Some applications of the (bary)center method to the study of cocycles of isometries

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Cylindrical cascades

Poincaré

Given $\alpha \notin \mathbb{Q}/\mathbb{Z}$ and $\psi\colon \mathrm{S}^1 \to \mathbb{R}$, we consider

$$F\colon \mathrm{S}^1 \times \mathbb{R} \to \mathrm{S}^1 \times \mathbb{R}$$

$$(x,t) \mapsto \big(x+\alpha, t+\psi(x)\big)$$

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$$S_n(\psi)(x) = \psi(x) + \psi(T(x)) + \cdots + \psi(T^{n-1}(x))$$

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If T is minimal, the existence of a continuous solution φ is equivalent to that of a point $x_0 \in X$ for which the Birkhoff sums $S_n(\psi)(x_0)$ are uniformly bounded.

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If T is a C^2 top. transitive, Anosov diffeomorphism and ψ is Hölder, then the existence of a Hölder solution φ is equivalent to that for each periodic point $x = T^n(x)$ one has $S_n(\psi)(x) = 0$.

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Regularity: Livšic, Guillemin-Kazhdan, Hurder-Katok, de la Llave, Nitika-Torok,...

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Γ: semigroup acting continuously on X and for each $g \in \Gamma$ there is a map $I(g, \cdot) \colon X \to Isom(H)$ such that

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Example: $\Gamma \sim \mathbb{Z} = \langle T \rangle, H = \mathbb{R} \implies I(n,x)(t) := t + S_n(\psi)(x)$

Example: $X =$ one point $\implies$ action of Γ by isometries on H

The general cohomological equation

There exist $\Theta(g,x) \in U(H)$ and $\psi(g,x) \in H$ such that

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We search for $\varphi\colon X \to H$ such that

$$\varphi(g(x)) - \Theta(g,x)\varphi(x) = \psi(x)$$

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Whenever X = point, Bruhat-Tits' lemma applies.

Theorem (Coronel, N., Ponce)

If the Γ-action on X is minimal, then the existence of a bounded orbit for the skew dynamics implies (is equivalent to) the existence of a continuous solution to the cohomological equation.

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$H \rightsquigarrow$ CAT(0) or uniformly locally convex space.

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– In finite dimension, this map is always continuous (elementary, but nontrivial).
– In infinite dimension, this map may fail to be continuous. However a careful choice of S yields a continuous section (still elementary, but more subtle).

Isometric actions on $C(X,H)$, $L^\infty(X,H)$, and $L^1(X,R)$

• A skew-action I leads to an action on a function space (graph transform):

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Every action by isometries on $L^1(X, R)$ with a bounded orbit has a fixed point. The same holds for preduals of von Newmann algebras.

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A Hedlund type theorem for matrix cocycles

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- We take $H = Pos(n)$ on which $GL(n, R)$ acts by isometries:

$$g : A \mapsto gAg^t$$

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Let $A : \Gamma \mapsto GL(n, R)$ be a cocycle with respect to a minimal Γ-action on X. Assume there exists $C > 0$ such that

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Then A is cohomologous to a cocycle taking values in $O(n,R)$: there exists a continuous $B : X \to GL(n,R)$ such that

$$B(g(x))A(x)B(x)^{-1} \in O(n,R), \qquad \text{for all } g \in \Gamma, x \in X.$$

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Almost-invariant sections: reduced cohomology

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Theorem (Bochi-N., extending Avila-Bochi-Damanik)

Let $I : X \to Isom(H)$ be a cocycle such that the drift $n \to dist(v_0, I(n,x)v_0)$ is uniformly sublinear in x. Then there exist almost-solutions.

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Using Furman's theorem we conclude:

Theorem (Bochi-N.)

Let $A : X \to GL(n,R)$ be a cocycle all of whose Lyapunov exponents are zero. Then A is C^0-close to a cocycle that is cohomologous to a cocycle taking values in $O(n)$.

Proof à la Følner

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$$\varphi_n(x) := bar\{v_0, I(1, T^{-1}(x))v_0, ..., I(n-1, T^{-(n-1)}(x))v_0\}$$

$$\varphi_n(T(x)) = \varphi_n(x) + \psi(x) - \frac{S_{n-1}(\psi)(T^{-(n+1)}(x))}{n}$$

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X : nonpositively curved in the sense of Buseman

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$$d\left(bar_n(x_1, \ldots, x_n), bar_n(y_1, \ldots, y_n)\right) \leq \frac{1}{n} \sum_{i=1}^{n} d(x_i, y_i)$$

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An L^1 ergodic theorem in nonpositive curvature

In general:

$$d\big(bar^*(\mu_1), bar^*(\mu_2)\big) \leq W_1(\mu_1, \mu_2) := \inf_{\nu\text{:joining}} \int_{X\times X} d(x,y)\, d\nu(x,y)$$

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If T on $(\Omega, \mathbb{P})$ is measure preserving $\varphi\colon \Omega \to X$ lies in $L^1(\Omega, X)$, then almost surely and in $L^1(\Omega, X)$ one has the convergence of

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This "extends" a theorem of T. Austin (previous results by Es-Sahib-Heinich and K.-T. Sturm)

Many thanks

Many thanks

• D. Coronel, A. Navas, M. Ponce. *Bounded orbits versus invariant curves for cocycles of affine isometries.*
• D. Coronel, A. Navas, M. Ponce. *On the dynamics of non-reducible cylindrical vortices.*
• J. Bochi, A. Navas. *A geometric path from zero Lyapunov exponents to invariant sections for cocycles.*
• A. Navas. *An L^1-ergodic theorem in nonpositively curved spaces via a canonical barycenter map.*