Rigidity for C^1 actions on the interval arising from hyperbolicity I: solvable groups

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Abstract: We consider Abelian-by-cyclic groups for which the cyclic factor acts by hyperbolic automorphisms on the Abelian subgroup. We show that if such a group acts faithfully by C^1 diffeomorphisms of the closed interval with no global fixed point at the interior, then the action is topologically conjugate to that of an affine group. Moreover, in case of non-Abelian image, we show a rigidity result concerning the multipliers of the homotheties, despite the fact that the conjugacy is not necessarily smooth. Some consequences for non-solvable groups are proposed. In particular, we give new proofs/examples yielding the existence of finitely-generated, locally-indicable groups with no faithful action by C^1 diffeomorphisms of the interval.

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1 Introduction

1.1 General panorama

The dynamics of (non-)solvable groups of germs of diffeomorphisms around a fixed point is an important subject that has been studied by many authors in connexion to foliations and differential equations. There is, however, a natural group-theoretical aspect of this study of large interest. In this direction, the classification of solvable groups of diffeomorphisms in dimension 1 has been completed, at least in large regularity: see [6, 16] for the real-analytic case and [29] for the C^2 case; see also [3, 30] for the piecewise-affine case. (For the higher-dimensional case, see [1, 21].)

In the C^1 context, this issue was indirectly addresed by Cantwell and Conlon in [8]. Indeed, although they were interested on problems concerning smoothing of some codimension-1 foliations, they dealt with a particular one for which the holonomy pseudo-group turns to be the Baumslag-Solitar group. In concrete terms, they proved that a certain natural (non-affine) action of $BS(1,2)$ on the closed interval is non-smoothable. Later, using the results of topological classification of general actions of $BS(1,2)$ on the interval contained in [33], the whole[1] picture was completed in [18]: every C^1 action of $BS(1,n)$ on the closed interval with no global fixed point inside is semiconjugate to the standard affine action.

Cantwell-Conlon's proof uses exponential growth of the orbit of certain intervals to yield a contradiction (such a behaviour is impossible close to a parabolic fixed point). This clever argument was later used in [26] to give a counter-example to the converse of the Thurston stability theorem: there exists a finitely-generated, locally indicable[2] group with no faithful action by C^1 diffeomorphisms of the interval. (See also [7].) As we will see, the relation with Thurston's stability arises

[1] Some of the results of this work strongly complements this. For instance, as we state below, the semiconjugacy is necessarily a (topological) conjugacy.

[2] Recall that a group is said to be *locally indicable* if every nontrivial, finitely-generated subgroup has a surjective homomorphism onto $\mathbb{Z}$. Every such group admits a faithful action by homeomorphisms of the interval provided it is countable; see [27].

not only at the level of results. Indeed, although Cantwell-Conlon's argument is very different, an arsenal of techniques close to Thurston's that may be applied in this context and related ones (see *e.g.* [21]) was independently developed in [4] (see also [5]). The aim of this work is to put together all these ideas (and to introduce new ones) to get a quite complete picture of all possible C^1 actions of a very large class of solvable groups, namely the Abelian-by-cyclic ones. We will show that these actions are rigid provided the cyclic factor acts hyperbolically on the Abelian subgroup, and that this rigidity dissapears in the non-hyperbolic case.

The idea of relating a certain notion of hyperbolicity (or at least, of growth of orbits) to C^1 rigidity phenomena for group actions on 1-dimensional spaces has been proposed –though not fully developed– by many authors. This is explicitly mentioned in [26], while it is implicit in the examples of [34]. More evidence is provided by the examples in [9, 12, 28] relying on the original constructions of Pixton [31] and Tsuboi [37]. All these works suggest that actions with orbits of (uniformly bounded) subexponential growth should be always C^1-smoothable[3] (compare [8, Conjecture 2.3]) and realizable in any neighborhood of the identity/rotations [24]. Despite this evidence and the results presented here, a complete understanding of all rigidity phenomena arising in this context remains far from being reached. More generally, the full picture of groups of homeomorphisms that can/cannot act faithfully by C^1 diffeomorphisms remains obscure. A particular case that is challenging from both the dynamical and the group-theoretical viewpoints can be summarized in the next

Question 1.1. What are the subgroups of the group of piecewise affine homeomorphisms of the circle/interval that are topologically conjugate to groups of C^1 diffeomorphisms ?

For simplicity, in this work, we will only consider actions by orientation-preserving maps.

1.2 Statements of Results

Given a matrix $A = (\alpha_{i,j}) \in M_d(\mathbb{Z}) \cap GL_d(\mathbb{R})$, $d \geq 1$, let us consider the meta-Abelian group G_A with presentation

$$G_A := \big\langle a, b_1, \ldots, b_d \mid b_i\, b_j = b_j\, b_i \ , \ \ ab_i a^{-1} = b_1^{\alpha_{1,i}} \cdots b_d^{\alpha_{d,i}} \big\rangle. \tag{1}$$

It is known that every finitely-presented, torsion-free, Abelian-by-cyclic group has this form [2] (see also [13]).

It is quite clear that $M_d(\mathbb{Z}) \cap GL_d(\mathbb{R}) \subset GL_d(\mathbb{Q})$. In particular, the group G_A above is isomorphic to a subgroup of $\mathbb{Z} \ltimes_A \mathbb{Q}^d$. In a slightly more general way, from now on we consider $A \in GL_d(\mathbb{Q})$, and we let $G = \mathbb{Z} \ltimes_A H$ be a non-Abelian finitely generated subgroup of $\mathbb{Z} \ltimes_A \mathbb{Q}^d$ such that $rank_{\mathbb{Q}}(H) = d$. Observe that G may fail to be finitely-presented. We can easily describe the groups G as above admitting a faithful affine action.

Proposition 1.2. *Suppose that the matrix $A \in GL_d(\mathbb{Q})$ is $\mathbb{Q}$-irreducible and that the $\mathbb{Q}$-rank of $H \subset \mathbb{Q}^d$ equals d. Then $\mathbb{Z} \ltimes_A H$ has a faithful affine action if and only if A has a positive real eigenvalue.*

Next, we assume that A has all its eigenvalues of norm $\neq 1$. Our main result is the following

[3]Actually, they should be $C^{1+\tau}$-smoothable in case of polynomial growth, with τ depending on the degree of the polynomial; see [9, 11, 23].

Theorem 1.3. *Assume $A \in GL_d(\mathbb{Q})$ has no eigenvalue of norm 1, and let G be a subgroup of $\mathbb{Z} \ltimes_A \mathbb{Q}^d$ of the form $G = \mathbb{Z} \ltimes_A H$, where $rank_{\mathbb{Q}}(H) = d$. Then every representation of G into $\mathrm{Diff}^1_+([0,1])$ whose image group admits no global fixed point in $(0,1)$ is topologically conjugate to a representation into the affine group.*

For the proof of Theorem 1.3, let us begin by considering an action of a general group G as above by homeomorphisms of $[0,1]$. We have the next generalization of [33, §4.1]:

Lemma 1.4. *Let G be a group as in Theorem 1.3. Assume that G acts by homeomorphisms of the closed interval with no global fixed point in $(0,1)$. Then either there exists $b \in H$ fixing no point in $(0,1)$, in which case the action of G is semiconjugate to that of an affine group, or H has a global fixed point in $(0,1)$, in which case the element $a \in G$ acts without fixed points inside $(0,1)$.*

In virtue of this lemma, the proof of Theorem 1.3 reduces to the next two propositions.

Proposition 1.5. *Let G be a group as in Theorem 1.3. Assume that G acts by homeomorphisms of $[0,1]$. If the subgroup H acts nontrivially but has a global fixed point inside $(0,1)$, then the action of G cannot be by C^1 diffeomorphisms.*

Proposition 1.6. *Let G be a group as in Theorem 1.3. Then every representation of G into $\mathrm{Diff}^1_+([0,1])$ with non-Abelian image is minimal on $(0,1)$.*

The structure theorem for actions is complemented by a result of rigidity for the multipliers of the group elements mapping into homotheties. More precisely, we prove

Theorem 1.7. *Let $G = \mathbb{Z} \ltimes_A H$ be a group as in Theorem 1.3, with $a \in G$ being the generator of $\mathbb{Z}$ (whose action on H is given by A). Assume that G acts by C^1 diffeomorphisms of $[0,1]$ with no fixed point in $(0,1)$ and the image group is non-Abelian. Then the derivative of a at the interior fixed point coincides with the ratio of the homothety to which a is mapped under the homomorphism of G into the affine group given by Theorem 1.3. More generally, for each $k \neq 0$ and all $b \in H$, the derivative of $a^k b$ at its interior fixed point equals the k^{th}-power of the ratio of that homothety.*

Besides several consequences of the preceding theorem given in the next section, there is an elementary one of particular interest. Namely, if we consider actions as in Theorem 1.3 but allowing the possibility of global fixed points in $(0,1)$, then only finitely many components of the complement of the set of these points are such that the action restricted to them has non-Abelian image. Otherwise, the element a would admit a sequence of hyperbolic fixed points, all of them with the same multiplier, converging to a parabolic fixed point, which is absurd.

Theorem 1.7 could lead one to think that the topological conjugacy to the affine action is actually smooth at the interior.[4] (Compare [35].) Nevertheless, a standard application of the Anosov-Katok technique leads to C^1 (faithful) actions for which this is not the case. As we will see, in higher regularity, the rigidity holds: if $r \geq 2$, then for every faithful action by C^r diffeomorphisms with no interior global fixed point, the conjugacy is a C^r diffeomorphism at the interior. It seems to be an interesting problem to try to extend this rigidity to the class $C^{1+\tau}$. Another interesting problem is to construct actions by C^1 diffeomorphisms that are conjugate to actions of non-Abelian affine groups though they are non-ergodic with respect to the Lebesgue measure. (Compare [20].)

The hyperbolicity assumption for the matrix A is crucial for the validity of Theorem 1.3. Indeed, Abelian groups of diffeomorphisms acting nonfreely (as those constructed in [37]) provide easy counter-examples with all eigenvalues equal to 1. A more delicate construction leads to the next

[4] In general, the conjugacy above is not smooth at the endpoints even in the real-analytic case: see [6] for a very complete discussion on this.

Theorem 1.8. *Let $A \in GL_d(\mathbb{Q})$ be non-hyperbolic and $\mathbb{Q}$-irreducible. Then $G := \mathbb{Z} \ltimes_A \mathbb{Q}^d$ admits a faithful action by C^1 diffeomorphisms of the closed interval that is not semiconjugate to an affine one though has no global fixed point in $(0, 1)$.*

This work is closed by some extensions of our main theorem to actions by C^1 diffeomorphisms of the circle. Roughly, we rule out Denjoy-like actions in class C^1 for the groups G above, though such actions may arise in the continuous cathegory (and also in the Lipschitz one; see [25, Proposition 2.3.15]). In particular, we have:

Theorem 1.9. *Let G be a group as in Theorem 1.3. Assume that G acts by C^1 diffeomorphisms of the circle with non-Abelian image. Then the action admits a finite orbit.*

This theorem clarifies the whole picture. Up to a finite-index subgroup G_0, the action has global fixed points. The group G_0 can still be presented in the form $\mathbb{Z} \ltimes_{A^k} H_0$ for a certain $k \geq 1$; as A^k is hyperbolic, and application of Theorem 1.3 to the restriction of the action of G_0 to intervals between global fixed points shows that these are conjugate to affine actions. Thus, roughly, G is a finite (cyclic) extension of a subgroup of a product of affine groups acting on intervals with pairwise disjoint interior. Moreover, only finitely many of these affine groups can be non-Abelian. (Otherwise, by Theorem 1.7, there would be accumulation of hyperbolic fixed points of a^k with the same multiplier towards a parabolic fixed point.)

To conclude, let us mention that the examples provided by Theorem 1.8 can be adapted to the case of the circle. More precisely, if $A \in GL_d(\mathbb{Q})$ is non-hyperbolic and $\mathbb{Q}$-irreducible, then $G := \mathbb{Z} \ltimes_A \mathbb{Q}^d$ admits a faithful action by C^1 circle diffeomorphisms having no finite orbit.

1.3 Some comments and complementary results/examples

Although the results presented so far only concern certain solvable groups, they lead to relevant results for other classes of groups. Let us start with an almost direct consequence of Theorem 1.7. For any pair of positive integers m, n, let $BS(1, m; 1, n)$ be the group defined by

$$BS(1, m; 1, n) := \langle a, b, c \mid aba^{-1} = b^m,\, aca^{-1} = c^n \rangle = BS(1, m) *_{\langle a \rangle} BS(1, n).$$

In other words, the subgroups generated by a, b and a, c are isomorphic to $BS(1, m)$ and $BS(1, n)$, respectively, and no other relation is assumed.

Notice that every element $\omega \in BS(1, m; 1; n)$ can be written in a unique way as $\omega = a^k \omega_0$, where $k \in \mathbb{Z}$ and ω_0 belongs to the subgroup generated by b, c and their roots. One easily deduces that $BS(1, m; 1, n)$ is locally indicable, hence it admits a faithful action by homeomorphisms of the interval (see the second footnote in page 1). However, it is easy to give a more explicit embedding of $BS(1, m; 1, , n)$ into $\mathrm{Homeo}_+([0, 1])$. Indeed, start by associating to a a homeomorphism f without fixed points in $(0, 1)$. Then choose a fundamental domain I of f and homeomorphisms g_0, h_0 supported on I and generating a rank-2 free group. Finally, extend g_0 and h_0 into homeomorphisms g, h of $[0, 1]$ so that $fgf^{-1} = g^m$ and $fhf^{-1} = h^n$ hold. Then the action of $B(1, m; 1, n)$ defined by associating g to b and h to c is faithful.

In what concerns smooth actions of $BS(1, m; 1, n)$ on the interval, we have:

Theorem 1.10. *Let m, n be distinct positive integers. Given a representation of $B(1, m; 1, n)$ into $\mathrm{Diff}^1_+([0, 1])$, let us denote by f, g, h the images of a, b, c, respectively. Then, the interiors of the supports of g and h are disjoint. In particular, g and h commute, hence the action is not faithful.*

Proof: The supports of g and h consist of unions of segments bounded by successive non-repelling fixed points of f; in particular, any two of these segments either coincide or have disjoint interior. If one of these segments is contained in the support of g (resp., h), then Theorem 1.7 asserts that its interior contains a unique hyperbolically-repelling fixed point of f with derivative equal to m (resp., n). Since $m \neq n$, the open segments in the supports of g and h must be disjoint. $\square$

Remark 1.11. Theorem 1.10 admits straightforward generalizations replacing the Baumslag-Solitar groups $BS(1,m)$ and $BS(1,n)$ by groups associated (as in Theorem 1.3) to matrices A and B that are hyperbolically expanding (*i.e.* with every eigenvalue of modulus > 1) and have distinct eigenvalues.

Below we give two other results in the same spirit. The first of these is new, whereas the second is already known though our methods provide a new and somewhat more conceptual proof. More sophisticated examples will be treated elsewhere.

Let us consider the group $G_{\lambda,\lambda'}$ generated by the transformations of the real-line

$$c\colon x \mapsto x+1, \quad b\colon x \mapsto \lambda x, \quad a\colon x \mapsto sgn(x)|x|^{\lambda'},$$

where λ, λ' are positive numbers. These groups are known to be non-solvable for certain parameters λ'. Indeed, if λ' is a prime number, then the elements a and c generate a free group (see [10]).

Theorem 1.12. *For all integers m,n larger than 1, the group $G_{m,n}$ does not embed into the group C^1 diffeomorphisms of the closed interval.*

Proof: Assume that $G_{m,n}$ can be realized as a group of C^1 diffeomorphisms of $[0,1]$. Then Theorem 1.3 applies to both subgroups $\langle b,c\rangle$ and $\langle a,b\rangle$ (which are isomorphic to $BS(1,m)$ and $BS(1,n)$, respectively). Let us consider a maximal open subinterval $I = (x_0,x_1)$ that is invariant under c and where the dynamics of c has no fixed point. The relation $bcb^{-1} = c^m$ shows that the action of b on I is nontrivial. Proposition 1.5 then easily implies that b preserves I, and by Theorem 1.3, the restriction of the action of $\langle b,c\rangle$ to I is conjugate to an affine action. Let y be the fixed point of b inside I. As before, the relation $aba^{-1} = b^n$ forces a to fix all points x_0, y, x_1; moreover, the actions of $\langle a,b\rangle$ on both intervals (x_0,y) and (y,x_1) are conjugate to affine actions. Finally, notice that the relation $aba^{-1} = b^n$ forces the derivative of b to be equal to 1 at y. However, this contradicts Theorem 1.7 when applied to $\langle b,c\rangle$. $\square$

As another application of our results, we give an alternative proof of a theorem from [26]:

Theorem 1.13. *If Γ is a non-solvable subgroup of $SL_2(\mathbb{R})$, then $\Gamma \ltimes \mathbb{Z}^2$ does not embed into* $\mathrm{Diff}^1_+([0,1])$.

Proof: Since Γ is non-solvable, it must contain two hyperbolic elements A, B generating a free group. Theorem 1.3 applied to $\mathbb{Z} \ltimes_A \mathbb{Z}^2 \subset \Gamma \ltimes \mathbb{Z}^2$ implies that the action restricted to $\langle A, \mathbb{Z}^2\rangle$ is topologically conjugate to an affine action with dense translation part on each connected component I fixed by $\langle A, \mathbb{Z}^2\rangle$ and containing no point that is globally fixed by this subgroup. As B normalizes $\mathbb{Z}^2$, it has to be affine in the coordinates induced by this translation part. As a consequence, the action of $\Gamma \ltimes \mathbb{Z}^2$ is that of an affine group on each interval I as above. We thus conclude that the action factors throughout that of a solvable group, hence it is unfaithful. $\square$

Remark 1.14. It is not hard to extend the previous proof to show that $\Gamma \ltimes \mathbb{Z}^2$ does not embed into the group of C^1 diffeomorphisms of neither the open interval nor the circle. (Compare [26, §4.2] and [26, §4.3], respectively.)

Remark 1.15. All groups discussed in this section are locally indicable. We thus get different infinite families of finitely-generated, locally-indicable groups with no faithful actions by C^1 diffeomorphisms of the closed interval. The existence of such groups was first established in [26]; the examples considered therein correspond to those of Theorem 1.13.

2 On affine actions

In this section, we prove Proposition 1.2. To simplify, vectors $(t_1, \ldots, t_d) \in \mathbb{R}^d$ will be denoted horizontally, though must be viewed as vertical ones. We begin with

Proposition 2.1. *Given $A \in GL_d(\mathbb{Q})$, let G be a subgroup of $\mathbb{Z} \ltimes_A \mathbb{Q}^d$ of the form $\mathbb{Z} \ltimes_A H$, with $rank_{\mathbb{Q}}(H) = d$.*
(i) If $(t_1, ..., t_d) \in \mathbb{R}^d$ is an eigenvector of the transpose A^T with eigenvalue $\lambda \in \mathbb{R}_+ \setminus \{1\}$, then there exists a homomorphism $\psi : G \to \mathrm{Aff}_+(\mathbb{R})$ with non-Abelian image defined by $\psi(h_i) := T_{t_i}$ and $\psi(a) := M_\lambda$, where T_t and M_λ stand for the translation by an amplitude t and the multiplication by a factor λ, respectively. This homomorphism is injective if and only if $\{t_1, ..., t_d\}$ is a $\mathbb{Q}$-linearly-independent subset of $\mathbb{R}$.
(ii) Every homomorphism $\psi : G \to \mathrm{Aff}_+(\mathbb{R})$ with non-Abelian image is conjugate to one as those described in (i).

Proof: The first claim of item (i) is obvious. For the other claim, notice that injectivity of ψ is equivalent to injectivity of its restriction to H. We fix a $\mathbb{Q}$-basis $\{b_1, \ldots, b_d\} \subset H$ of $\mathbb{Q} \otimes H$, and we let a be the generator of the $\mathbb{Z}$-factor of G. Assume there is an element $b = b_1^{\beta_1} \cdots b_d^{\beta_d} \in H$ mapping into the trivial translation. Then $\sum_i \beta_i t_i = 0$, which implies that the t_i's are linearly dependent over $\mathbb{Q}$. Conversely, assume $\sum_i \beta_i t_i = 0$ holds for certan rational numbers β_i that are not all equal to zero. Up to multiplying them by a common integer factor, we may assume that such a relation holds with all β_i's integer. Then $b := b_1^{\beta_1} \cdots b_d^{\beta_d}$ is a nontrivial element of H mapping into the trivial translation under ψ.

For (ii), suppose $\psi : G \to \mathrm{Aff}_+(\mathbb{R})$ is a homomorphism with non-Abelian image. Then we have

$$\{id\} \subsetneq \psi([G,G]) \subseteq [\mathrm{Aff}_+(\mathbb{R}), \mathrm{Aff}_+(\mathbb{R})] = \{T_t, \ t \in \mathbb{R}\}.$$

Fix $b \in [G,G]$ such that $\psi(b)$ is a nontrivial translation. As $b \in H$, we have that $\psi(b)$ commutes with every element in $\psi(H)$. Therefore, $\psi(H)$ is a subgroup of the group of translations.

Let $t_1, \ldots, t_d$ in $\mathbb{R}$ be such that $\psi(b_i) = T_{t_i}$. As $\psi(G)$ is non-Abelian, we have $\psi(a) = T_t M_\lambda$ for certain $\lambda \neq 1$ and $t \in \mathbb{R}$. We may actually suppose that $t = 0$ just by conjugating ψ by $T_{\frac{t}{\lambda - 1}}$. Then, for each $i \in \{1, \ldots, d\}$,

$$T_{\lambda t_i} = \psi(a)\psi(b_i)\psi(a)^{-1} = \psi(a b_i a^{-1}) = \psi(b_1^{\alpha_{1,i}} \ldots b_d^{\alpha_{d,i}}) = T_{\alpha_{1,i}t_1 + \ldots + \alpha_{d,i}t_d}.$$

Thus, $\lambda t_i = \alpha_{1,i}t_1 + \ldots + \alpha_{d,i}t_d$, which implies that $(t_1, \ldots, t_d)$ is an eigenvector of A^T with eigenvalue λ. $\square$

Remark 2.2. The preceding proposition implies in particular that if A^T has no real eigenvalue, then there is no representation of G in $\mathrm{Aff}_+(\mathbb{R})$ with non-Abelian image. As a consequence, due to Theorem 1.3, if moreover the eigenvalues of A^T all have modulus different from 1, then every representation of G in $\mathrm{Diff}^1_+([0,1])$ has Abelian image.

As a matter of example, given positive integers m, n, let A be the matrix

$$A = A_{m,n} := \begin{pmatrix} m & -n \\ n & m \end{pmatrix}.$$

Then the group $G(m,n) := \mathbb{Z} \ltimes_A \mathbb{Q}^2$ has no inyective representation into $\mathrm{Diff}_+^1([0,1])$. Notice that each of these groups $G(m,n)$ is locally indicable. Hence, this produces still another infinite family of finitely-generated, locally-indicable groups with no faithful action by C^1 diffeomorphisms of the closed interval. (Compare Remark 1.15.)

In view of the discussion above, the proof of Proposition 1.2 is closed by the next

Lemma 2.3. *Suppose that the matrix $A \in GL_d(\mathbb{Q})$ is $\mathbb{Q}$-irreducible. If $\lambda \in \mathbb{R}$ is an eigenvalue of A^T and $v := (t_1, \ldots, t_d) \in \mathbb{R}^d$ is such that $A^T v = \lambda v$, then $\{t_1, \ldots, t_d\}$ is a $\mathbb{Q}$-linearly-independent subset of $\mathbb{R}$.*

Proof: If v is an eigenvector of A^T, then the subspace $v^\perp \subseteq \mathbb{R}^d$ is invariant under A. Since A is $\mathbb{Q}$-irreducible, we have $v^\perp \cap \mathbb{Q}^d = \{0\}$. Therefore, if $v := (t_1, \ldots, t_d)$ and $\beta_1, \ldots, \beta_d$ in $\mathbb{Q}$ verify $\beta_1 t_1 + \cdots + \beta_d t_d = 0$, then we have $\beta_1 = \ldots = \beta_d = 0$. $\square$

3 On continuous actions on the interval

In this section, we deal with actions by homeomorphisms. The proof below was given in [33] for the Baumslag-Solitar group $B(1,2)$. As we next see, the argument can be adapted to the group G.

Proof of Lemma 1.4: The Lemma will easily follow if we show that if G acts by homeomorphisms of $[0,1]$ in such a way that H admits no global fixed point on $(0,1)$, then the action is semiconjugate to that of an affine group.

We let $N \subseteq H$ be the set of elements having a fixed point inside $(0,1)$; as H is Abelian, N is easily seen to be a subgroup. Since H has finite $\mathbb{Q}$-rank, we have $N \neq H$. (This easily follows along the lines of [25, Exercise 2.2.47] just noticing that every homeomorphism of the interval has the same fixed points as each of its rational powers.) Therefore, there is an H-invariant infinite measure ν on $(0,1)$ that is finite on compact subsets [25, Proposition 2.2.48]. We claim that ν has no atoms, and that it is unique up to scalar multiple. Indeed, by [32], this holds whenever H/N is not isomorphic to $\mathbb{Z}$, and here we are in this case because N is A^T-invariant and A^T has no eigenvalue of modulus 1.

Now, as H is normal in G, we have that $a_*(\nu)$ is also invariant by H. By uniqueness, this implies that $a_*(\nu) = \lambda \nu$ for some $\lambda \in \mathbb{R}^+$. More generally, for every $b \in G$, there exists $\lambda_b \in \mathbb{R}^+$ such that $b_*(\nu) = \lambda_b \nu$. The map $b \mapsto \lambda_b$ is a homomorphism from G into $\mathbb{R}^+$. It is then easy to check that the map $\psi \colon G \to \mathrm{Aff}_+(\mathbb{R})$ defined by

$$\psi(b)(x) := \frac{1}{\lambda_b} x + \nu\big([1/2, b(1/2)]\big),$$

is a representation. Moreover, the map $F \colon \mathbb{R} \to \mathbb{R}$ defined by $F(x) := \nu(1/2, x)$ if $x > 1/2$, and $F(x) := -\nu(x, 1/2)$ for $x < 1/2$, semiconjugates the action of G with ψ. $\square$

In the statement of Lemma 1.4, the semiconjugacy is not necessarily a conjugacy. This easily follows by applying a Denjoy-like technique replacing the orbit of a single point by that of a wandering interval. See also Theorem 4.10 below, where this procedure is carried out smoothly on the open interval.

4 On C^1 actions on the interval

4.1 All actions are semiconjugate to affine ones

In this section, we show Proposition 1.5. Suppose for a contradiction that there is an action of G such that the subgroup H acts nontrivially on $(0,1)$ but having a fixed point inside. For each $x \in (0,1)$ which is not fixed by H, let us denote by I_x the maximal interval containing x such that H has no fixed point inside. Since G has no global fixed point in $(0,1)$ and H is normal in G, we must have $a^n(I_x) \cap I_x = \emptyset$ for all $n \neq 0$. In particular, I_x is contained in $(0,1)$. Moreover, a has no fixed point in $(0,1)$, and up to changing it by its inverse, we may suppose that $a(z) > z$ for all $z \in (0,1)$.

The rough idea now is, for a point x not fixed by H as above, to apply a^{-1} iteratively at x and examine the behavior of an appropriately defined *displacement vector* (see (2) below). Our first lemmata (4.2 to 4.6) build the groundwork needed to show that the direction of this vector nearly converges along a subsequence (Lemma 4.7). That A is hyperbolic them implies that, along this subsequence, the magnitude of the vector is uniformly expanded (Lemma 4.9), giving a contradiction.

To implement the strategy above, we first recall a useful tool that arises in this context, namely, there is an H-invariant infinite measure μ_x supported on I_x which is finite on compact subsets. This measure is not unique, but independently of the choice, we can define the *translation homomorphism* $\tau_{\mu_x} : H \to \mathbb{R}$ by $\tau_{\mu_x}(h) := \mu_x([z, h(z)))$ (here and in all what follows, we use the convention $\mu([y,z)) := -\mu((z,y])$ for $z < y$). The value of this morphism is independent of $z \in I_x$, and its kernel K_x, coincides with the set of elements of H having fixed points inside I_x. See [25, Section 2.2.5] for all of this.

From now on, we fix a $\mathbb{Q}$-basis $\{b_1, \ldots, b_d\} \subset H$ of $\mathbb{Q} \otimes H$. Although unnatural, this choice equips $\mathbb{R} \otimes H$ with an inner product, which yields to the following crucial notion.

Definition 4.1. For every I_x as above, we define the *translation vector* $\vec{\tau}_{\mu_x} \in \mathbb{R}^d$ as the unit vector pointing in the direction $(t_1, \ldots, t_d)$, where $t_i := \mu_x([z, b_i(z)))$.

We have

Lemma 4.2. *If we identify each $b = b_1^{\beta_1} \cdots b_d^{\beta_d} \in H$ with the vector $(\beta_1, \ldots, \beta_d) \in \mathbb{R} \otimes H$, then $\vec{\tau}_{\mu_x} \in \mathbb{R}^d$ is orthogonal to the hyperplane $\mathbb{R} \otimes K_x$.*

Proof: For $b = b_1^{\beta_1} \cdots b_d^{\beta_d} \in H$ we have that $\tau_{\mu_x}(b) = \sum_i \beta_i \mu_x([z, b_i(z)))$ equals zero if and only if the vector $(\beta_1, \ldots, \beta_d)$ is ortoghonal to $\vec{\tau}_{\mu_x}$. $\square$

In the sequel, we will denote $\vec{\tau}_{\mu_x}$ simply by $\vec{\tau}_x$. We have

Lemma 4.3. *For each $x \in (0,1)$, we have $K_{a^{-1}(x)} = A^{-1} K_x$. Moreover, the directions of $\vec{\tau}_{a^{-1}(x)}$ and $A^T \vec{\tau}_x$ coincide.*

Proof: A vector $v = (\beta_1, ..., \beta_d) \in \mathbb{R} \otimes H$ gives a positive (resp., negative) value under $\tau_{\mu_{a^{-1}(x)}}$ if and only if

$$b_1^{\beta_1} \cdots b_d^{\beta_d}(a^{-1}(z)) > a^{-1}(z)$$

(resp., $b_1^{\beta_1} \cdots b_d^{\beta_d}(a^{-1}(z)) < a^{-1}(z)$) holds for every $z \in I_x$, that is

$$a^{-1} b_1^{\alpha_{1,1}\beta_1 + \cdots + \alpha_{1,d}\beta_d} \cdots b_d^{\alpha_{d,1}\beta_1 + \cdots + \alpha_{d,d}\beta_d}(z) > a^{-1}(z)$$

(resp., $a^{-1}b_1^{\alpha_{1,1}\beta_1+\cdots+\alpha_{1,d}\beta_d}\cdots b_d^{\alpha_{d,1}\beta_1+\cdots+\alpha_{d,d}\beta_d}(z)<a^{-1}(z)$). This happens if and only if

$$b_1^{\alpha_{1,1}\beta_1+\cdots+\alpha_{1,d}\beta_d}\cdots b_d^{\alpha_{d,1}\beta_1+\cdots+\alpha_{d,d}\beta_d}(z)>z$$

(resp., $b_1^{\alpha_{1,1}\beta_1+\cdots+\alpha_{1,d}\beta_d}\cdots b_d^{\alpha_{d,1}\beta_1+\cdots+\alpha_{d,d}\beta_d}(z)<z$). This directly yields the first assertion of the lemma. The second one is an easy consequence. Indeed, $\vec{\tau}_{a^{-1}x}$ generates the subspace

$$(\mathbb{R}\otimes K_{a^{-1}x})^{\perp}=(A^{-1}(\mathbb{R}\otimes K_x))^{\perp}=A^T(\mathbb{R}\otimes K_x)^{\perp},$$

and the last subspace is generated by the vector $A^T\vec{\tau}_x$. □

We now state our main tool to deal with C^1 diffeomorphisms. Roughly, it says that diffeomorphisms that are close-enough to the identity in the C^1 topology behave like translations under composition. For each $\delta>0$, we denote $U_\delta(id)$ the neighborhood of the identity in $\mathrm{Diff}^1_+([0,1])$ given by

$$U_\delta(id):=\Big\{f\in\mathrm{Diff}^1_+([0,1])\colon \sup_{z\in[0,1]}\big|Df(z)-1\big|<\delta\Big\}.$$

Proposition 4.4 ([4])**.** *For each $\eta>0$ and all $k\in\mathbb{N}$, there exists a neighborhood U of the identity in $\mathrm{Diff}^1_+([0,1])$ such that for all $f_1,\ldots,f_k$ in U, all $\epsilon_1,\ldots,\epsilon_k$ in $\{-1,1\}$ and all $x\in[0,1]$, we have*

$$\Big|[f_k^{\epsilon_k}\circ\ldots\circ f_1^{\epsilon_1}(x)-x]-\sum_i\epsilon_i(f_i(x)-x)\Big|\le\eta\max_j\big\{|f_j(x)-x|\big\}.$$

Proof: First of all, observe that if $g\in\mathrm{Diff}^1_+([0,1])$ satisfies $|Dg(z)-1|<\lambda$ for all $z\in[0,1]$, then for all x,y,

$$\big|(g(x)-x)-(g(y)-y)\big|<\lambda|x-y|.$$

Next, notice that for every $f\in U_\delta(id)$ and all $x\in[0,1]$, there exists $y\in[0,1]$ such that

$$\big|(f_i^{-1}(x)-x)-(x-f_i(x))\big|=\big|(f_i(x)-x)-(f_i(f_i^{-1}(x))-f_i^{-1}(x))\big|=\big|Df_i(y)-1\big|\cdot\big|x-f_i^{-1}(x)\big|\le\delta|x-f_i^{-1}(x)|.$$

Using this, it is not hard to see that we may assume that $\epsilon_i=1$ for all i.

We proceed by induction on k. The case $k=1$ is trivial. Suppose the lemma holds up to $k-1$, and choose $\delta>0$ so that the lemma applies to any $k-1$ diffeomorphisms in the neighborhood $U=U_\delta(id)$ for the constant $\eta/2$. We may suppose δ is small enough to verify $\delta(k-1+\eta/2)<\eta/2$. Now take $f_1,\ldots,f_k$ in $U_\delta(id)$ and $x\in[0,1]$. Then the value of the expression

$$\Big|f_k\circ\ldots\circ f_1(x)-x-\sum_{i=1}^{k}(f_i(x)-x)\Big|$$

is smaller than or equal to

$$\Big|f_k\circ\ldots\circ f_1(x)-f_{k-1}\circ\ldots\circ f_1(x)-(f_k(x)-x)\Big|+\Big|f_{k-1}\circ\ldots\circ f_1(x)-x-\sum_{i=1}^{k-1}(f_i(x)-x)\Big|.$$

Now notice that, by the inductive hypothesis, the second term in the sum above is bounded from above by $\eta/2\max_j|f_j(x)-x|$. Moreover, the observation at the beginning of the proof and the inductive hypothesis yield

$$\big|f_k(f_{k-1}\circ\ldots\circ f_1(x))-f_{k-1}\circ\ldots\circ f_1(x)-(f_k(x)-x)\big|\le\delta\big|f_{k-1}\circ\ldots\circ f_1(x)-x\big|\le\delta\Big(\sum_{i=1}^{k-1}|f_i(x)-x|+\varepsilon\Big),$$

with $\varepsilon < \eta/2 \max_j |f_j(x) - x|$. By the choice of δ, the last expression is bounded from above by $\eta/2 \max_j |f_j(x) - x|$, thus finishing the proof. $\square$

We next deduce three consequences from this proposition.

Lemma 4.5. *Given $\eta > 0$, there exists $\delta > 0$ satisfying the following: if $f \in \mathrm{Diff}^1_+([0,1])$ and $q \in \mathbb{N}$ are such that $f^{1/q} \in U_\delta(id)$, then for all $x \in [0,1]$,*

$$\big|(f(x) - x) - q(f^{1/q}(x) - x)\big| \leq \eta\, |f^{1/q}(x) - x|.$$

In particular,

$$\frac{|f(x) - x|}{q + \eta} \leq |f^{1/q}(x) - x| \leq \frac{|f(x) - x|}{q - \eta}.$$

Proof: This directly follows from the proposition by letting $k = q$ and $f_1 = \ldots = f_k = f^{1/q}$. Details are left to the reader. $\square$

Notice that the lemma above does not state that if f is close to the identity, then its roots (whenever they exist) remain close to the identity. Indeed, this is known to be false in general. Nevertheless, as we will see along the proof of the lemma below, this turns to be partially true in the group G. For the statement, notice that the set of global fixed points of H accumulate at the origin. Hence, given an element $b \in H$, for each $\delta > 0$, there is $\sigma > 0$ which is fixed by H and such that b restricted to $[0, \sigma]$ belongs to the $U_\delta(id)$-neighborhood of the identity in $\mathrm{Diff}^1_+([0,\sigma])$ (the latter group is being identified with $\mathrm{Diff}^1_+([0,1])$ just by rescaling the interval). Let us hence consider the displacement vector $\triangle(x)$ defined by

$$\triangle(x) := \big(b_1(x) - x, \ldots, b_d(x) - x\big) \in \mathbb{R}^d, \tag{2}$$

and let us denote by $\|\triangle(x)\|$ its max norm. Notice that $\|\triangle(x)\| \leq 1$ for all $x \in [0,1]$.

Lemma 4.6. *For all $r > 0$, there exists $\sigma > 0$ such that*

$$\triangle(a^{-1}(x)) = Da^{-1}(0)\, A^T \triangle(x) + \epsilon(x) \quad \textit{for all} \quad x \in (0, \sigma),$$

and

$$\triangle(a(x)) = Da(1)\, (A^T)^{-1} \triangle(x) + \hat{\epsilon}(x) \quad \textit{for all} \quad x \in (1 - \sigma, 1),$$

where $\|\epsilon(x)\| \leq r \|\triangle(x)\|$ and $\|\hat{\epsilon}(x)\| \leq r \|\triangle(x)\|$.

Proof: Both assertions being analogous, we will prove only the first one. Let us write $\alpha_{i,j} := p_{i,j}/q_{i,j}$, with $(p_{i,j}, q_{i,j}) = 1$ and $q_{i,j} > 0$. Then each $b_j^{1/q_{i,j}}$ is an element of H. Indeed, letting $m = m_{i,j}$ and $n = n_{i,j}$ be integers such that $mp_{i,j} + nq_{i,j} = 1$, we have $b_j^{1/q_{i,j}} = (b_j^{p_{i,j}/q_{i,j}})^m (b_j)^n$.

Let $k := \max_i \{\sum_j |p_{j,i}|\}$, and let $\eta > 0$ be small enough so that

$$\eta \, \max_i \left(\frac{D}{\min_j q_{j,i} - \eta} \Big(1 + \sum_j |\alpha_{j,i}|\Big) + \sum_j |\alpha_{j,i}| \right) \leq r,$$

where $D = \max_z Da^{-1}(z)$. Let U be a neighborhood of the identity in $\mathrm{Diff}^1_+([0,1])$ for which both Proposition 4.4 and Lemma 4.5 hold simultaneously, and let $\sigma > 0$ be fixed by H such that the restrictions to $[0,\sigma]$ of all the $b_j^{1/q_{j,i}}$ belong to U and $|Da^{-1}(z) - Da^{-1}(0)| \leq \eta$ holds for all $z \in [0,\sigma]$. Then

$$\begin{aligned}
\Big|(ab_i a^{-1}(x) - x) - \sum_j \alpha_{j,i}(b_j(x) - x)\Big| &= \Big|(b_1^{\alpha_{1,i}} \circ \cdots \circ b_d^{\alpha_{d,i}}(x) - x) - \sum_j \alpha_{j,i}(b_j(x) - x)\Big| \\
&= \Big|((b_1^{1/q_{1,i}})^{p_{1,i}} \circ \cdots \circ (b_d^{1/q_{d,i}})^{p_{d,i}}(x) - x) - \sum_j \frac{p_{j,i}}{q_{j,i}}(b_j(x) - x)\Big| \\
&\leq \Big|((b_1^{1/q_{1,i}})^{p_{1,i}} \circ \cdots \circ (b_d^{1/q_{d,i}})^{p_{d,i}}(x) - x) - \sum_j p_{j,i}(b_j^{1/q_{j,i}}(x) - x)\Big| \\
&\qquad + \sum_j |p_{j,i}| \Big|(b_j^{1/q_{j,i}}(x) - x) - \frac{1}{q_{j,i}}(b_j(x) - x)\Big| \\
&\leq \eta \max_j |b_j^{1/q_{j,i}}(x) - x| + \eta \max_j |b_j^{1/q_{j,i}}(x) - x| \sum_j \frac{|p_{j,i}|}{|q_{j,i}|} \\
&\leq \frac{\eta}{\min_j q_{j,i} - \eta} \left(1 + \sum_j |\alpha_{j,i}|\right) \max_j |b_j(x) - x|.
\end{aligned}$$

The i-th entry in the vector $\triangle(a^{-1}(x))$ is

$$b_i a^{-1}(x) - a^{-1}(x) = a^{-1} a b_i a^{-1}(x) - a^{-1}(x) = Da^{-1}(z_i)\big(ab_i a^{-1}(x) - x\big),$$

where the last equality holds for a certain point $z_i \in I_x$. By the estimate above, for $x \in (0,\sigma)$, this expression equals

$$Da^{-1}(z_i) \sum_j \alpha_{j,i}\big(b_j(x) - x\big)$$

up to an error $\tilde{\varepsilon}_i(x)$ satisfying

$$|\tilde{\epsilon}_i(x)| \leq Da^{-1}(z_i) \frac{\eta}{\min_j q_{j,i} - \eta} \left(1 + \sum_j |\alpha_{j,i}|\right) \max_j |b_j(x) - x|.$$

Moreover, by the choice of σ, the value of $Da^{-1}(z_i) \sum_j \alpha_{j,i}(b_j(x) - x)$ equals

$$Da^{-1}(0) \sum_j \alpha_{j,i}(b_j(x) - x)$$

up to an error bounded from above by $\eta \big| \sum_j \alpha_{j,i}(b_j(x) - x) \big|$. Summarizing, $b_i a^{-1}(x) - a^{-1}(x)$ coincides with $Da^{-1}(0) \sum_j \alpha_{j,i}\big(b_j(x) - x\big)$ up to an error $\varepsilon_i(x)$ satisfying

$$|\varepsilon_i(x)| \leq \eta \left(\frac{D}{\min_j q_{j,i} - \eta} \Big(1 + \sum_j |\alpha_{j,i}|\Big) + \sum_j |\alpha_{j,i}| \right) \max_j |b_j(x) - x| \leq r \|\triangle(x)\|.$$

Since this holds for every $i \in \{1, \ldots, d\}$, this finishes the proof. $\square$

Before stating our third lemma, we observe that Lemma 4.3 and the compactness of the unit sphere $S^{d-1} \subset \mathbb{R}^d$ imply that for each point x_0 not fixed by H, the vectors $\vec{\tau}_{a^{-n}(x_0)}$ (resp., $\vec{\tau}_{a^n(x_0)}$) accumulate at some $\vec{\tau} \in S^d$ (resp., $\vec{\tau}_*$) as $n \to \infty$. For each $n \in \mathbb{Z}$, we let $x_n := a^{-n}(x_0)$, and we choose a sequence of positive integers n_k such that $\vec{\tau}_{x_{n_k}} \to \vec{\tau}$ and $\vec{\tau}_{x_{-n_k}} \to \vec{\tau}_*$ as $k \to \infty$.

Lemma 4.7. *For every $\eta > 0$, there exists K such that $k \geq K$ implies*

$$\frac{\triangle(x_{n_k})}{\|\triangle(x_{n_k})\|} = \vec{\tau} + \epsilon(k) \quad \textit{and} \quad \frac{\triangle(x_{-n_k})}{\|\triangle(x_{-n_k})\|} = \vec{\tau}_* + \epsilon_*(k),$$

where $\|\epsilon(k)\| \leq \eta$ and $\|\epsilon_*(k)\| \leq \eta$.

Proof: Up to passing to a subsequence if necessary, there is $b_* \in \{b_1, \ldots, b_d\}$ such that for all k,

$$|b_*(x_{n_k}) - x_{n_k}| \geq \max_i \big\{|b_i(x_{n_k}) - x_{n_k}|\big\}.$$

Then the functions $\psi_k : H \to \mathbb{R}$ defined by

$$\psi_k(b) = \frac{b(x_{n_k}) - x_{n_k}}{b_*(x_{n_k}) - x_{n_k}}$$

converge as $k \to \infty$ to a homomorphism $\psi : H \to \mathbb{R}$ which is normalized, in the sense that $\max_i |\psi(b_i)| = 1$. Indeed, this is the content of the Thurston's stability theorem [36] (which in its turn can be easily deduced from Proposition 4.4).

The vectors $\vec{\tau}_k$ and $\vec{\tau}$ naturally induce normalized homomorphisms from H into $\mathbb{R}$, denoted $\vec{\tau}_k$ and $\vec{\tau}$ as well. For these homomorphisms and any b, c in H, the inequality $\vec{\tau}(b) < \vec{\tau}(c)$ implies $\vec{\tau}_k(b) < \vec{\tau}_k(c)$ for k larger than a certain K_0, which implies $b(z) < c(z)$ for all $z \in I_{x_{n_k}}$ and all $k > K_0$. By evaluating at $z = x_{n_k}$, this yields $\psi_k(b) < \psi_k(c)$ for $k > K_0$. Passing to the limit, we finally obtain $\psi(b) \leq \psi(c)$. As a consequence, there must exist a constant λ for which $\vec{\tau} = \lambda\psi$. Nevertheless, since both homomorphisms are normalized (and point in the same direction), we must have $\lambda = 1$, which yields the convergence of $\triangle(x_{n_k})/\|\triangle(x_{n_k})\|$ towards $\vec{\tau}$. The second convergence is proved in an analogous way. $\square$

Henceforth, and in many other parts of this work, we will use a trick due to Muller and Tsuboi that allows reducing to the case where all group elements are tangent to the identity at the endpoints. This is achieved after conjugacy by an appropiate homeomorphism that is smooth at the interior and makes flat the germs at the enpoints. In concrete terms, we have:

Lemma 4.8 ([22, 38])**.** *Let us consider the germ at the origin of the local (non-differentiable) homeomorphism $\varphi(x) := sgn(x)\exp(-1/|x|)$. Then for every germ of C^1 diffeomorphism f (resp. vector field $\mathcal{X}$) at the origin, the germ of the conjugate $\varphi^{-1} \circ f \circ \varphi$ (resp., push-forward $\varphi_*(\mathcal{X})$) is differentiable and flat in a neighborhood of the origin.*

We should stress, however, that although this lemma simplifies many computations, in what follows it may avoided just noticing that, as Da is continuous, the element a behaves like an affine map close to each endpoint.

Recall that $\mathbb{R}^d$ decomposes as $E^s \oplus E^u$, where E^s (resp. E^u) stands for the stable (unstable) subspace of A^T. We denote by π_s and π_u the projections onto E^s and E^u, respectively. We let $\|\cdot\|_*$ be the natural norm on $\mathbb{R}^d$ associated to this direct-sum structure, namely,

$$\|v\|_* := \max\big\{\|\pi_s(v)\|, \|\pi_u(v)\|\big\}.$$

Lemma 4.9. *For any neighborhood $V \subset S^d$ of $E^u \cap S^d_*$ in the unit sphere $S^d_* \subset \mathbb{R}^d$ (with the norm $\|\cdot\|_*$), there is $\sigma > 0$ such that for all $x \in (0, \sigma)$ not fixed by H,*

$$\frac{\triangle(x)}{\|\triangle(x)\|_*} \in V \implies \frac{\triangle(a^{-1}(x))}{\|\triangle(a^{-1}(x))\|_*} \in V.$$

Moreover, if V is small enough, then there exists $\kappa > 1$ such that

$$\frac{\triangle(x)}{\|\triangle(x)\|_*} \in V \implies \|\triangle(a^{-1}x)\|_* \geq \kappa \|\triangle(x)\|_*.$$

Proof: For the first statement, we need to show that for every prescribed positive $\varepsilon < 1$, for points x close to the origin and not fixed by H, we have

$$\frac{\|\pi_s \triangle(a^{-1}(x))\|}{\|\pi_u \triangle(a^{-1}(x))\|} < \varepsilon \quad \text{provided} \quad \frac{\|\pi_s \triangle(x)\|}{\|\pi_s \triangle(x)\|} < \varepsilon.$$

Let $\lambda > 1$ (resp., $\lambda' < 1$) be such that the norm of nonzero vectors in E^u (resp., E^s) are expanded by at least λ (resp., contracted by at least λ') under the action of A^T. Choose $r < \lambda/2$ small enough so that

$$\frac{(\lambda' + r)\varepsilon}{\lambda - 2r} + \frac{r}{\lambda - 2r} < \varepsilon,$$

and consider a point x not fixed by H lying in the interval $(0, \sigma)$ given by Lemma 4.6. Then from

$$\|\pi_s \triangle(a^{-1}(x))\| \leq \|\pi_s A^T \triangle(x)\| + r\|\triangle(x)\| \leq \lambda' \|\pi_s \triangle(x)\| + r\|\triangle(x)\| \leq (\lambda' + r)\|\pi_s \triangle(x)\| + r\|\pi_u \triangle(x)\|,$$

$$\|\pi_u \triangle(a^{-1}(x))\| \geq \|\pi_u A^T \triangle(x)\| - r\|\triangle(x)\| \geq \lambda \|\pi_u \triangle(x)\| - r\|\triangle(x)\| \geq (\lambda - 2r)\|\pi_u \triangle(x)\|,$$

we obtain

$$\frac{\|\pi_s \triangle(a^{-1}(x))\|}{\|\pi_u \triangle(a^{-1}(x))\|} \leq \frac{(\lambda' + r)\varepsilon}{\lambda - 2r} + \frac{r}{\lambda - 2r} < \varepsilon,$$

as desired.

Moreover, by the estimates above,

$$\|\triangle(a^{-1}(x))\|_* \geq \|\pi_u \triangle(a^{-1}(x))\| \geq (\lambda - 2r)\|\pi_u \triangle(x)\| = (\lambda - 2r)\|\triangle(x)\|_*,$$

which yields the second statement for r small enough. $\square$

Now we can easily finish the proof of Proposition 1.5. To do this, choose a point $x_0 \in (0,1)$ that is not fixed by H. We need to consider two cases:

Case 1: $\vec{\tau}_{x_0} \notin E^s$

In this case, we first observe that Lemma 4.3 implies that any accumulation point of $\vec{\tau}_{a^n(x_0)}$ (in particular, $\vec{\tau}$) must belong to E^u. Let V be a small neighborhood around $E^u \cap S^d_*$ in S^d_* so that both statements of Lemma 4.9 hold. Then, by Lemma 4.7, the vector $\triangle(x_k)/\|\triangle(x_k)\|_*$ belongs to V starting from a certain $k = K$. This allows applying Lemma 4.9 inductively, thus showing that for all $n \geq 0$,

$$1 \geq \|\triangle(x_{n+k})\|_* \geq \kappa^n \|\triangle(x_k)\|_*.$$

Letting n go to infinity, this yields a contradiction.

Case 2: $\vec{\tau}_{x_0} \in E^s$.

In this case, Lemma 4.3 yields $\vec{\tau}_* \in E^s$. We then proceed as above but on a neighborhood of 1 working with a instead of a^{-1} and with $(A^T)^{-1}$ instead of A^T. Details are left to the reader. (This requires for instance an analog of Lemma 4.9 for the dynamics close to 1.)

4.2 Minimality of affine-like actions

In this section, we begin by showing Proposition 1.6. Let $\phi : G \to \mathrm{Diff}^1_+([0,1])$ be a representation with non-Abelian image. We know from Proposition 1.5 that ϕ is semiconjugate to a representation $\psi\colon G \to \mathrm{Aff}_+([0,1])$ in the affine group. The elements in the commutator subgroup $[\psi(G),\psi(G)]$ are translations. In what follows, we will assume that the right endpoint is topologically attracting for $\psi(a)$, hence $\psi(a)$ is conjugate to an homothety $x \to \lambda x$ with $\lambda > 1$ (the other case is analogous). Up to changing a by a positive power, we may assume that $\lambda \geq 2$. We fix $b \in H$ such that $\psi(b)$ is a non-trivial translation. Up to changing b by its inverse and conjugating ψ by an appropriate homothety, we may assume that $\psi(b) = T_1$. We consider a finite system of generators of G that contains both a and b.

Suppose for a contradiction that $\phi(G)$ does not act minimally. Then there is an interval I that is wandering for the action of $[\phi(G),\phi(G)]$. As before, we may assume that $D\phi(c)(1) = 1$ for all $c \in G$. Fix $\varepsilon > 0$ such that $(1-\varepsilon)^3 > 1/2$, and let $\delta > 0$ be such that

$$1-\epsilon \leq D\phi(c)(x) \leq 1+\epsilon \quad \text{for each } c \in \{a^{\pm 1}, b\} \text{ and all } x \in [1-\delta, 1]. \tag{3}$$

Clearly, we may assume that $I \subset [1-\delta, 1]$.

Notice that $\psi(a^{-k}ba^k) = T_{\lambda^{-k}}$ for all $k \in \mathbb{Z}$. We consider the following family of translations

$$h_{(\varepsilon_i)} := (a^{-n}b^{\varepsilon_n}a^n)\cdots(a^{-2}b^{\varepsilon_2}a^2)(a^{-1}b^{\varepsilon_1}a),$$

where $(\varepsilon_i) = (\varepsilon_1,\ldots,\varepsilon_n) \in \{0,1\}^n$. These satisfy the following properties:

(i) We have that $(\varepsilon_i) \neq (\tilde{\varepsilon}_i)$ implies $h_{(\varepsilon_i)} \neq h_{(\tilde{\varepsilon}_i)}$: this easily follows from that $\lambda \geq 2$.

(ii) We have $\phi(h_{(\varepsilon_i)})(1-\delta) \geq 1-\delta$: this follows from that $\phi(b)$ attracts towards 1 and that $\varepsilon_i \geq 0$ for all i.

(iii) The element $h_{(\varepsilon_i)} = a^{-n}(b^{\varepsilon_n}a)\cdots(b^{\varepsilon_2}a)(b^{\varepsilon_1}a)$ belongs to the ball of radius $3n$ in G. In particular, due to (3) and the preceding claim, we have $D\phi(b_{(\varepsilon_i)})(x) \geq (1-\epsilon)^{3n}$ for all $x \in [1-\delta, 1]$.

Since for each $c \in G$ there exists $x_I \in I$ for which $|c(I)| = Dc(x_I)|I|$ (where $|\cdot|$ stands for the length of the corresponding interval), putting together the assertions above we conclude

$$1 \geq \sum_{(\varepsilon_i)} \left|h_{(\varepsilon_i)}(I)\right| \geq 2^n(1-\epsilon)^{3n}|I| > 1,$$

where the last inequality holds for n large enough. This contradiction finishes the proof of Proposition 1.6.

It should be emphasized that Proposition 1.6 is no longer true for C^1 (even real-analytic) actions on the real line (equivalently, on the open interval). Indeed, this issue was indirectly adressed by Ghys and Sergiescu in [17, section III], as we next state and explain.

Theorem 4.10. ([17]). *The Baumslag-Solitar group $BS(1,2) := \langle a, b \mid aba^{-1} = b^2 \rangle$ embeds into $\mathrm{Diff}^1_+(\mathbb{R})$ via an action that is semiconjugate, but not conjugate, to the canonical affine action and such that the element $a \in B(1,2)$ acts with two fixed points.*

Recall that $BS(1,2)$ is isomorphic to the group of order-preserving affine bijections of $\mathbb{Q}_2$, where $\mathbb{Q}_2$ is the group of diadic rationals. Hence, every element in $BS(1,2)$ may be though as a pair $\left(2^n, \frac{p}{2^q}\right)$, which identifies to the affine map

$$\left(2^n, \frac{p}{2^q}\right)\colon\ x \to 2^n x + \frac{p}{2^q}.$$

Notice that $\mathbb{Q}_2$ corresponds to the subgroup of translations inside $BS(1,2)$.

Next, following [17], we consider a homeomorphism $f : \mathbb{R} \to \mathbb{R}$ satisfying the following properties:

(I) For every $x \in \mathbb{R}$, we have $f(x+1) = f(x) + 2$.

(II) $f(0) = 0$.

Lemma 4.11 ([17])**.** *The map* $\theta_f : \frac{p}{2^q} \in \mathbb{Q}_2 \to f^{-q} T_p f^q \in \mathrm{Homeo}_+(\mathbb{R})$ *is a well-defined homomorphism.*

Lemma 4.12 ([17])**.** The map $\left(2^n, \frac{p}{2^q}\right) \in BS(1,2) \to \theta_f(\frac{p}{2^q}) \circ f^n \in \mathrm{Homeo}_+(\mathbb{R})$ is a group homomorphism.

The homomorphism provided by the last lemma above will still be denoted by θ_f. Notice that $\theta_f(a) = f$.

Next, for $1 \leq r \leq \infty, \omega$, we impose a third condition on f:

(III_r) The map f is of class C^r.

We have

Lemma 4.13 ([17])**.** The image $\theta_f(BS(1,2))$ is a subgroup of $\mathrm{Diff}^r_+(\mathbb{R})$.

We end with

Lemma 4.14 ([17])**.** Suppose that the function f has at least two fixed points. Then $\theta_f(BS(1,2))$ has an exceptional minimal set (*i.e.* a minimal invariant closed set locally homeomorphic to the Cantor set).

To close this section, we point out that a similar construction can be carried out for all Baumslag-Solitar's groups $BS(1,n) := \langle a, b \mid bab^{-1} = a^n \rangle$. Roughly, we just need to replace condition (I) by:

$(\mathrm{I})_n$ For every $x \in \mathbb{R}$, we have $f(x+1) = f(x+n)$.

4.3 Rigidity of multipliers

We start by dealing with the Baumslag-Solitar group $BS(1,2)$. Let us consider a faithful action of this group by C^1 diffeomorphisms of the closed interval. We known that such an action must be topologically conjugate to an affine action, hence to the standard affine action given by $a \colon x \mapsto 2x$ and $b \colon x \mapsto x+1$. (It is not hard to check that all faithful affine actions of $B(1,2)$ are conjugate inside $\mathrm{Aff}(\mathbb{R})$.) Let $\varphi : (0,1) \to \mathbb{R}$ denote the topological conjugacy. Our goal is to show

Proposition 4.15. The derivative of a at the interior fixed point equals 2.

Proof: For the proof, we let $I := \varphi^{-1}([0,1])$. Notice that for all positive integers n, N, the intervals

$$(a^{-n} b^{\varepsilon_n} a^n) \cdots (a^{-2} b^{\varepsilon_2} a^2)(a^{-1} b^{\varepsilon_1} a) b^N a^{-n}(I), \quad \varepsilon_i \in \{0,1\},$$

have pairwise disjoint interiors. Indeed, these intervals are mapped by φ into the dyadic intervals of length $1/2^n$ contained in $[N, N+1]$. For simplicity, we assume below that both a and b have derivative 1 at the endpoints. (As before, this may be performed via the Muller-Tsuboi trick; *c.f.* Lemma 4.8).

Assume first that $Da(x_0) < 2$, where x_0 is the interior fixed point of a. Then there are $C > 0$ and $\varepsilon > 0$ such that for all $n \geq 1$,

$$\big|a^{-n}(I)\big| \geq C\Big(\frac{1}{2} + \varepsilon\Big)^n |I|.$$

Fix $\delta > 0$ such that

$$\big(1 - \delta\big)^3 \Big(\frac{1}{2} + \varepsilon\Big) > 1/2. \tag{4}$$

Let $\sigma > 0$ be small enough so that

$$Da(x) \geq 1 - \delta, \quad Da^{-1}(x) \geq 1 - \delta \quad \text{and} \quad Db(x) \geq 1 - \delta \quad \text{for all} \quad x \in [1 - \sigma, 1].$$

Finally, let $N \geq 1$ be such that $b^N(x_0) \geq 1 - \sigma$. Similarly to the proof of Proposition 1.6, for such N and all $n \geq 1$, we have for all choices $\varepsilon_i \in \{0, 1\}$,

$$\big|(a^{-n} b^{\varepsilon_n} a^n) \cdots (a^{-2} b^{\varepsilon_2} a^2)(a^{-1} b^{\varepsilon_1} a) b^N a^{-n}(I)\big| \geq (1 - \delta)^{3n} DC\Big(\frac{1}{2} + \varepsilon\Big)^n \big|I\big|,$$

where $D := \min_x Db^N(x)$. As there are 2^n of these intervals, we have

$$1 \geq 2^n (1 - \delta)^{3n} DC\Big(\frac{1}{2} + \varepsilon\Big)^n |I|,$$

which is impossible for a large-enough n due to (4).

Assume next that $Da(x_0) > 2$. Then there are $C' > 0$ and $\varepsilon' > 0$ such that for all $n \geq 1$,

$$\big|a^{-n}(I)\big| \leq C'\Big(\frac{1}{2} - \varepsilon'\Big)^n.$$

Fix $\delta' > 0$ such that

$$\big(1 + \delta'\big)^3 \Big(\frac{1}{2} - \varepsilon'\Big) < 1/2. \tag{5}$$

Let $\sigma' > 0$ be small enough so that

$$Da(x) \leq 1 + \delta, \quad Da^{-1}(x) \leq 1 + \delta \quad \text{and} \quad Db(x) \leq 1 + \delta \quad \text{for all} \quad x \in [1 - \sigma', 1].$$

Finally, let $N' \geq 1$ be such that $b^{N'}(x_0) \geq 1 - \sigma'$. Proceeding as before, we see that for such N' and all $n \geq 1$, we have for all choices $\varepsilon_i \in \{0, 1\}$,

$$\big|(a^{-n} b^{\varepsilon_n} a^n) \cdots (a^{-2} b^{\varepsilon_2} a^2)(a^{-1} b^{\varepsilon_1} a) b^N a^{-n}(I)\big| \leq (1 + \delta')^{3n} D'C'\Big(\frac{1}{2} - \varepsilon'\Big)^n \big|I\big|,$$

where $D' := \max_x Db^N(x)$. However, the involved intervals cover $I_{N'} := b^{N'}(I) = \varphi^{-1}\big([N', N'+1]\big)$. Thus,

$$|I_{N'}| \leq 2^n (1 + \delta')^{3n} D'C'\Big(\frac{1}{2} - \varepsilon'\Big)^n \big|I\big|,$$

which is again impossible for a large-enough n due to (5). □

Remark 4.16. The action of the Baumslag-Solitar group by C^1 diffeomorphisms of the real line constructed in the preceding section can be easily modified into a minimal one for which the derivative of a at the fixed point equals 1. Roughly, we just need to ask for the map f along the construction to have a single fixed point, with derivative 1 at this point. This shows that Theorem 1.7 is no longer true for actions by C^1 diffeomorphisms of the open interval.

The preceding proposition corresponds to a particular case of Theorem 1.7 but illustrates the technique pretty well. Below we give the proof of the general case along the same ideas. First, as A is supposed to be hyperbolic, we know that the action of G is topologically conjugate to an affine one. Moreover, Proposition 2.1 completely describes such an action: up to a topological conjugacy φ, it is given by correspondences $a \mapsto M_\lambda$ and $h_i \mapsto T_{t_i}$, where $(t_1, \ldots, t_d)$ is an eigenvector of A with eigenvalue λ. Up to conjugacy in $\mathrm{Aff}(\mathbb{R})$, we may assume that one of the $t_i's$ equals 1, hence $b := b_i$ is sent into $T_t := T_1$.

Next, we proceed as above, but with a little care. Notice that changing a by an integer power if necessary, we may assume that $\lambda \geq 2$.

Assume first that $Da(x_0) < \lambda$, where x_0 is the interior fixed point of a. Then there are $C > 0$ and $\varepsilon > 0$ such that for all $n \geq 1$,

$$\big|a^{-n}(I)\big| \geq C\Big(\frac{1}{\lambda} + \varepsilon\Big)^n.$$

Fix $\delta > 0$ such that $(1-\delta)^3(\frac{1}{\lambda}+\varepsilon) > \frac{1}{\lambda}$. Let $\sigma > 0$ be small so that $Da(x) \geq 1-\delta$, $Da^{-1}(x) \geq 1-\delta$ and $Db(x) \geq 1-\delta$ hold for all $x \in [1-\sigma, 1]$. Finally, let $N \geq 1$ be such that $b^N(x_0) \geq 1-\sigma$. Given $n \geq 1$, we consider for all choices $\varepsilon_i \in \{0, 1, \ldots, [\lambda]\}$, the intervals $(a^{-n}b^{\varepsilon_n}a^n)\cdots(a^{-2}b^{\varepsilon_2}a^2)(a^{-1}b^{\varepsilon_1}a)b^N a^{-n}(I)$, where I is the preimage of $[0,1]$ under the topological conjugacy into the affine action. As before, we have for each such choice

$$\big|(a^{-n}b^{\varepsilon_n}a^n)\cdots(a^{-2}b^{\varepsilon_2}a^2)(a^{-1}b^{\varepsilon_1}a)b^N a^{-n}(I)\big| \geq (1-\delta)^{3n} DC\Big(\frac{1}{\lambda}+\varepsilon\Big)^n \big|I\big|,$$

where $D := \min_x Db^N(x)$. These intervals do not necessarily have pairwise disjoint interiors, but their union covers I with multiplicity at most 2. As there are $\big([\lambda]+1\big)^n$ of these intervals, we have

$$2 \geq \big([\lambda]+1\big)^n (1-\delta)^{3n} DC\Big(\frac{1}{\lambda}+\varepsilon\Big)^n |I|,$$

which is impossible for large enough n.

Assume next that $Da(x_0) > \lambda$. Then there are $C' > 0$ and $\varepsilon' > 0$ such that for all $n \geq 1$,

$$\big|a^{-n}(I)\big| \leq C'\Big(\frac{1}{\lambda} - \varepsilon'\Big)^n.$$

Fix $\delta' > 0$ such that $(1+\delta')^3(\frac{1}{\lambda}-\varepsilon') < \frac{1}{\lambda}$. Let $\sigma' > 0$ be small enough so that $Da(x) \leq 1+\delta'$, $Da^{-1}(x) \leq 1+\delta'$ and $Db(x) \leq 1+\delta'$ hold for all $x \in [1-\sigma', 1]$. Finally, let $N' \geq 1$ be such that $b^{N'}(x_0) \geq 1-\sigma'$. As before, given $n \geq 1$, for all choices $\varepsilon_i \in \{0, 1, \ldots, [\lambda]\}$, we have

$$\big|(a^{-n}b^{\varepsilon_n}a^n)\cdots(a^{-2}b^{\varepsilon_2}a^2)(a^{-1}b^{\varepsilon_1}a)b^{N'} a^{-n}(I)\big| \leq (1+\delta')^{3n} D'C'\Big(\frac{1}{\lambda}-\varepsilon'\Big)^n \big|I\big|,$$

where $D' := \max_x Db^N(x)$. These intervals cover $I_{N'} := b^{N'}(I)$ for each $n \geq 1$. As there are $([\lambda]+1)^n$ of these intervals, we have

$$|I_{N'}| \leq \big([\lambda]+1\big)^n (1+\delta')^{3n} DC\Big(\frac{1}{\lambda}-\varepsilon'\Big)^n |I|.$$

Although this is not enough to conclude, we notice that we may replace a by a^k along the preceding computations, now yielding

$$|I_{N'}| \leq \big([\lambda^k]+1\big)^n (1+\delta')^{3n} DC\Big(\frac{1}{\lambda}-\varepsilon'\Big)^{kn} |I|.$$

Choosing k large enough so that

$$\big([\lambda^k]+1\big)\Big(\frac{1}{\lambda}-\varepsilon'\Big)^k(1+\delta')^3<1$$

and then letting n go to infinity, this gives the desired contradiction.

We have hence showed that $Da(x_0)=\lambda$. To show that the derivative of a^kb at the interior fixed point equals λ^k for each $k\neq 0$ and all $b\in H$, just notice that the associated affine action can be conjugate in $\mathrm{Aff}(\mathbb{R})$ so that a^kb is mapped into T_{λ^k}. Knowing this, we may proceed in the very same way as above.

4.4 On the smoothness of conjugacies

As we announced in the Introduction, actions by C^1 diffeomorphisms are rarely rigid in what concerns the regularity of conjugacies. In our context, this is actually never the case, as it is shown by the next

Proposition 4.17. *Let G be a group of the form $\mathbb{Z}\ltimes_A H$, where $A\in GL_d(\mathbb{Q})$ and $rank_{\mathbb{Q}}(H)=d$. Then every faithful action of G by C^1 diffeomorphisms of $[0,1]$ can be approximated in the C^1 topology by actions by C^1 diffeomorphisms that are topologically conjugate to it but for which no Lipschitz conjugacy exists.*

Proof: This follows by an standard application of the Anosov-Katok method (see [14, 15] for a general panorama on this).

Start with G viewed as a group of C^1 diffeomorphisms of $[0,1]$. Fix two points x_0,x_1 in $(0,1)$, and denote by a the generator of the $\mathbb{Z}$-factor of G and by $\{b_1,\ldots,b_d\}$ a $\mathbb{Q}$-basis of H. Consider a sequence of diffeomorphisms φ_k of $[0,1]$ such that for all $c\in\{a^{\pm1},b_i^{\pm1}\}$, where $i\in\{1,\ldots,d\}$, we have for $\tilde\varphi_k:=\varphi_1\circ\cdots\circ\varphi_k$:

(i) $\big\|\tilde\varphi_{k+1}-\tilde\varphi_k\big\|_{C^0}\le\frac{1}{2^k}$,

(ii) $\big\|\tilde\varphi_{k+1}\circ c\circ(\tilde\varphi_{k+1})^{-1}-\tilde\varphi_k\circ c\circ(\tilde\varphi_k)^{-1}\big\|_{C^1}\le\frac{1}{2^k}$,

(iii) x_0,x_1 are both fixed by φ_k,

(iv) $D\varphi_k(x_0)>2^k\min_y D\tilde\varphi_{k-1}(y)$, and if we denote by I_k the connected component of the set $\big\{x\mid D\varphi_k(x)>2^k\min_y D\tilde\varphi_{k-1}(y)\big\}$ containing x_0, then the support of φ_{k+1} has measure $<|I_k|/2$.

This may be easily achieved inductively by making φ_{k+1} almost commute with the action of G conjugate by $\tilde\varphi_k$ along a very small neighborhood of a large but finite part of the orbit of x_0.

By (i), we have that the sequence $(\tilde\varphi_k)$ converges to a homeomorphism $\tilde\varphi_\infty$. By (ii), the sequence of the actions conjugated by $\tilde\varphi_k$ converge in the C^1 topology to the action conjugated by $\tilde\varphi_\infty$. Due to (iii), each $\tilde\varphi_k$ fixes x_0 and x_1, hence the same holds for $\tilde\varphi_\infty$. As conjugacies to affine groups with dense translation subgroup are unique up to right composition with an affine map, we deduce that $\tilde\varphi_\infty$ is the unique conjugacy between G and $\tilde\varphi_\infty G\tilde\varphi_\infty^{-1}$ fixing these two points. Finally, using (iv), it is not hard to see that the derivative of $\tilde\varphi_k$ is larger than 2^k on certain intervals that remain disjoint from the supports of $\varphi_{k+1},\varphi_{k+2},\ldots$ As a consequence, the limit homeomorphism $\tilde\varphi_\infty$ is not Lipschitz. Because of the uniqueness up to affine transformations previously discussed, this implies that G and $\tilde\varphi_\infty G\tilde\varphi_\infty^{-1}$ cannot be conjugated by any Lipschitz homeomorphism. $\square$

Next, we deal with the C^r case, where $r\ge 2$.

Proposition 4.18. *Let G be a group of the form $\mathbb{Z} \ltimes_A H$, where $A \in GL_d(\mathbb{Q})$ has no eigenvalue of norm 1 and $rank_{\mathbb{Q}}(H) = d$. Then for all $r \geq 2$, every faithful action of G by C^r diffeomorphisms of $[0,1]$ with no global fixed point in $(0,1)$ is conjugate to an affine action by a homeomorphism that restricted to $(0,1)$ is a C^r diffeomorphism.*

Proof: We know from Theorem 1.3 that the action is conjugate to an affine action via a homeomorphism φ. The image of H is a subgroup of the group of translations which is necessarily dense; otherwise, H would have rank 1 and A^2 would stabilize it pointwise, thus contradicting hyperbolicity. As g is assumed to be C^r, $r \geq 2$, and has no fixed point in $(0,1)$, Szekeres' theorem implies that the restrictions of g to $[0,1)$ and $(0,1]$ are the time-one map of the flows of vector fields $\mathcal{X}_-$ and $\mathcal{X}_+$, respectively, that are C^1 on their domains and C^{r-1} at the interior. Futhermore, Kopell's lemma implies that the C^1 centralizer of g is contained in the intersection of the flows of $\mathcal{X}_-$ and $\mathcal{X}_+$. Therefore, the flows coincide for a dense subset of times, hence $\mathcal{X}_- = \mathcal{X}_+$ on $(0,1)$. We denote this vector field by $\mathcal{X}$ and we call it the Szekeres vector field associated to b. (See [25, §4.1.3] for the details.)

The homeomorphism φ must send this flow into that of the translations. Since $\mathcal{X}$ is of class C^{r-1} on $(0,1)$, we have that φ is a C^{r-1} diffeomorphism of $(0,1)$. To see that φ is actually a C^r diffeomorphism, we use Theorem 1.7, which says that the interior fixed point x_0 of the element a is hyperbolic. Indeed, this implies that φ is a C^1 diffeomorphism that conjugates two germs of hyperbolic diffeomorphisms. By a well-known application of (the sharp version of) Sternberg's linearization theorem, such a diffeomorphism has to be of class C^r in a neighborhood of x_0 (see [25, Corollary 3.6.3]). Since the action is minimal on $(0,1)$ due to Proposition 1.6, this easily implies that φ is of class C^r on the whole open interval. □

5 Examples involving non-hyperbolic matrices

We next consider the situation where $A \in GL_d(\mathbb{Q})$ has some eigenvalues of modulus $= 1$ and some others of modulus $\neq 1$. Our goal is to prove Theorem 1.8, according to which the group $\mathbb{Z} \ltimes_A \mathbb{Q}^d$ has an action by C^1 diffeomorphisms of the closed interval that is not semiconjugate to an affine action provided A is irreducible. In particular, this is the case for the matrix

$$A := \begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & -4 \end{pmatrix} \in SL_4(\mathbb{Z}).$$

Indeed, A has characteristic polynomial $p(x) = x^4 + 4x^3 + 4x^2 + 4x + 1 = p_1(x)p_2(x)$, where $p_1(x) := x^2 + (2+\sqrt{2})x + 1$ and $p_2(x) := x^2 + (2-\sqrt{2})x + 1$. Notice that $p(x)$ has no rational root, neither a decomposition into two polynomial of rational coefficients of degree two; hence, it is irreducible over $\mathbb{Q}$. Moreover, the roots λ and $1/\lambda$ of p_1 are real numbers of modulus different from 1, while the roots w, $\overline{w}$ of p_2 are complex numbers of modulus 1, where

$$w = \frac{\sqrt{2} - 2 + i\sqrt{4\sqrt{2} - 2}}{2}, \qquad \lambda = \frac{-\sqrt{2} - 2 + \sqrt{4\sqrt{2} + 2}}{2}.$$

Given any $A \in GL_d(\mathbb{Q})$, we begin by constructing an action of $G := \mathbb{Z} \ltimes_A \mathbb{Q}^d$ by homeomorphisms of the interval that is not semiconjugate to an affine action. To do this, we consider a decomposition $[0,1] = \overline{\bigcup_{k\in\mathbb{Z}} I_k}$, where the I_k's are open intervals disposed on $[0,1]$ in an ordered way and such that the right endpoint of I_k coincides with the left endpoint of I_{k+1}, for all $k \in \mathbb{Z}$. Let f be a homeomorphism of $[0,1]$ sending each I_k into I_{k+1}. For each $(t_1, \ldots, t_d) \in \mathbb{Q}^d$ and $k \in \mathbb{Z}$, denote

$$(t_{1,k}, \ldots, t_{d,k}) := A^k(t_1, \ldots, t_d).$$

Let ξ^t be a nontrivial topological flow on I_0. Next, fix $(s_1, \ldots, s_d) \in \mathbb{R}^d$, and for each $(t_1, \ldots, t_d) \in \mathbb{Q}^d$, define $g := g_{(t_1,\ldots,t_d)}$ on I_0 by $g|_{I_0} := \xi^{\sum_i s_i t_i}$. Extend g to the whole interval by letting

$$g\big|_{I_{-k}} = f^{-k} \circ \xi^{\sum_i s_i t_{i,k}}\big|_{I_0} \circ f^k. \tag{6}$$

It is not hard to see that the correspondences $a \mapsto f$, $(t_1, \ldots, t_d) \mapsto g_{(t_1,\ldots,t_d)}$, define a representation of G, where a stands for the generator of the $\mathbb{Z}$-factor of G.

Lemma 5.1. *If A is $\mathbb{Q}$-irreducible and $(s_1, \ldots, s_d)$ is nonzero, then the action constructed above is faithful.*

Proof: Denote by $b_1, \ldots, b_d$ the canonical basis of $H := \mathbb{Q}^d$. We need to show that for a given nontrivial $b := b_1^{t_1} \cdots b_d^{t_d} \in H$, the associated map $g := g_{(t_1,\ldots,t_d)}$ acts nontrivially on $[0,1]$. Assume otherwise. Then according to (6), for all $k \in \mathbb{Z}$,

$$0 = \sum_i s_i t_{i,k} = \big\langle (s_1, \ldots, s_d), A^k(t_1, \ldots, t_d) \big\rangle.$$

As a consequence, the $\mathbb{Q}$-span of $A^k(t_1, \ldots, t_d), k \in \mathbb{Z}$, is a $\mathbb{Q}$-invariant subspace orthogonal to $(s_1, \ldots, s_d)$. However, as A is $\mathbb{Q}$-irreducible, the only possibility is $(t_1, \ldots, t_d) = 0$, which implies that b is the trivial element in H. $\square$

Assume next that A is not hyperbolic. Associated to the transpose matrix A^T, there is a decomposition $\mathbb{R}^d = E^s \oplus E^u \oplus E^c$ into stable, unstable, and central subspaces, respectively. The space E^c necessarily contains a subspace E^c_* of dimension 1 or 2 that is completely invariant under A^T and such that for each nontrivial vector therein, all vectors in its orbit under A^T have the same norm. Our goal is to prove

Proposition 5.2. *If $(s_1, \ldots, s_d)$ belongs to E^c_*, then the action above is C^1 smoothable.*

This will follow almost directly from the next

Proposition 5.3. *The map f and the subintervals I_k of the preceding construction can be taken so that f is a C^1 diffeomorphism that commutes with a C^1 vector field whose support in $(0,1)$ is nontrivial and contained in the union of the interior of the I_k's.*

Using f and the vector field above, we may perform the construction taking ξ^t as being the flow associated to it. Indeed, since the vector field is C^1 on the whole interval, equation (6) implies that for a given $(t_1, \ldots, t_d)$, the corresponding $g_{(t_1,\ldots,t_d)}$ is a C^1 diffeomorphism provided the expressions $\sum_i s_i t_{i,k}$ remain uniformly bounded on k. However, as $(s_1, \ldots, s_d)$ belongs to E^c_*, this is always the case, because

$$\sum_i s_i t_{i,k} = \big\langle (s_1, \ldots, s_d), A^k(t_1, \ldots, t_d) \big\rangle = \big\langle (A^T)^k(s_1, \ldots, s_d), (t_1, \ldots, t_d) \big\rangle$$

and $\{(A^T)^k(s_1, \ldots, s_d), k \in \mathbb{Z}\}$ is a bounded subset of $\mathbb{R}^d$.

To conclude the proof of Theorem 1.8, we need to show Proposition 5.3. Although at this point we could refer to the classical construction of Pixton [31], we prefer to give a simpler argument that decomposes into two elementary parts given by the next lemmas.

Lemma 5.4. *There exists a vector field $\mathcal{X}_0$ on $[0,1]$ with compact support in $(0,1)$ and a sequence (φ_k) of C^∞ diffeomorphisms of $[0,1]$ with compact support inside $(0,1)$ that converges to the identity in the C^1 topology and such that the diffeomorphisms $\tilde{\varphi}_k := \varphi_k \circ \cdots \circ \varphi_1$ satisfy $(\tilde{\varphi}_k)_*(\mathcal{X}_0) = t_k \mathcal{X}_0$ for a certain sequence (t_k) of positive numbers converging to zero.*

Proof: Sart with the flow of translations on the real line and the corresponding (constant) vector field. Any two positive times of this flow are smoothly conjugate by appropriate affine transformations. Now, map the real-line into the interval by a projective map. This yields the desired vector field and diffeomorphisms, except for that the supports are not contained in $(0,1)$. To achieve this, just start by performing the Muller-Tsuboi trick (*c.f.* Lemma 4.8) in order to make everything flat at the endpoints, then extend everything trivially in both directions by slightly enlarging the interval, and finally renormalize the resulting interval into $[0,1]$. $\square$

Given a diffeomorphism φ of (resp., vector field $\mathcal{X}$ on) an interval I, we denote by $\varphi^\vee$ (resp., $\mathcal{X}^\vee$) the diffeomorphism of (resp., vector field on) $[0,1]$ obtained after conjugacy (resp., push forward) by the unique affine map sending I into $[0,1]$. Proposition 5.3 is a direct consequence of the next

Lemma 5.5. *There exists a C^1 diffeomorphism f of $[0,1]$ fixing only the endpoints (with the origin as a repelling fixed point) as well as a C^1 vector field $\mathcal{Y}$ on $[0,1]$ such that $f_*(\mathcal{Y}) = \mathcal{Y}$ and so that for a certain $x_0 \in (0,1)$, we have $(\mathcal{Y}|_{[x_0,f(x_0)]})^\vee = \mathcal{X}_0$.*

Proof: Start with a C^∞ diffeomorphism g of $[0,1]$ that has no fixed point at the interior, and has the origin as a repelling fixed point. Fix any $x_0 \in (0,1)$, and let $\mathcal{Z}$ be a vector field on $[x_0, g(x_0)]$ such that $\mathcal{Z}^\vee = \mathcal{X}_0$. A moment's reflexion shows that this construction can be performed so that g is affine close to each endpoint.

For each $k \in \mathbb{Z}$, let $I_k := g^k([x_0, f(x_0)])$. Let $\varphi_k^\wedge$ be a diffeomorphism of I_k into itself such that $(\varphi_k^\wedge)^\vee = \varphi_k$. Now let f be defined by letting $f\big|_{I_{|k|}} := \varphi_{|k|}^\wedge \circ g\big|_{I_{|k|}}$. Extend $\mathcal{Z}$ to the whole interval $[0,1]$ by making it commute with g. Finally, define $\mathcal{Y}$ by letting $\mathcal{Y}\big|_{I_{|k|}} := t_{|k|}\mathcal{Z}\big|_{I_{|k|}}$ for every $k \in \mathbb{Z}$. One easily checks that f and $\mathcal{Y}$ satisfy the desired properties. $\square$

To close this section, we remark that similar ideas yield to faithful actions by C^1 circle diffeomorphisms without finite orbits for the groups considered here. Indeed, it suffices to consider f as being a Denjoy counter-example and then proceed as before along the intervals $I_k := f^k(I)$, where I is a connected component of the complement of the exceptional minimal set of f. We leave the details of this construction to the reader.

6 Actions on the circle

Recall the next folklore (and elementary) result: For every group of circle homeomorphisms, one of the next three possibilities holds:

(i) there is a finite orbit,

(ii) all orbits are dense,

(iii) there is a unique minimal invariant closed set that is homeomorphic to the Cantor set. (This is usually called an *exceptional minimal set.*)

Moreover, a result of Margulis states that in case of a minimal action, either the group is Abelian and conjugate to a group of rotations, or it contains free subgroups in two generators. (See [25, Chapter 2] for all of this.)

Assume next that a non-Abelian, solvable group acts faithfully by circle homeomorphisms. By the preceding discussion, such an action cannot be minimal. As we next show, it can admit an exceptional minimal set. For concreteness, we consider the group $G := \mathbb{Z} \ltimes_A \mathbb{Q}^d$, with $A \in GL_d(\mathbb{Q})$. Start with a Denjoy counter-example $g \in \mathrm{Homeo}_+(\mathrm{S}^1)$, that is, a circle homeomorphism of irrational rotation number that is not minimal. Let Λ be the exceptional minimal set of g. Let I be one of the connected components of $\mathrm{S}^1 \setminus \Lambda$, and for each $n \in \mathbb{Z}$, denote $I_n := g^n(I)$. Consider any representation $\phi_I : \mathbb{Q}^d \to \mathrm{Homeo}(I)$. (Such an action can be taken faithful just by integrating a topological flow up to rationally independent times and associating the resulting maps to the generators of $\mathbb{Q}^d$.) Then extend ϕ_I into $\phi : G \to \mathrm{Homeo}_+(\mathrm{S}^1)$ on the one hand by letting $\phi(a) := g$, and on the other hand, for each $b \in H$, letting the restriction of $\phi(b)$ to $\mathrm{S}^1 \setminus \bigcup_n I_n$ being trivial, and setting $\phi(h)|_{I_n} = g^{-n} \circ \phi_I(A^{-n}(h)) \circ g^n$ for each $n \in \mathbb{Z}$. It is easy to check that ϕ is faithful. Part of the content of Theorem 1.9 is that in case A is hyperbolic, such an action cannot be by C^1 diffeomorphisms. (Compare [18], where Cantwell-Conlon's argument is used to prove this for the case of the Baumslag-Solitar group.)

We next proceed to the proof of Theorem 1.9. Let again denote by G a subgroup of $\mathbb{Z} \ltimes_A \mathbb{Q}^d$ of the form $H \times_A \mathbb{Z}$, with $rank_{\mathbb{Q}}(H) = d$ and $A \in GL_d(\mathbb{Z})$. Fix a $\mathbb{Q}$-basis $\{b_1, \ldots, b_d\}$ of H, and denote by a the generator of the cyclic factor (induced by A). We start with the next

Lemma 6.1. *Suppose A has no eigenvalue equal to 1. Then for every representation of G into* $\mathrm{Homeo}_+(\mathrm{S}^1)$*, the set* $\bigcap \mathrm{Per}(b_i)$ *of common periodic points of the b_i's is nonempty and G-invariant.*

Proof: Let $\rho_i \in \mathbb{R}/\mathbb{Z}$ be the rotation number of b_i. Since H is Abelian and $ab_ia^{-1} = b_1^{\alpha_{1,i}} \cdots b_d^{\alpha_{d,i}}$, we have

$$\rho_i = \alpha_{1,i}\rho_1 + \cdots + \alpha_{d,i}\rho_d \quad (mod\ \mathbb{Z}).$$

If we denote $v := (\rho_1, \ldots, \rho_d)$, this yields $A^T v = v\ (mod\ \mathbb{Z}^d)$. Hence, $v \in (A^T - I)^{-1}(\mathbb{Z}^d) \subseteq \mathbb{Q}^d$. Therefore, all the rotation numbers ρ_i are rational, thus all the b_i's have periodic points. Next, notice that for every family of commuting circle homeomorphisms each of which has a fixed point, there must be common fixed points. Indeed, they all necessarily fix the points in the support of a common invariant probability measure. To show the invariance of $\bigcap \mathrm{Per}(b_i)$, notice that H-invariance is obvious by commutativity. Next, let p be fixed by $b_1^{k_1}, \ldots, b_d^{k_d}$. Take $N \in \mathbb{N}$ such that $N\alpha_{i,j}$ is an integer for all i, j. Then

$$ab_i^{Nk_i}a^{-1}(p) = b_1^{k_iN\alpha_{1,i}} \cdots b_d^{k_iN\alpha_{d,i}}(p) = p,$$

hence $b_i^{Nk_i}a^{-1}(p) = a^{-1}(p)$. We thus conclude that $a^{-1}(p)$ is a common periodic point of the b_i's, as desired. □

Lemma 6.2. *If a has periodic points, then there exists a finite orbit for G.*

Proof: If a has periodic points, then every probability measure μ that is invariant by a must be supported at these points. Since G is solvable (hence amenable), such a μ can be taken invariant by the whole group. The points in the support of this measure must have a finite orbit. □

Summarizing, for every faithful action of G by circle homeomorphisms, the nonexistence of a finite orbit implies that a admits an exceptional minimal set, say Λ. In what follows, we will show that this last possibility cannot arise for representations into $\mathrm{Diff}^1_+(S^1)$ with non-Abelian image.

As the set $\bigcap \mathrm{Per}(b_i)$ is invariant under a, closed, and nonempty, we must have $\Lambda \subseteq \bigcap \mathrm{Per}(b_i)$. Changing each b_i by b_i^k for some $k \in \mathbb{N}$, we may assume that the periodic points of the b_i's are actually fixed. (Observe that the map sending b_i into b_i^k and fixing a is an automorphism of G.) Given a point x in the complement of $\bigcap \mathrm{Fix}(b_i)$ (which is nonempty due to the hypothesis), denote by I_x the connected component of the complement of $\bigcap \mathrm{Fix}(b_i)$ containing x. Then there is an H-invariant measure μ_x supported on I_x associated to which there is a translation vector τ_x; moreover, Lemma 4.3 still holds in this context.

If I is any connected component of the complement of $\bigcap \mathrm{Fix}(b_i)$, then there are points $z_1, \dots z_d$ in I such that $Db_i(z_i) = 1$. Therefore, for every $\varepsilon > 0$, there exists $\delta > 0$ such that if $|I| < \delta$, then $1 - \varepsilon \leq Db_i(z) \leq 1 + \varepsilon$ holds for all $z \in I$ and all $i \in \{1, \dots, d\}$. By decreasing δ if necessary, we may also assume that

$$1 - \varepsilon \leq \frac{Da(y)}{Da(z)} \leq 1 + \varepsilon \text{ for all } y, z \text{ at distance } dist(z, y) \leq \delta. \tag{7}$$

As I_x is a wandering interval for a, we have that there exists $k_0 \in \mathbb{N}$ such that $|a^k(I_x)| < \delta$ and $|a^{-k}(I_x)| < \delta$ for all $k \geq k_0$. Together with (7), this allows to show the next analogue of Lemma 4.6 for the translation vectors $\Delta(x) := \big(b_1(x) - x, \dots, b_d(x) - x\big)$.

Lemma 6.3. *For every $\eta > 0$, there exists $k_0 \in \mathbb{N}$ such that if we denote by y_k the left endpoint of $a^k(I)$ and we let $\varepsilon, \hat{\varepsilon}$ be defined by*

$$\triangle(a^{-1}(x)) = Da^{-1}(y_{-k})\, A^T \triangle(x) + \epsilon(x), \quad x \in I_{a^{-k}(x_0)}$$

and

$$\triangle(a(x)) = Da(y_k)\, (A^T)^{-1} \triangle(x) + \hat{\epsilon}(x), \quad x \in I_{a^k(x_0)},$$

then $\|\epsilon(x)\| \leq \eta \|\triangle(x)\|$ and $\|\hat{\epsilon}(x)\| \leq \eta \|\triangle(x)\|$ do hold for all $k \geq k_0$.

Again, the normalized translation vectors $\vec{\tau}_{a^{-n}(x_0)}$ (resp., $\vec{\tau}_{a^n(x_0)}$) accumulate at some $\vec{\tau} \in S^d$ (resp., $\vec{\tau}_*$) as $n \to \infty$. For each $n \in \mathbb{Z}$, we let $x_n := a^{-n}(x_0)$, and we choose a sequence of positive integers n_k such that $\vec{\tau}_{x_{n_k}} \to \vec{\tau}$ and $\vec{\tau}_{x_{-n_k}} \to \vec{\tau}_*$ as $k \to \infty$. With this notation, Lemma 4.7 remains true.

Finally, Lemma 4.9 is easily adapted to this case:

Lemma 6.4. *For any neighborhood $V \subset S^d$ of $E^u \cap S^d_*$ in the unit sphere $S^d_* \subset \mathbb{R}^d$ (with the norm $\| \cdot \|_*$), there is $K_0 \in \mathbb{N}$ such that for all $k \geq K_0$ and all $x \in a^{-k}(I_{x_0})$ not fixed by H,*

$$\frac{\triangle(x)}{\|\triangle(x)\|_*} \in V \implies \frac{\triangle(a^{-1}(x))}{\|\triangle(a^{-1}(x))\|_*} \in V.$$

Moreover, if V is small enough, then there exists $\kappa > 1$ such that

$$\frac{\triangle(x)}{\|\triangle(x)\|_*} \in V \implies \|\triangle(a^{-1}x)\|_* \geq \kappa\, Da^{-1}(y_{-k}) \|\triangle(x)\|_*.$$

Now, we may conclude as in the proof of Proposition 1.5 up to a small detail. Namely, suppose $\vec{\tau}_{x_0} \notin E^s$. Then $\vec{\tau} \in E^u$. Using Lemmas 4.7 and 6.4, we get for $k \geq K_0$ and all $n \in \mathbb{N}$,

$$\|\triangle(x_{n+k})\|_* \geq \kappa^n Da^{-n}(y_{-k}) \|\triangle(x_k)\|_*.$$

Now, using the fact that the growth of Da^n is uniformly sub-exponential,[5] we get a contradiction as n goes to infinity. In the case where $\vec{\tau}_{x_0} \in E^s$, we have $\vec{\tau}_{x_0} \notin E^s$, and we may proceed as before using a^{-1} instead of a.

This closes the proof of the absence of an exceptional minimal set, hence of the existence of a finite orbit for G.

[5] This is well-known and follows from the unique ergodicity of a together with that the mean of $\log(Da)$ with resepect to the unique invariant probability measure equals zero; see [19, Proposition I.I, Chapitre VI].

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