffeomorphisms defined on the complex neighbourhood $U_r^{\mathbf{C}}(x_0)$ of radius $r > 0$ of $x_0 \in \mathbf{C}$:

Proposition 3. *There exists $\varepsilon_0 > 0$ with the following property: Assume that the complex analytic local diffeomorphisms f_1, $f_2 : U_r^{\mathbf{C}}(x_0) \to \mathbf{C}$ are ε_0-close (in the C^0 topology) to the identity, and let the sequence f_k be defined by the recurrence relation*

$$f_{k+2} = [f_k, f_{k+1}], \quad k = 1, 2, 3, \ldots$$

Then all the maps f_k are defined on the disc $U_{r/2}^{\mathbf{C}}(x_0)$ of radius $1/2$, and f_k converges to the identity in the C^1 topology on $U_{r/2}^{\mathbf{C}}(x_0)$.

The main point of this proposition is that if the sequence of iterated commutators $(f_k)_{k\in\mathbf{N}}$ is not eventually trivial, then f_1 and f_2 generate a group which is not locally discrete.

From the previous steps it is not difficult to find elements f_1, f_2 of the form $g_{E(m)}$ which are very close to the identity in some neighbourhood of x_0, but we must exhibit explicit f_1 and f_2 for which we are able to show that the sequence of iterated commutators f_k does not stop eventually to the identity (Proposition 9).

3.4 Step 1: Getting close to the identity

Here we review the argument given in [8, §3.2] and [12, §2.5], which explains how we can find elements which are close to the identity in a neighbourhood of a non expandable point. The result is stated quite in a general form, because of the algebraic issues that we have to overcome in §3.6. The main result of this section is a variation of [8, Lemma 3.12]:

Proposition 4. *Let $(E(n))_{n\in\mathbf{N}}$ be a sequence of subsets of G containing the identity. If*

$$\frac{\rho(E(n))^2}{S_{E(n)}}=o(1),$$

then the sequence $g_{F(n)}$ for $F(n)=E(n)^{-1}E(n)$ converges C^1 to the identity over a complex disc of size $o(1/\rho(E(n)))$ around x_0. More precisely, considering $r_n=o\left(1/\rho(E(n))\right)$ such that

$$\frac{\rho(E(n))}{S_{E(n)}}=o\left(\frac{1}{r_n}\right),$$

the (affinely) rescaled sequence

$$\widetilde{g}_{F(n)}(t)=\frac{g_{F(n)}(x_0+r_n\,t)-x_0}{r_n}$$

converges to the identity on $C^1([-1,1])$ as well as on $C^0(U_1^{\mathbf{C}}(0))$.

We avoid the (somehow technical) details of the proof and prefer to explain the relevant ideas, which mostly rely on the classical technique of *control of the affine distortion* (see [8, Lemma 3.6]). If $J\subset\mathbf{S}^1$ is an interval, the *distortion coefficient* of a diffeomorphism $g:J\to g(J)$ on J is defined as

$$\varkappa(g;J)=\sup_{x,y\in J}\left|\log\frac{g'(x)}{g'(y)}\right|.$$

Not only it measures how far is g to be an affine map, but also it well behaves under compositions:

$$\varkappa(gh;J)\le\varkappa(g;h(J))+\varkappa(h;J),\quad \varkappa(g;J)=\varkappa(g^{-1};g(J)).$$

If we fix the finite generating system $\mathcal{G}$ of the group G, we can find a finite constant $C_{\mathcal{G}}$ such that

$$\varkappa(g;J)\le C_{\mathcal{G}}\,|J|\quad\text{for every } g\in\mathcal{G}.$$

This implies that if $g=g_n\cdots g_1$ belongs to the ball of radius n in G, $g_i\in\mathcal{G}$, then

$$\varkappa(g_n\cdots g_1;J)\le C_{\mathcal{G}}\sum_{i=0}^{n-1}|g_i\cdots g_1(J)|,\tag{2}$$

where $g_i \cdots g_1 = id$ for $i = 0$.

The inequality (2) suggests that the control of the affine distortion of g on some small interval J can be controlled by the *intermediate compositions* $g_i \cdots g_1$. This is better explained in the following way (which goes back to Sullivan): if

$$S = \sum_{i=0}^{n-1} (g_i \cdots g_1)'(x_0) \tag{3}$$

is the sum of the intermediate derivatives at some *single* point $x_0 \in \mathbf{S}^1$, then the affine distortion of g can be controlled in a (complex) neighbourhood of radius $\sim 1/S$ about x_0. More precisely, we have:

Proposition 5. *For a point $x_0 \in \mathbf{S}^1$ and $g \in B(n)$, let S be as in (3) and $c = \log 2/4C_{\mathcal{G}}$. For every $r \leq c/S$, we have the following bound on the affine distortion of g:*

$$\varkappa(g; U_r^{\mathbf{C}}(x_0)) \leq 4C_{\mathcal{G}} S r.$$

The key observation in our framework (and originally of [8, 12]) is that at non expandable points $x_0 \in \mathrm{NE}$ we always have $S \leq n$ for $g \in B(n)$. Therefore, for a very large n, in a neighbourhood of size $r \ll 1/n$ about x_0, the maps in $B(n)$ are almost affine. In particular the element $g_{F(n)}$ (resp. $\widetilde{g}_{F(n)}$) is almost affine on a neighbourhood of radius $r_n = o\left(1/\rho(E(n))\right)$ (resp. 1) about x_0 (resp. 0).

To see that the derivative of $g_{F(n)}$ (and $\widetilde{g}_{F(n)}$) is close to 1, we consider the inverse map $g_{F(n)}^{-1}$, which satisfies

$$(g_{F(n)}^{-1})'(x_0) \leq 1 \quad \text{and} \quad (g_{F(n)}^{-1})'(x_{F(n)}) = \frac{1}{g'_{F(n)}(x_0)} \geq 1.$$

The point $x_{F(n)}$ is at distance $\ell_{F(n)}$ from x_0. If $\ell_{F(n)} = o(r_n)$, then the control on the affine distortion would guarantee that the derivative of $g_{F(n)}$ is close to 1 on the neighbourhood of radius r_n. This asymptotic condition also assures that the map $\widetilde{g}_{F(n)}$ is almost the identity, since $\widetilde{g}_{F(n)}(0) = \ell_{F(n)}/r_n$.

Therefore the conclusion is enclosed in the following estimate:

Lemma 1. *Let $E \subset G$ be a finite subset of G containing the identity and define $F = E^{-1}E$. The length ℓ_F verifies*

$$\ell_F \leq C\,\frac{c_E}{S_E},$$

where the constant $C > 0$ does not depend on E.

Sketch of the proof. We observe that any two intervals $g(J_F)$ and $h(J_F)$, for $g, h \in E$, are either disjoint or one contained into the other, with equality if and only if $g \in h\,\mathrm{Stab}_G(x_0)$. Indeed, suppose that the left endpoint of $g(J_F)$ belongs to $h(J_F)$. Then $g^{-1}h(x_0)$ is closer than x_F on the right of x_0 and since $g^{-1}h \in E^{-1}E = F$, we must have $g^{-1}h(x_0) = x_0$, that is $h \in g\,\mathrm{Stab}_G(x_0)$.

Therefore the union of the intervals $g(J_F)$, for $g \in E$, covers the circle $\mathbf{S}^1$ at most c_E times. With the (quite subtle) argument in [8, Lemma 3.12] relying on the control of the affine distortion, we find

$$\ell_F \leq C\,\frac{c_E}{S_E}.$$

$\square$

3.5 Step 2: Exponential lower bound

Let us recall [20] that every element in an amalgamated product can be written in a *normal form*. Fix transversal sets of cosets $T_1 \subset G_1$ and $T_2 \subset G_2$ for $Z \backslash G_1$ and $Z \backslash G_2$ respectively, both containing the identity. Then every element $g \in G$ has a unique factorization as $g = \gamma\, t_n \cdots t_1$, with $\gamma \in Z$ and $t_j \in T_{i_j}$, with none of two consecutive i_j's equal. We shall detail more on this in §3.6 (Step 3). We can now use some of the tools provided in [8] for free groups, using the normal form of the elements. The aim of this step is to find a sequence of subsets $A(n)$ with an exponential lower bound on $S_{A(n)}$. We actually prove more, giving an exponential lower bound for the sum of the derivatives *at any point* $x \in \mathbf{S}^1$. On the one hand, this turns out to be very useful, since it gives exponential lower bounds for the sums $S_{\psi A(n) \psi^{-1}}$ associated to any conjugated set of $A(n)$ by some element $\psi \in G$. On the other side, the price we pay to have this more general statement is simply a compactness argument: since G_1 acts minimally, the same proof of [8, Proposition 2.5] yields

Proposition 6. *Let $G_1 \subset \mathrm{Diff}^2_+(\mathbf{S}^1)$ be a f.g. group whose action on $\mathbf{S}^1$ is minimal. For any $M > 0$, there exists $R_1 \in \mathbf{N}$ such that for every $x \in S^1$ we have*

$$\sum_{g \in B_1^{\times}(R_1)} g'(x) > M,$$

where $B_1^{\times}(R_1)$ is the ball $B_1(R_1)$ in G_1, but with the identity excluded.

Next we rule out possible problems given by the finite group Z:

Proposition 7. *Let $c_0 = \sup_{\gamma \in Z} \|\gamma'\|_0$ and take $M > 2c_0$ and the associated R_1. Then*

$$\sum_{g \in B_1^{\times}(R_1) \cap T_1} g'(x) > \frac{M}{c_0} - 1.$$

Proof. Using the chain rule and taking care of the identity, we obtain

$$\begin{aligned} 1 + \sum_{g_1 \in B_1^{\times}(R_1)} g_1'(x) &= \sum_{\gamma \in Z,\, t \in T_1 \,:\, \gamma t \in B_1(R_1)} (\gamma t)'(x) \\ &\leq \sup_{\gamma \in Z} \|\gamma'\|_0 \left(\sum_{t \in B_1^{\times}(R_1) \cap T_1} t'(x) + 1 \right). \end{aligned} \tag{4}$$

□

Considering now the products by representatives in T_2, it is easy to find sequence of sets $A(n)$ with an exponential lower bound on the sum of the derivatives. We actually need to consider a fixed element $\sigma \in T_2 - \{id\}$ only. Then we define the product set

$$A(n) = \sigma \left(B_1^{\times}(R_1) \cap T_1 \right) \cdots \sigma \left(B_1^{\times}(R_1) \cap T_1 \right),$$

where the product of $\sigma \left(B_1^{\times}(R_1) \cap T_1 \right)$ is repeated n times. Notice that $A(n)$ is contained in the ball of radius $n(R_1 + 1)$ in G.

Lemma 2. *Let $c_0 = \sup_{\gamma \in Z} \|\gamma'\|_0$, $s_0 = \inf \sigma'$, and take $M > (s_0^{-1} + 1)c_0$ and the associated R_1. Then there exists $a > 1$ such that for any $n \in \mathbf{N}$ and $x \in \mathbf{S}^1$,*

$$\sum_{g \in A(n)} g'(x) \geq a^{\rho(A(n))}.$$

Proof. Let us choose $\sigma \in T_2 - \{id\}$ and consider all the products σt_1, with $t_1 \in B_1^\times(R_1) \cap T_1$. We define $\bar{M} = (M/c_0 - 1) \cdot \inf \sigma'$, which is larger than 1 by assumption. With this choice, we have

$$\begin{aligned}\sum_{g \in A(n)} g'(x) &= \sum_{t_1,\ldots,t_n \in B_1^\times(R_1)\cap T_1} (\sigma t_n \cdots \sigma t_1)'(x)\\ &\geq \bar{M} \cdot \sum_{t_1,\ldots,t_{n-1} \in B_1^\times(R_1)\cap T_1} (\sigma t_{n-1} \cdots \sigma t_1)'(x),\end{aligned}$$

so that $S_{A(n)}(x) \geq \bar{M}^n$. Defining $a = \bar{M}^{\frac{1}{R_1+1}}$ we obtain the exponential lower bound, as desired. $\square$

Finally, we have:

Proposition 8. *For any $\psi \in G$, there exists a constant $C(\psi)$ such that*

$$S_{\psi A(n) \psi^{-1}} \geq C(\psi)\, a^{\rho(\psi A(n) \psi^{-1})}.$$

Proof. For $\psi \in G$, let $\lambda = \lambda(\psi)$ denote its length in the generating system $\mathcal{G}$. Then for any $n \in \mathbf{N}$, we have

$$\rho(\psi A(n) \psi^{-1}) \leq \rho(A(n)) + 2\lambda.$$

We can easily compare the sum $S_{\psi A(n)\psi^{-1}}$ with the sum of the derivatives of elements in $A(n)$:

$$\begin{aligned}S_{\psi A(n)\psi^{-1}} &= \sum_{g \in \psi A(n) \psi^{-1}} g'(x_0)\\ &= \sum_{h \in A(n)} (\psi^{-1} h \psi)'(x_0)\\ &\geq \inf(\psi^{-1})' \cdot \sum_{h \in A(n)} h'(\psi(x_0)) \cdot \psi'(x_0).\end{aligned}$$

Hence by Lemma 2, we have the inequality

$$S_{\psi A(n)\psi^{-1}} \geq \Big(\psi'(x_0) \inf(\psi^{-1})'\Big)\, a^{\rho(A(n))}.$$

With $C(\psi) = a^{2\lambda}\, \psi'(x_0) \inf(\psi^{-1})'$, the proof is over. $\square$

After the results of §3.4, we deduce the following important result. Before stating it, let us set $E(n) = \{id\} \cup A(n)$ and $F(n) = E(n)^{-1}E(n)$. We have:

Corollary 1. *Given $\varepsilon_0 > 0$ and $k \in G$, there exists $m = m(k)$ such that the element $g_{\psi F(m)\psi^{-1}}$ is locally ε_0-close to the identity in the C^0 topology in restriction to a certain complex neighbourhood of $x_0 \in$ NE.*

3.6 Step 3: Chain of commutators

As we have already explained, Proposition 3 implies that if two diffeomorphisms f_1, f_2 in G are ε_0-close to the identity over a small interval, then the sequence of commutators f_k must be eventually trivial, since G is locally discrete. We want to contradict this consequence, finding two elements f_1 and f_2 which are locally ε_0-close to id, though generate a subgroup in G which is almost free. This seems very likely since an amalgamated product over a finite group is highly similar to a free group. However, we need less: we shall not try to investigate further properties of the group generated by f_1 and f_2 other than looking at the sequence of iterated commutators.

By Corollary 1, for any $\psi \in G$ there exist $m_1 = m(id)$ and $m_2 = m(\psi)$ such that the elements $f_1 = g_{F(m_1)}$ and $f_2 = g_{\psi F(m_2)\psi^{-1}}$ are both locally ε_0-close to the identity in the C^0 topology in restriction to some complex neighbourhood of $x_0 \in \mathrm{NE}$.

Remark that by construction, the element $g_{F(m_1)}$ belongs to the set

$$F(m_1) = A(m_1) \cup A(m_1)^{-1} \cup A(m_1)^{-1}A(m_1).$$

Notice also that the intersections $F(n) \cap G_1 = \left(B_1^{\times}(R_1) \cap T_1\right)^{-1} \left(B_1^{\times}(R_1) \cap T_1\right) \subset B_1(2R_1)$ are finite and do not depend on n. Since the elements $g_{F(n)}$ do not belong to a finite set (the length $\ell_{F(n)}$ goes to zero), up to consider a larger m_1, we can suppose that $g_{F(m_1)} \notin G_1$.

Elements in $A(m_1)$ are in normal form in the amalgamated product $G = G_1 *_Z G_2$, though those in $A(m_1)^{-1}$ and $A(m_1)^{-1}A(m_1)$ could be not. In order to be able to compose different $g \in A(m_1)$ and take their inverses, we need to recall some consequences of the normal forms in amalgamated products.

If an element $g \in G = G_1 *_Z G_2$ writes in normal form as $g = \gamma\, t_n \cdots t_1$, we say that the *reduced length* $\varrho(g)$ of g is equal to n. It is easy to check that if g writes differently as $g = s_k \cdots s_1$ with every $s_j \in G_{i_j} - Z$ and none of two consecutive i_j's equal, then $k = n = \varrho(g)$ and for every $j = 1, \ldots, n$, the factor t_j belongs to G_{i_j}. In particular this implies $\varrho(g) = \varrho(g^{-1})$.

Let us consider an element $\psi \in G$ of the form $u\sigma v$, where σ is our fixed element in T_2 and u and v belong to $T_1 - B_1(2R_1)$.

We take $g_{\psi F(m_2)\psi^{-1}}$ locally ε_0-close to the identity and not in $\psi G_1 \psi^{-1}$. Then there exists $h_{F(m_2)}$ such that

$$g_{\psi F(m_2)\psi^{-1}} = \psi h_{F(m_2)} \psi^{-1} = u\sigma v\, h_{F(m_2)}\, v^{-1}\sigma^{-1}u^{-1}. \tag{5}$$

By our choice for u and v, it is clear that $g_{\psi F(m_2)\psi^{-1}}$ does not commute with $h_{F(m_1)}$. To prepare the end of the proof of Theorem A, we show this by making use of the reduced length.

Using the writing (5), the reader can easily realize that

$$\varrho(g_{\psi F(m_2)\psi^{-1}}) \geq 4,$$

since the letters $\sigma^{\pm 1}$ which appear in (5) "survive" in the normal form.

In a similar way, we can see that the commutator $[g_{\psi F(m_2)\psi^{-1}}, g_{F(m_1)}]$ verifies (we set $f_1 = g_{F(m_1)}$ and $f_2 = \psi h_2 \psi^{-1}$, $h_2 = h_{F(m_2)}$):

$$\varrho\left(\psi h_2 \psi^{-1}\, f_1\, \psi h_2^{-1} \psi^{-1}\, f_1^{-1}\right) \geq 2\varrho(f_1) + 2\varrho(h_2) + 5;$$

in particular, it is not trivial.

Proceeding by induction, we consider for $k \geq 1$ the diffeomorphism $f_{k+2} = [f_{k+1}, f_k]$. Using the combinatorics of the commutators in a free group, we get:

Proposition 9. *The series of commutators $f_{k+2} = [f_{k+1}, f_k]$, $f_1 = g_{F(m_1)}$, $f_2 = g_{\psi F(m_2)\psi^{-1}}$, never ends to id in G.*

Proof. Consider first a series of iterated commutators $(f_k)_{k\in\mathbf{N}}$ in the free group of rank 2 generated by two elements f_1 and f_2. Since a subgroup of a free group is free, the element f_k is never the identity. This implies that when writing f_{k+2} as a reduced word in the alphabet $\{f_1, f_2\}$, at least a subword of the form $f_1^{\alpha} f_2^{\beta}$ appears (α and $\beta \neq 0$).

In the group G, this subword writes as

$$f_1^{\alpha} f_2^{\beta} = f_1^{\alpha}\, u\sigma v\, h_2^{\beta}\, (u\sigma v)^{-1},$$

which has reduced length greater than 4 and therefore it is not trivial in G, unless f_1 and f_2 have torsion. However by our choice for m_1 and m_2, we know that f_1 and f_2 are not torsion elements, since otherwise they would belong to G_1 and $\psi G_1 \psi^{-1}$ respectively.

The estimate on the reduced length of $f_1^\alpha f_2^\beta$ also implies that the reduced length $\varrho(f_{k+2})$ is at least 4. Thus f_{k+2} is not the identity in G. □

4 Theorem B: Duminy revisited

The purpose of this Section is to give the proof of Theorem B, which is a version of Duminy's Theorem in the context of minimal actions satisfying property $(\star)$.

4.1 Duminy's theorem in analytic regularity

Duminy's original work deals with pseudogroups of class C^2 that act on the circle with exceptional minimal sets: a proof can be found in [24, §3]. We will mainly discuss it in the case of f.g. groups of analytic diffeomorphisms, because in this context the proof is relatively short and gives a good idea of the proof of our Theorem B.

Theorem 8 (Duminy)**.** *Let $G \leq \mathrm{Diff}_+^2(\mathbf{S}^1)$ be a f.g. group acting on $\mathbf{S}^1$ with an exceptional minimal set Λ and consider a semi-exceptional point $x \in \Lambda \cap \overline{\mathbf{S}^1 - \Lambda}$. Then the Schreier graph of the orbit $G.x$ has infinitely many ends.*

In the particular case $G \leq \mathrm{Diff}_+^\omega(\mathbf{S}^1)$, this implies that the group G itself has infinitely many ends.

Idea of the proof (in C^ω regularity). Let us give a sketch of the proof of Duminy's Theorem: we want to prove that if the conclusion fails to be true, then the group G has to preserve an *affine structure* on $\mathbf{S}^1$. This is done by using control of the affine distortion of well chosen maps: the relevant quantity here the *nonlinearity* of diffeomorphisms of the line. If $f : I \to J$ is a C^2 diffeomorphism of one dimensional manifolds, set:

$$\mathcal{N}(f) = \frac{f''}{f'}.$$

The nonlinearity of a map vanishes if and only if it is affine. Moreover, this nonlinearity operator satisfies the cocycle relation $\mathcal{N}(f \circ g) = g'\mathcal{N}(f) \circ g + \mathcal{N}(g)$.

The first step of the proof is to use Sacksteder's theorem: there exists $f \in G$, $I \subset \mathbf{S}^1$ and $p \in I$ with $f' < 1$ in I and $f(p) = p$. A way to describe an end of $\mathrm{Sch}(J, \mathcal{G})$, the Schreier graph of a *gap* $J \subset I$ (a connected component of $\mathbf{S}^1 \setminus \Lambda$), is to iterate it by f. If $g \in G$ maps J inside I, the iterates of $g(J)$ by f determine another end of $\mathrm{Sch}(J, \mathcal{G})$, which coincide with the first end if and only if the following property is satisfied: There exists a sequence of elements $h_n \in G$ with arbitrarily small nonlinearity satisfying $h_n f^n(J) = f^n g(J)$. In particular, the conjugate of g by f^{-n} is obtained by pre-composition of h_n by an element u_n of the stabilizer of the gap $f^n(J)$, which is cyclic by Hector's Lemma (Theorem 7). Looking at the derivatives of u_n, it is easy to deduce that the u_n are iterations of a single diffeomorphism, with uniformly bounded exponent: in particular, its nonlinearities are uniformly bounded.

The second step of the proof is the use of Sternberg's (or in this case Kœnigs-Poincaré's) linearization theorem: f is conjugated to a homothety h_μ of ratio $\mu = f'(p)$ by an analytic diffeomorphism $\phi : (I, p) \to (\mathbf{R}, 0)$. We are going to prove that it provides a chart with *affine holonomy* in the sense that the conjugate by ϕ^{-1} of every holonomy map of I, *i.e.* of every element

$g \in G$ satisfying $g^{-1}(I) \cap I \neq \emptyset$ has to be an affine map. Assuming for example that $\mathrm{Sch}(J, \mathcal{G})$ has one end, we use for that purpose the maps h_n and u_n previously constructed, and read them in the chart given by φ: the key points are the cocycle relation satisfied by the nonlinearity, the fact that $\mathcal{N}(h_n) \to 0$ and the fact that both u_n' and $\mathcal{N}(u_n)$ are uniformly bounded.

The third step is to use the minimality of G on Λ in order to extend the chart I to an invariant affine structure. There is a finite number of intervals I_j and of elements $g_j \in G$ satisfying $g_j(I_j) \subset I$, and the post-composition by ϕ defines charts $\phi_j : I_j \to I$. The "affine holonomy" property implies that transition functions are affine, and that any element of the group, when read in the charts, is affine. We have thus constructed an invariant affine structure.

The final step is to remark that a subgroup of automorphisms of some affine structure on $\mathbf{S}^1$ has to have a finite number of globally periodic points and thus cannot preserve a Cantor set, leading to a contradiction. Therefore, the Schreier graph $\mathrm{Sch}(J, \mathcal{G})$ has infinitely many ends. Since the stabilizer of the gap is cyclic, the group itself has infinitely many ends [13, Corollaire 2.6]. $\square$

Strategy of the proof of Theorem B – In the setting of minimal actions with non-expandable points, the strategy we adopt is similar to that of the proof of Duminy's Theorem described above.

However, in our setting, it is not an invariant affine structure, but an invariant *projective structure* that we intend to build. The relevant quantity is no longer the nonlinearity, but the *Schwarzian derivative* of diffeomorphisms of the line.

The first step of the proof will be to use a control of the *projective* distortion. Instead of using gaps of Cantor sets, we substitute them by considering the orbit of a non expandable point, and take advantage of the Markov partition for groups acting minimally with property $(\star)$. The criterion for two points of the orbit to determine the same end will be given in Lemma 7.

Then, the same argument, using Sternberg's linearization theorem, allows one to construct a chart with projective holonomy (see Lemma 8). Finally, using the minimality of the action, we extend this chart to a projective structure, just as we did above: this is Lemma 9.

4.2 Markov partition and expansion procedure

Partitions of higher level – We consider the Markov partition

$$\mathcal{I} = \left\{ I_1, \ldots, I_k, I_1^+, I_1^-, \ldots, I_\ell^+, I_\ell^- \right\}$$

the expansion constant $\lambda > 1$ and the elements $g_I \in G$, $I \in \mathcal{I}$ given by Theorem 5. In order to encode the dynamics within the orbit of the set of non expandable points, it is appropriate to define subpartitions of $\mathcal{I}$.

We define the endpoints of the atoms of the partition of level k by the following inductive procedure.

First, Δ_0 is the set of endpoints of atoms of the partition $\mathcal{I}$. Now, if Δ_k is constructed, consider $\Delta_k(I) = \Delta_k \cap I$, where $I \in \mathcal{I}$, so that $\Delta_k = \bigcup_{I \in \mathcal{I}} \Delta_k(I)$. We distinguish two possibilities:

- if I is not adjacent to a non expandable point, set

$$\Delta_{k+1}(I) = g_I^{-1}(\Delta_k \cap g_I(I));$$

- for $I \in \mathcal{I}$ adjacent to one of the non expandable points, set

$$\Delta_{k+1}(I) = \bigcup_{j=1}^{\infty} g_I^{-j}(\Delta_k \cap (g_I(I) \setminus I)).$$

Definition 6. The connected components of $\mathbf{S}^1 \setminus \Delta_k$ form a partition called the partition of level k that we denote by $\mathcal{I}_k$.

Expansion of a non expandable point − We start by the following result describing the orbits of non expandable points (see for instance [24, Lemma 3.5.14]).

Lemma 3. *Let $G \leq \mathrm{Diff}^2_+(\mathbf{S}^1)$ be a f.g. group whose action is minimal and satisfies property $(\star)$. Then a point $x \in \mathbf{S}^1$ belongs to the orbit of a non expandable point if and only if the set $\{g'(x) \mid g \in G\}$ is bounded.*

The tool of the proof is a process of expansion that we describe below. Assume that $x \in G.\mathrm{NE}$. Consider the *expansion sequence* of x, denoted by $(x_k)_{k=0}^{\infty}$, consisting of points $x_k \in G.\mathrm{NE}$ and defined as follows: First, set $x_0 = x$. Now assume that x_k has been constructed. Then there exists $I \in \mathcal{I}$ such that $x_k \in \bar{I}$ (if x_k is one of the endpoints of I, one can always ask that it is the left one). Then we have three possibilities:

- if $x_k \in \mathrm{NE}$, then the procedure stops;
- if I is not adjacent to a non expandable point, we set $x_{k+1} = g_{k+1}(x_k)$, where $g_k = g_I$;
- if the right endpoint of I is an non expandable point we set $x_{k+1} = g_{k+1}(x_k)$, where $g_{k+1} = g_{I_{j(x_k)}} g_I^{k_I(x_k)}$. Here k_I and j are the numbers defined in Theorem 5.

Proposition 10. *Let $G \leq \mathrm{Diff}^2_+(\mathbf{S}^1)$ be a f.g. group whose action is minimal, satisfies property $(\star)$ and such that $\mathrm{NE} \neq \emptyset$. Let $x \in G.\mathrm{NE}$. Then the following assertions hold true.*

i. *There exists a number $k = k(x)$, called the level of x, such that the procedure stops after k steps.*

ii. *Let $\mathbf{g}_x$ denote the composition $g_k\, g_{k-1} \cdots g_1$ and J_x^+ denote $\mathbf{g}_x^{-k}(I_j^+(x_k))$ (note that x is the left endpoint of J_x^+ and $x_k = \mathbf{g}_x(x) \in \mathrm{NE}$). Then there exists a number $\kappa = \kappa(x) \geq k$ such that J_x^+ is an atom of $\mathcal{I}_\kappa$, the partition of level κ.*

iii. *There exists a constant $C_0 > 0$ which does not depend on $x \in G.\mathrm{NE}$ such that $\varkappa(\mathbf{g}_x, J_x^+) \leq C_0$.*

Proof. Note that $g_j'(x) \geq \lambda > 1$ for every j. Since $x \in G.\mathrm{NE}$, by Lemma 3, the quantity $(g_j\, g_{j-1} \cdots g_1)'(x) \geq \lambda^j$ has to be bounded. This is possible if and only if the expansion procedure described above stops at a step k.

That the intervals J_x^+ are atoms of the partition of some level κ is clear from the definition of the two procedures.

The map $\mathbf{g}_x$ is precisely the *expansion map* of J_x^+, in the sense of [11, Definition 7]. Thus the third assertion follows from [11, Proposition 2], because the size of $\mathbf{g}_x(J_x^+)$ is uniformly bounded from below. ◻

Lemma 4. *The following assertions hold true.*

i. *The family $(J_x^+)_{k(x)=k}$ consists of disjoint intervals.*

ii. *There exists a constant $C > 1$ which does not depend on $x \in G.\mathrm{NE}$ such that*

$$\frac{C^{-1}}{|J_x^+|} \leq \mathbf{g}_x'(x) \leq \frac{C}{|J_x^+|}.$$

Proof. By Proposition 10.ii, each interval J_x^+ is an atom of some partition of level $\kappa(x)$. This implies that two different intervals J_x^+ either are disjoint, or one is contained into the other.

Assume for example that J_x^+ contains J_y^+ for some $x, y \in G.\mathrm{NE}$. Then we claim that $k(x) < k(y)$. Indeed the maps g_i defined by the expansion procedure of x and y must coincide at least before the procedure stops for x. It stops for x when $i = k$, and $x = x_k$. Then $\mathbf{g}_x(y) = y_k$ lies strictly inside $I^+_{j(x_k)}$, which contains no non expandable point. Hence the expansion procedure of y must continue after the k-th step, and we have $k(x) < k(y)$ as desired.

The second assertion directly follows from Proposition 10.iii. □

4.3 Control of the projective distortion

Distortion control – From [11, Lemma 5] we have:

Lemma 5. *The stabilizer* $\mathrm{Stab}_G(x_0)$ *(in the C^2 setting, considered in the group of germs) is an infinite cyclic group, generated by some $h \in G$.*

We introduce a function $\mathcal{E} : X \to (0, 1]$, that we will call the *energy* (and which is, in fact, the inverse of the function defined in [11]), defined on the orbit $X = G.x_0$ as

$$\mathcal{E}(g(x_0)) = g'(x_0) \quad \text{for every } g \in G. \tag{6}$$

The map is well defined. Indeed, assume that $x = g_1(x_0) = g_2(x_0)$ for $g_1, g_2 \in G$. Then the element $g_2^{-1}g_1$ fixes x_0. Since this point is non expandable, we must have $(g_2^{-1}g_1)'(x_0) = 1$, hence $g_1'(x_0) = g_2'(x_0)$.

Lemma 6. *The series* $\sum_{x\in X} \mathcal{E}(x)^2$ *converges.*

Proof. Let $x \in X$, and let $\mathbf{g}_x$ be the map obtained in Proposition 10. We have $\mathcal{E}(x) = \mathbf{g}_x'(x)^{-1}$. By Lemma 4, the ratio between $\mathcal{E}(x)$ and $|J_x^+|$ is uniformly bounded away from 0 and ∞. Therefore, it is enough to prove that the series $\sum_{x\in X} |J_x^+|^2$ is convergent.

We can decompose this sum as

$$\sum_{k=0}^{\infty} \sum_{k(x)=k} |J_x^+|^2 \le \sum_{k=0}^{\infty} \left(\left(\max_{x\,:\,k(x)=k} |J_x^+| \right) \sum_{k(x)=k} |J_x^+| \right). \tag{7}$$

We first note that $|J_x^+|$ can be controlled by a term of the order of $\lambda^{-k(x)}$, because by construction we have $\mathbf{g}_x'(x) \ge \lambda^{k(x)}$.

Using Lemma 4, we get the following inequality holding for every $k \in \mathbf{N}$:

$$\sum_{k(x)=k} |J_x^+| \le |\mathbf{S}^1| = 1.$$

This suffices to prove that the upper bound in (7) is controlled by a converging geometric sum. □

The Schwarzian energy – If $f \in \mathrm{Diff}^3_+(\mathbf{S}^1)$, we define its *Schwarzian derivative* by the classical expression

$$\mathcal{S}(f) = \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2.$$

We have the following cocycle formula:

$$\mathcal{S}(f \circ g) = (g')^2 \cdot \mathcal{S}(f) \circ g + \mathcal{S}(g). \tag{8}$$

Recall that the stabilizer of x_0 is generated by some $h \in G$, which moreover verifies $h'(x_0) = 1$; we set $b = \mathcal{S}(h)(x_0)$. From this we can define a new function on the orbit X of x_0:

Proposition - Definition. *The* Schwarzian energy *is the function*

$$\begin{array}{rccc} Q: & X & \longrightarrow & \mathbf{R}/b\mathbf{Z} \\ & g(x_0) & \longmapsto & \mathcal{S}(g)(x_0) \end{array}$$

(where the quotient $\mathbf{R}/b\mathbf{Z}$ can possibly be trivial, if $b = 0$).

Proof. We have to check that the function Q is well-defined. Assume that $x = g_1(x_0) = g_2(x_0)$ for some $g_1, g_2 \in G$. By Lemma 5, we have $g_1 = g_2 h^k$ for some $k \in \mathbf{Z}$. Using the cocycle relation (8) and the fact that $h'(x_0) = 1$, we find

$$\mathcal{S}(g_1)(x_0) = \mathcal{S}(g_2)(x_0) + k\,\mathcal{S}(h)(x_0).$$

which is equal to $\mathcal{S}(g_2)(x_0) \pmod b$. □

An immediate corollary of (8) is

$$Q(f(x)) = \mathcal{E}(x)^2 \cdot \mathcal{S}(f)(x) + Q(x). \tag{9}$$

Distinguishing the ends – The following lemma provides a criterion to distinguish ends of the Schreier graph of x_0, that we identified with the orbit X. It is much in Duminy's spirit: see for example [24, Lemma 3.4.2].

Lemma 7.

i. *Let $(x_n)_{n\in\mathbf{N}}$ be a sequence of points in X which goes to infinity. Then $\lim_{n\to\infty} Q(x_n)$ exists.*

ii. *If $(x_n)_{n\in\mathbf{N}}, (y_n)_{n\in\mathbf{N}}$ determine the same ends in X, then*

$$\lim_{n\to\infty} Q(x_n) = \lim_{n\to\infty} Q(y_n).$$

Proof. It is enough to prove the first assertion in the following case: $x_n = g_n \cdots g_1(x_0)$, where $(g_n)_{n\in\mathbf{N}}$ is a sequence of elements of the (symmetric) system of generators of $\mathcal{G}$ (recall that the notions of going to infinity and of ends are independent of the system of generators).

Using (9), we get

$$Q(x_{n+1}) - Q(x_n) = \mathcal{E}(x_n)^2 \cdot \mathcal{S}(g_{n+1})(x_n).$$

Using Lemma 6 and an upper bound for the Schwarzian derivatives of the generators, we easily get that the sequence $(Q(x_n))_{n\in\mathbf{N}}$ is a Cauchy sequence, and hence converges.

We have the convergence of the sequence $(Q(x_n) - Q(y_n))_{n\in\mathbf{N}}$, and we have to prove that the limit is 0 in the case where x_n and y_n converge to the same end. Let $\varepsilon > 0$ and n_0 such that $\sum_{x\notin X(n_0)} \mathcal{E}(x)^2 < \varepsilon$, where $X(n_0)$ denotes the set of those $x \in X$ at distance no greater than n_0 to x_0 for the word distance in X.

Assume that x_n and y_n converge to the same end. When n is large enough, there exists a path linking x_n and y_n which avoids $X(n_0)$. Using the same type of argument as above, we get that $|Q(x_n) - Q(y_n)|$ is smaller than ε times a uniform constant which only depends on the system of generators. Since ε is arbitrary, this concludes the proof of the lemma. □

4.4 Invariant projective structure

After all the preliminaries, we intend here to prove Theorems B and C. Even if the statement of Theorem C involves *pseudogroups* of diffeomorphisms of a one dimensional manifold, we rather prefer to consider *groups* of circle diffeomorphisms only: the arguments are exactly the same, though probably the reader will be more acquainted with the latter.

Consider a f.g. subgroup $G \leq \mathrm{Diff}^r_+(\mathbf{S}^1)$, $r \geq 3$, which acts minimally, possesses property $(\star)$, and has at least one non expanding point x_0. *We will assume that the Schreier graph of x_0 has finitely many ends.* Indeed, if there were infinitely many, since the stabilizer of x_0 is cyclic (Lemma 5), the argument explained by Ghys in [13, Corollaire 2.6] implies that the group itself has infinitely many ends.

The goal is to produce a projective structure which is invariant for the action of G.

A projective chart – We begin the proof of Theorem C by the construction of a single projective chart. We will next use the minimality of the action to construct a projective atlas.

The action of G on $\mathbf{S}^1$ is at least C^2, minimal and non-elementary (it does not preserve any measure). The usual Sacksteder's theorem, which classically holds is the context of groups acting with exceptional minimal sets, applies: see [9, Theorem 1.2.7] for a proof due to Ghys, see also [6, Proposition 4.1] for a probabilistic proof. According to this version, the group G acts on $\mathbf{S}^1$ with *hyperbolic holonomy*.

More precisely, there exists a point $p \in \mathbf{S}^1$ and an element $f \in G$ with $f(p) = p$ and $\mu = f'(p) < 1$. Sternberg's linearization theorem [24, Section 3.6.1] provides an interval I about p, as well as a C^r-diffeomorphism $\varphi : (I, p) \to (\mathbf{R}, 0)$, with $\varphi(p) = 0$ and

$$\varphi\, f\, \varphi^{-1} = h_\mu,$$

where h_μ denotes the homothety $x \mapsto \mu x$.

Lemma 8 (Projective holonomy)**.** *Assume that the Schreier graph of x_0 has finitely many ends. Then the chart (I, φ) has projective holonomy. More precisely, for every $\gamma \in G$ such that $J = \gamma^{-1}(I) \cap I \neq \emptyset$, the following equality holds on $\varphi(J)$:*

$$\mathcal{S}(\varphi\gamma\varphi^{-1}) = 0.$$

Proof. Assume that G has finitely many ends. It comes from Lemma 7 that for every $x \in I \cap X$, the limit $\lim_{n\to\infty} Q(f^n(x))$ exists and there is a finite set $\mathbf{q} = \{q_1, \ldots, q_\ell\}$ such that

$$\lim_{n\to\infty} Q(f^n(x)) \in \mathbf{q} + b\mathbf{Z}.$$

Now let $x = g(x_0) \in I \cap X$. Note that any homothety has zero Schwarzian derivative. Hence the cocycle relation (8) implies the following equality:

$$\begin{aligned} Q(f^n(x)) &= \mathcal{S}(\varphi^{-1}\, h_\mu^n\, \varphi g)(x_0) \\ &= \mu^{2n} (\varphi g)'(x_0)^2 \cdot \mathcal{S}(\varphi^{-1})(\mu^n \varphi g(x_0)) + \mathcal{S}(\varphi g)(x_0). \end{aligned}$$

Letting n go to infinity, we find $\lim_{n\to\infty} Q(f^n(x)) = \mathcal{S}(\varphi g)(x_0)$. The latter shows that for every $g \in G$ satisfying $g(x_0) \in I$, we have that the Schwarzian derivative $\mathcal{S}(\varphi g)(x_0)$ belongs to the discrete set $\mathbf{q} + b\mathbf{Z}$.

Now consider a holonomy map of I, *i.e.* an element $\gamma \in G$ satisfying $J = \gamma^{-1}(I) \cap I \neq \emptyset$. Note that by minimality, the set $J \cap X$ is dense in J. So let $x \in J \cap X$: we can write $x = g(x_0)$ for some $g \in G$. Since $x \in J$, we also have $\gamma g(x_0) = \gamma(x) \in I$. We deduce that both $\mathcal{S}(\varphi g)(x_0)$ and $\mathcal{S}(\varphi\,\gamma g)(x_0)$ are in $\mathbf{q} + b\mathbf{Z}$. By (9), their difference is

$$\mathcal{S}(\varphi\,\gamma g)(x_0) - \mathcal{S}(\varphi g)(x_0) = \varphi'(x)^2\,\mathcal{E}(x)^2 \cdot \mathcal{S}(\varphi\,\gamma\varphi^{-1})(\varphi(x)) \in \mathbf{q} - \mathbf{q} + b\mathbf{Z}.$$

The set $\mathbf{q} - \mathbf{q} + b\mathbf{Z}$ is discrete in $\mathbf{R}$ and contains 0, so there is $\delta > 0$ such that if

$$\left|\varphi'(x)^2\,\mathcal{E}(x)^2 \cdot \mathcal{S}(\varphi\,\gamma\varphi^{-1})(\varphi(x))\right| < \delta$$

then $\varphi'(x)^2\,\mathcal{E}(x)^2 \cdot \mathcal{S}(\varphi\,\gamma\varphi^{-1})(\varphi(x)) = 0$. Since $\varphi'(x)^2\,\mathcal{E}(x)^2 > 0$, the latter condition implies $\mathcal{S}(\varphi\,\gamma\varphi^{-1})(\varphi(x)) = 0$.

By compactness, there is $M > 0$ such that

$$\sup_J \left|\varphi' \cdot \mathcal{S}(\varphi\,\gamma\varphi^{-1}) \circ \varphi\right| \leq M.$$

Consider the set X' of points $x \in X$ such that $\mathcal{E}(x)^2 < \frac{\delta}{M}$, which contains all but finitely many points of X. The set $X' \cap J$ is dense in J and the condition that its points verify implies that $\mathcal{S}(\varphi\,\gamma\varphi^{-1})(\varphi(x)) = 0$ for every $x \in X' \cap J$. Since the orbit $X \cap J$ is dense in J, so is $X' \cap J$. We have just shown that the Schwarzian derivative of $\varphi\,\gamma\,\varphi^{-1}$ vanishes on a dense set of $\varphi(J)$ and hence $\varphi\,\gamma\,\varphi^{-1}$ is projective on $\varphi(J)$. □

Invariant projective structure – By compactness of $\mathbf{S}^1$ and minimality of the action of G, there exists a finite number of open intervals $(I_j)_{j=1}^m$ and a finite number of elements of the group $(g_j)_{j=1}^m$ such that:

1. the family $(I_j)_{j=1}^m$ is an open cover of $\mathbf{S}^1$,
2. for every $j = 1, \ldots, k$, we have $g_j(I_j) \subset I$.

Lemma 9 (Invariant projective structure)**.** *For $j = 1, \ldots, m$, we set $\varphi_j = \varphi \circ g_j : I_j \to \mathbf{R}$.*

i. *The atlas $(I_j, \varphi_j)_{j=1}^m$ defines a projective structure on $\mathbf{S}^1$, i.e. for every j, k with $I_j \cap I_k \neq \emptyset$, we have:*
$$\mathcal{S}(\varphi_k \varphi_j^{-1}) = 0.$$

ii. *The projective structure is G-invariant, i.e. for every $g \in G$ and j, k satisfying $g^{-1}(I_k) \cap I_j \neq \emptyset$, we have:*
$$\mathcal{S}(\varphi_k\, g\, \varphi_j^{-1}) = 0.$$

Proof. For every $g \in G$, when $g^{-1}(I_k) \cap I_j \neq \emptyset$, the map $g_k g g_j^{-1}$ is a holonomy map of I.

Hence this lemma is a direct application of the fact that (I, φ) has projective holonomy (see Lemma 8). □

Rigidity – On the circle, there is a canonical projective structure which is given by that of $\mathbf{RP}^1$, and whose group of automorphisms is $PSL_2(\mathbf{R})$.

In the proof of [14, Lemme 5.1], Ghys establishes the following alternative for the group of automorphisms of a projective structure on $\mathbf{S}^1$:

- either it is abelian;

- or it is isomorphic to $PSL_2^{(k)}(\mathbf{R})$, the k-fold covering of the canonical projective structure for some $k \in \mathbf{N}$.

The projective structure we constructed in Lemma 9 cannot have an abelian group of automorphism, since G realizes as a subgroup and is not abelian.

Hence, the group of automorphism of our invariant projective structure has to be conjugated to $PSL_2^{(k)}(\mathbf{R})$. In order to see that the conjugacy is C^r, notice that it is given by the developing map, which is C^r by construction.

We conclude that G is C^r-conjugated to a (discrete) subgroup of $PSL_2^{(k)}(\mathbf{R})$. This immediately gives us the desired conclusion: by our assumption, there are non expandable points, which means that there are parabolic elements in G, hence G is virtually the fundamental group of an hyperbolic surface with non-empty boundary and so virtually free and with infinitely many ends.

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The alternative approach to Duminy's Theorem probably arose some time ago after Deroin's result about minimal expanding actions of locally discrete groups.

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