Some remarks and questions concerning orderable groups

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The Unique Product Property (UPP) vs orderability

For all finite subsets A, B, there is an element

$$f \in AB := \{gh : g \in A, h \in B\}$$

that can be written uniquely as $f = gh$, with $g \in A$ and $h \in B$.

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There exist torsion-free groups without the UPP.

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- The UPP fails for $A = B$ being equal to

$$\{(ba)^2, (ab)^2, a^2b, aba^{-1}, b, ab^{-1}a, b^{-1}, aba, ab^{-2}, b^2a^{-1}, a(ba)^2, bab, a, a^{-1}\}$$

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Consider triplets $(u\,v\,w)$, where each u,v,w is either an integer or of the form $\hat{m}$.

Multiplication rule:

$$(u_1\,v_1\,w_1)(u_2\,v_2\,w_2) := (u_1 \oplus u_2 \;\; v_1 \oplus v_2 \;\; w_1 \oplus w_2)$$

where for m, n in $\mathbb{Z}$:

$m \oplus n := m+n, \quad m \oplus \hat{n} := \widehat{m+n}, \quad \hat{m} \oplus n := \widehat{m-n}, \quad \hat{m} \oplus \hat{n} := m-n$

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$(002), (00\underline{2}), (\hat{2}1\hat{1}), (\hat{2}\underline{1}\underline{\hat{1}}), (\hat{0}1\hat{1}), (\hat{0}1\underline{\hat{1}}), (\hat{0}\underline{1}\hat{1}),$

$(\hat{0}\underline{1}\underline{\hat{1}}), (1\hat{2}\hat{0}), (\underline{1}\hat{2}\hat{0}), (1\hat{0}\underline{\hat{2}}), (\underline{1}\hat{0}\hat{2}), (1\hat{0}\hat{0}), (\underline{1}\hat{0}\hat{0})$

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	(002)	$(00\underline{2})$	$(\hat{2}1\hat{1})$	$(\hat{2}\underline{1\hat{1}})$	$(\hat{0}1\hat{1})$	$(\hat{0}1\underline{\hat{1}})$	$(\hat{0}\underline{1}\hat{1})$	$(\hat{0}\underline{1\hat{1}})$	$(1\hat{2}\hat{0})$	$(\underline{1}\hat{2}\hat{0})$	$(1\hat{0}\underline{\hat{2}})$	$(\underline{1}\hat{0}\hat{2})$	$(1\hat{0}\hat{0})$	$(\underline{1}\hat{0}\hat{0})$
(002)	(004)	(000)	$(\hat{2}1\hat{3})$	$(\hat{2}\underline{1}\hat{1})$	$(\hat{0}1\hat{3})$	$(\hat{0}1\hat{1})$	$(\hat{0}\underline{1}\hat{3})$	$(\hat{0}\underline{1}\hat{1})$	$(1\hat{2}\hat{2})$	$(\underline{1}\hat{2}\hat{2})$	$(1\hat{0}\hat{0})$	$(\underline{1}\hat{0}\hat{4})$	$(1\hat{0}\hat{2})$	$(\underline{1}\hat{0}\hat{2})$
$(00\underline{2})$	(000)	$(00\underline{4})$	$(\hat{2}1\underline{\hat{1}})$	$(\hat{2}\underline{1\hat{3}})$	$(\hat{0}1\underline{\hat{1}})$	$(\hat{0}1\underline{\hat{3}})$	$(\hat{0}\underline{1\hat{1}})$	$(\hat{0}\underline{1\hat{3}})$	$(1\hat{2}\underline{\hat{2}})$	$(\underline{1}\hat{2}\underline{\hat{2}})$	$(1\hat{0}\underline{\hat{4}})$	$(\underline{1}\hat{0}\hat{0})$	$(1\hat{0}\underline{\hat{2}})$	$(\underline{1}\hat{0}\underline{\hat{2}})$
$(\hat{2}1\hat{1})$	$(\hat{2}1\underline{\hat{1}})$	$(\hat{2}1\hat{3})$	(020)	(002)	(220)	(222)	(200)	(202)	$(\hat{1}\hat{3}1)$	$(\hat{3}\hat{3}1)$	$(\hat{1}\hat{1}3)$	$(\hat{3}\hat{1}\underline{1})$	$(\hat{1}\hat{1}1)$	$(\hat{3}\hat{1}1)$
$(\hat{2}\underline{1\hat{1}})$	$(\hat{2}\underline{1\hat{3}})$	$(\hat{2}\underline{1}\hat{1})$	$(00\underline{2})$	$(0\underline{2}0)$	$(20\underline{2})$	(200)	$(2\underline{22})$	$(2\underline{2}0)$	$(\hat{1}\hat{1}\underline{1})$	$(\hat{3}\hat{1}\underline{1})$	$(\hat{1}\underline{\hat{1}}1)$	$(\hat{3}\underline{\hat{1}3})$	$(\hat{1}\underline{\hat{1}1})$	$(\hat{3}\underline{\hat{1}1})$
$(\hat{0}1\hat{1})$	$(\hat{0}1\underline{\hat{1}})$	$(\hat{0}1\hat{3})$	$(\underline{2}20)$	$(\underline{2}02)$	(020)	(022)	(000)	(002)	$(\underline{\hat{1}}\hat{3}1)$	$(\hat{1}\hat{3}1)$	$(\underline{\hat{1}}\hat{1}3)$	$(\hat{1}\hat{1}\underline{1})$	$(\underline{\hat{1}}\hat{1}1)$	$(\hat{1}\hat{1}1)$
$(\hat{0}1\underline{\hat{1}})$	$(\hat{0}1\underline{\hat{3}})$	$(\hat{0}1\hat{1})$	$(\underline{2}2\underline{2})$	$(\underline{2}00)$	$(02\underline{2})$	(020)	$(00\underline{2})$	(000)	$(\underline{\hat{1}}\hat{3}\underline{1})$	$(\hat{1}\hat{3}\underline{1})$	$(\underline{\hat{1}}\hat{1}1)$	$(\hat{1}\hat{1}\underline{3})$	$(\underline{\hat{1}}\hat{1}\underline{1})$	$(\hat{1}\hat{1}\underline{1})$
$(\hat{0}\underline{1}\hat{1})$	$(\hat{0}\underline{1\hat{1}})$	$(\hat{0}\underline{1}\hat{3})$	$(\underline{2}00)$	$(\underline{22}2)$	(000)	(002)	$(0\underline{2}0)$	$(0\underline{2}2)$	$(\underline{\hat{1}}\hat{1}1)$	$(\hat{1}\hat{1}1)$	$(\underline{\hat{1}\hat{1}}3)$	$(\hat{1}\underline{\hat{1}1})$	$(\underline{\hat{1}\hat{1}}1)$	$(\hat{1}\underline{\hat{1}}1)$
$(\hat{0}\underline{1\hat{1}})$	$(\hat{0}\underline{1\hat{3}})$	$(\hat{0}\underline{1}\hat{1})$	$(\underline{2}0\underline{2})$	$(\underline{22}0)$	$(00\underline{2})$	(000)	$(0\underline{22})$	$(0\underline{2}0)$	$(\underline{\hat{1}}\hat{1}\underline{1})$	$(\hat{1}\hat{1}\underline{1})$	$(\underline{\hat{1}\hat{1}}1)$	$(\hat{1}\underline{\hat{1}3})$	$(\underline{\hat{1}\hat{1}1})$	$(\hat{1}\underline{\hat{1}1})$
$(1\hat{2}\hat{0})$	$(1\hat{2}\underline{\hat{2}})$	$(1\hat{2}\hat{2})$	$(\hat{3}\hat{1}\underline{1})$	$(\hat{3}\hat{3}1)$	$(\hat{1}\hat{1}\underline{1})$	$(\hat{1}\hat{1}1)$	$(\hat{1}\hat{3}\underline{1})$	$(\hat{1}\hat{3}1)$	(200)	(000)	(222)	$(02\underline{2})$	(220)	(020)
$(\underline{1}\hat{2}\hat{0})$	$(\underline{1}\hat{2}\underline{\hat{2}})$	$(\underline{1}\hat{2}\hat{2})$	$(\hat{1}\hat{1}\underline{1})$	$(\hat{1}\hat{3}1)$	$(\underline{\hat{1}}\hat{1}\underline{1})$	$(\underline{\hat{1}}\hat{1}1)$	$(\underline{\hat{1}}\hat{3}\underline{1})$	$(\underline{\hat{1}}\hat{3}1)$	(000)	$(\underline{2}00)$	(022)	$(\underline{2}2\underline{2})$	(020)	$(\underline{2}20)$
$(1\hat{0}\underline{\hat{2}})$	$(1\hat{0}\underline{\hat{4}})$	$(1\hat{0}\hat{0})$	$(\hat{3}\underline{\hat{1}3})$	$(\hat{3}\hat{1}\underline{1})$	$(\hat{1}\underline{\hat{1}3})$	$(\hat{1}\underline{\hat{1}1})$	$(\hat{1}\hat{1}\underline{3})$	$(\hat{1}\hat{1}\underline{1})$	$(2\underline{22})$	$(0\underline{22})$	(200)	$(00\underline{4})$	$(20\underline{2})$	$(00\underline{2})$
$(\underline{1}\hat{0}\hat{2})$	$(\underline{1}\hat{0}\hat{0})$	$(\underline{1}\hat{0}\hat{4})$	$(\hat{1}\underline{\hat{1}}1)$	$(\hat{1}\hat{1}3)$	$(\underline{\hat{1}\hat{1}}1)$	$(\underline{\hat{1}\hat{1}}3)$	$(\underline{\hat{1}}\hat{1}1)$	$(\underline{\hat{1}}\hat{1}3)$	$(0\underline{2}2)$	$(\underline{22}2)$	(004)	$(\underline{2}00)$	(002)	$(\underline{2}02)$
$(1\hat{0}\hat{0})$	$(1\hat{0}\underline{\hat{2}})$	$(1\hat{0}\hat{2})$	$(\hat{3}\underline{\hat{1}1})$	$(\hat{3}\hat{1}1)$	$(\hat{1}\underline{\hat{1}1})$	$(\hat{1}\underline{\hat{1}}1)$	$(\hat{1}\hat{1}\underline{1})$	$(\hat{1}\hat{1}1)$	$(2\underline{2}0)$	$(0\underline{2}0)$	(202)	$(00\underline{2})$	(200)	(000)
$(\underline{1}\hat{0}\hat{0})$	$(\underline{1}\hat{0}\underline{\hat{2}})$	$(\underline{1}\hat{0}\hat{2})$	$(\hat{1}\underline{\hat{1}1})$	$(\hat{1}\hat{1}1)$	$(\underline{\hat{1}\hat{1}1})$	$(\underline{\hat{1}\hat{1}}1)$	$(\underline{\hat{1}}\hat{1}\underline{1})$	$(\underline{\hat{1}}\hat{1}1)$	$(0\underline{2}0)$	$(\underline{22}0)$	(002)	$(\underline{2}0\underline{2})$	(000)	$(\underline{2}00)$

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(002)	(004)	(000)	$(\hat{2}1\hat{3})$	$(\hat{2}\underline{1}\hat{1})$	$(\hat{0}1\hat{3})$	$(\hat{0}1\hat{1})$	$(\hat{0}\underline{1}\hat{3})$	$(\hat{0}\underline{1}\hat{1})$	$(1\hat{2}\hat{2})$	$(\underline{1}\hat{2}\hat{2})$	$(1\hat{0}\hat{0})$	$(\underline{1}\hat{0}\hat{4})$	$(1\hat{0}\hat{2})$	$(\underline{1}\hat{0}\hat{2})$
$(00\underline{2})$	(000)	$(00\underline{4})$	$(\hat{2}1\underline{\hat{1}})$	$(\hat{2}\underline{1}\underline{\hat{3}})$	$(\hat{0}1\underline{\hat{1}})$	$(\hat{0}1\underline{\hat{3}})$	$(\hat{0}\underline{1}\underline{\hat{1}})$	$(\hat{0}\underline{1}\underline{\hat{3}})$	$(1\hat{2}\underline{\hat{2}})$	$(\underline{1}\hat{2}\underline{\hat{2}})$	$(1\hat{0}\underline{\hat{4}})$	$(\underline{1}\hat{0}\hat{0})$	$(1\hat{0}\underline{\hat{2}})$	$(\underline{1}\hat{0}\underline{\hat{2}})$
$(\hat{2}1\hat{1})$	$(\hat{2}1\underline{\hat{1}})$	$(\hat{2}1\hat{3})$	(020)	(002)	(220)	(222)	(200)	(202)	$(\hat{1}\hat{3}1)$	$(\hat{3}\hat{3}1)$	$(\hat{1}\hat{1}3)$	$(\hat{3}\hat{1}\underline{1})$	$(\hat{1}\hat{1}1)$	$(\hat{3}\hat{1}1)$
$(\hat{2}\underline{1}\underline{\hat{1}})$	$(\hat{2}\underline{1}\underline{\hat{3}})$	$(\hat{2}\underline{1}\hat{1})$	$(00\underline{2})$	$(0\underline{2}0)$	$(20\underline{2})$	(200)	$(2\underline{2}\underline{2})$	$(2\underline{2}0)$	$(\hat{1}\hat{1}\underline{1})$	$(\hat{3}\hat{1}\underline{1})$	$(\hat{1}\underline{\hat{1}}1)$	$(\hat{3}\underline{\hat{1}}\underline{3})$	$(\hat{1}\underline{\hat{1}}\underline{1})$	$(\hat{3}\underline{\hat{1}}\underline{1})$
$(\hat{0}1\hat{1})$	$(\hat{0}1\underline{\hat{1}})$	$(\hat{0}1\hat{3})$	$(\underline{2}20)$	$(\underline{2}02)$	(020)	(022)	(000)	(002)	$(\underline{\hat{1}}\hat{3}1)$	$(\hat{1}\hat{3}1)$	$(\underline{\hat{1}}\hat{1}3)$	$(\hat{1}\hat{1}\underline{1})$	$(\underline{\hat{1}}\hat{1}1)$	$(\hat{1}\hat{1}1)$
$(\hat{0}1\underline{\hat{1}})$	$(\hat{0}1\underline{\hat{3}})$	$(\hat{0}1\hat{1})$	$(\underline{2}2\underline{2})$	$(\underline{2}00)$	$(02\underline{2})$	(020)	$(00\underline{2})$	(000)	$(\underline{\hat{1}}\hat{3}\underline{1})$	$(\hat{1}\hat{3}\underline{1})$	$(\underline{\hat{1}}\hat{1}1)$	$(\hat{1}\hat{1}\underline{3})$	$(\underline{\hat{1}}\hat{1}\underline{1})$	$(\hat{1}\hat{1}\underline{1})$
$(\hat{0}\underline{1}\hat{1})$	$(\hat{0}\underline{1}\underline{\hat{1}})$	$(\hat{0}\underline{1}\hat{3})$	$(\underline{2}00)$	$(\underline{2}\underline{2}2)$	(000)	(002)	$(0\underline{2}0)$	$(0\underline{2}2)$	$(\underline{\hat{1}}\hat{1}1)$	$(\hat{1}\hat{1}1)$	$(\underline{\hat{1}}\underline{\hat{1}}3)$	$(\hat{1}\underline{\hat{1}}\underline{1})$	$(\underline{\hat{1}}\underline{\hat{1}}1)$	$(\hat{1}\underline{\hat{1}}1)$
$(\hat{0}\underline{1}\underline{\hat{1}})$	$(\hat{0}\underline{1}\underline{\hat{3}})$	$(\hat{0}\underline{1}\hat{1})$	$(\underline{2}0\underline{2})$	$(\underline{2}\underline{2}0)$	$(00\underline{2})$	(000)	$(0\underline{2}\underline{2})$	$(0\underline{2}0)$	$(\underline{\hat{1}}\hat{1}\underline{1})$	$(\hat{1}\hat{1}\underline{1})$	$(\underline{\hat{1}}\underline{\hat{1}}1)$	$(\hat{1}\underline{\hat{1}}\underline{3})$	$(\underline{\hat{1}}\underline{\hat{1}}\underline{1})$	$(\hat{1}\underline{\hat{1}}\underline{1})$
$(1\hat{2}\hat{0})$	$(1\hat{2}\underline{\hat{2}})$	$(1\hat{2}\hat{2})$	$(\hat{3}\hat{1}\underline{1})$	$(\hat{3}\hat{3}1)$	$(\hat{1}\hat{1}\underline{1})$	$(\hat{1}\hat{1}1)$	$(\hat{1}\hat{3}\underline{1})$	$(\hat{1}\hat{3}1)$	(200)	(000)	(222)	$(02\underline{2})$	(220)	(020)
$(\underline{1}\hat{2}\hat{0})$	$(\underline{1}\hat{2}\underline{\hat{2}})$	$(\underline{1}\hat{2}\hat{2})$	$(\hat{1}\hat{1}\underline{1})$	$(\hat{1}\hat{3}1)$	$(\underline{\hat{1}}\hat{1}\underline{1})$	$(\underline{\hat{1}}\hat{1}1)$	$(\underline{\hat{1}}\hat{3}\underline{1})$	$(\underline{\hat{1}}\hat{3}1)$	(000)	$(\underline{2}00)$	(022)	$(\underline{2}2\underline{2})$	(020)	$(\underline{2}20)$
$(1\hat{0}\underline{\hat{2}})$	$(1\hat{0}\underline{\hat{4}})$	$(1\hat{0}\hat{0})$	$(\hat{3}\underline{\hat{1}}\underline{3})$	$(\hat{3}\hat{1}\underline{1})$	$(\hat{1}\underline{\hat{1}}\underline{3})$	$(\hat{1}\underline{\hat{1}}\underline{1})$	$(\hat{1}\hat{1}\underline{3})$	$(\hat{1}\hat{1}\underline{1})$	$(2\underline{2}\underline{2})$	$(0\underline{2}\underline{2})$	(200)	$(00\underline{4})$	$(20\underline{2})$	$(00\underline{2})$
$(\underline{1}\hat{0}\underline{\hat{2}})$	$(\underline{1}\hat{0}\hat{0})$	$(\underline{1}\hat{0}\hat{4})$	$(\hat{1}\underline{\hat{1}}1)$	$(\hat{1}\hat{1}3)$	$(\underline{\hat{1}}\underline{\hat{1}}1)$	**$(\underline{\hat{1}}\underline{\hat{1}}3)$**	$(\underline{\hat{1}}\hat{1}1)$	$(\underline{\hat{1}}\hat{1}3)$	$(0\underline{2}2)$	$(\underline{2}\underline{2}2)$	(004)	$(\underline{2}00)$	(002)	$(\underline{2}02)$
$(1\hat{0}\hat{0})$	$(1\hat{0}\underline{\hat{2}})$	$(1\hat{0}\hat{2})$	$(\hat{3}\underline{\hat{1}}\underline{1})$	$(\hat{3}\hat{1}1)$	$(\hat{1}\underline{\hat{1}}\underline{1})$	$(\hat{1}\underline{\hat{1}}1)$	$(\hat{1}\hat{1}\underline{1})$	$(\hat{1}\hat{1}1)$	$(2\underline{2}0)$	$(0\underline{2}0)$	(202)	$(00\underline{2})$	(200)	(000)
$(\underline{1}\hat{0}\hat{0})$	$(\underline{1}\hat{0}\underline{\hat{2}})$	$(\underline{1}\hat{0}\hat{2})$	$(\hat{1}\underline{\hat{1}}\underline{1})$	$(\hat{1}\hat{1}1)$	$(\underline{\hat{1}}\underline{\hat{1}}\underline{1})$	$(\underline{\hat{1}}\underline{\hat{1}}1)$	$(\underline{\hat{1}}\hat{1}\underline{1})$	$(\underline{\hat{1}}\hat{1}1)$	$(0\underline{2}0)$	$(\underline{2}\underline{2}0)$	(002)	$(\underline{2}0\underline{2})$	(000)	$(\underline{2}00)$

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	(002)	$(00\underline{2})$	$(\hat{2}1\hat{1})$	$(\hat{2}\underline{1}\underline{\hat{1}})$	$(\hat{0}1\hat{1})$	$(\hat{0}1\underline{\hat{1}})$	$(\hat{0}\underline{1}\hat{1})$	$(\hat{0}\underline{1}\underline{\hat{1}})$	$(1\hat{2}\hat{0})$	$(\underline{1}\hat{2}\hat{0})$	$(1\hat{0}\underline{\hat{2}})$	$(\underline{1}\hat{0}\hat{2})$	$(1\hat{0}\hat{0})$	$(\underline{1}\hat{0}\hat{0})$
(002)	(004)	(000)	$(\hat{2}1\hat{3})$	$(\hat{2}\underline{1}\hat{1})$	$(\hat{0}1\hat{3})$	$(\hat{0}1\hat{1})$	$(\hat{0}\underline{1}\hat{3})$	$(\hat{0}\underline{1}\hat{1})$	$(1\hat{2}\hat{2})$	$(\underline{1}\hat{2}\hat{2})$	$(1\hat{0}\hat{0})$	$(\underline{1}\hat{0}\hat{4})$	$(1\hat{0}\hat{2})$	$(\underline{1}\hat{0}\hat{2})$
$(00\underline{2})$	(000)	$(00\underline{4})$	$(\hat{2}1\underline{\hat{1}})$	$(\hat{2}\underline{1}\underline{\hat{3}})$	$(\hat{0}1\underline{\hat{1}})$	$(\hat{0}1\underline{\hat{3}})$	$(\hat{0}\underline{1}\underline{\hat{1}})$	$(\hat{0}\underline{1}\underline{\hat{3}})$	$(1\hat{2}\underline{\hat{2}})$	$(\underline{1}\hat{2}\underline{\hat{2}})$	$(1\hat{0}\underline{\hat{4}})$	$(\underline{1}\hat{0}\hat{0})$	$(1\hat{0}\underline{\hat{2}})$	$(\underline{1}\hat{0}\underline{\hat{2}})$
$(\hat{2}1\hat{1})$	$(\hat{2}1\underline{\hat{1}})$	$(\hat{2}1\hat{3})$	(020)	(002)	(220)	(222)	(200)	(202)	$(\hat{1}\hat{3}1)$	$(\hat{3}\hat{3}1)$	$(\hat{1}\hat{1}3)$	$(\hat{3}\hat{1}\underline{1})$	$(\hat{1}\hat{1}1)$	$(\hat{3}\hat{1}1)$
$(\hat{2}\underline{1}\underline{\hat{1}})$	$(\hat{2}\underline{1}\underline{\hat{3}})$	$(\hat{2}\underline{1}\hat{1})$	$(00\underline{2})$	$(0\underline{2}0)$	$(20\underline{2})$	(200)	$(2\underline{2}\underline{2})$	$(2\underline{2}0)$	$(\hat{1}\hat{1}\underline{1})$	$(\hat{3}\hat{1}\underline{1})$	$(\hat{1}\underline{\hat{1}}1)$	$(\hat{3}\underline{\hat{1}}\underline{3})$	$(\hat{1}\underline{\hat{1}}\underline{1})$	$(\hat{3}\underline{\hat{1}}\underline{1})$
$(\hat{0}1\hat{1})$	$(\hat{0}1\underline{\hat{1}})$	$(\hat{0}1\hat{3})$	$(\underline{2}20)$	$(\underline{2}02)$	(020)	(022)	(000)	(002)	$(\underline{\hat{1}}\hat{3}1)$	$(\hat{1}\hat{3}1)$	$(\underline{\hat{1}}\hat{1}3)$	$(\hat{1}\hat{1}\underline{1})$	$(\underline{\hat{1}}\hat{1}1)$	$(\hat{1}\hat{1}1)$
$(\hat{0}1\underline{\hat{1}})$	$(\hat{0}1\underline{\hat{3}})$	$(\hat{0}1\hat{1})$	$(\underline{2}2\underline{2})$	$(\underline{2}00)$	$(02\underline{2})$	(020)	$(00\underline{2})$	(000)	$(\underline{\hat{1}}\hat{3}\underline{1})$	$(\hat{1}\hat{3}\underline{1})$	$(\underline{\hat{1}}\hat{1}1)$	$(\hat{1}\hat{1}\underline{3})$	$(\underline{\hat{1}}\hat{1}\underline{1})$	$(\hat{1}\hat{1}\underline{1})$
$(\hat{0}\underline{1}\hat{1})$	$(\hat{0}\underline{1}\underline{\hat{1}})$	$(\hat{0}\underline{1}\hat{3})$	$(\underline{2}00)$	$(\underline{2}\underline{2}2)$	(000)	(002)	$(0\underline{2}0)$	$(0\underline{2}2)$	$(\underline{\hat{1}}\hat{1}1)$	$(\hat{1}\hat{1}1)$	**$(\underline{\hat{1}}\underline{\hat{1}}3)$**	$(\hat{1}\underline{\hat{1}}\underline{1})$	$(\underline{\hat{1}}\underline{\hat{1}}1)$	$(\hat{1}\underline{\hat{1}}1)$
$(\hat{0}\underline{1}\underline{\hat{1}})$	$(\hat{0}\underline{1}\underline{\hat{3}})$	$(\hat{0}\underline{1}\hat{1})$	$(\underline{2}0\underline{2})$	$(\underline{2}\underline{2}0)$	$(00\underline{2})$	(000)	$(0\underline{2}\underline{2})$	$(0\underline{2}0)$	$(\underline{\hat{1}}\hat{1}\underline{1})$	$(\hat{1}\hat{1}\underline{1})$	$(\underline{\hat{1}}\underline{\hat{1}}1)$	$(\hat{1}\underline{\hat{1}}\underline{3})$	$(\underline{\hat{1}}\underline{\hat{1}}\underline{1})$	$(\hat{1}\underline{\hat{1}}\underline{1})$
$(1\hat{2}\hat{0})$	$(1\hat{2}\underline{\hat{2}})$	$(1\hat{2}\hat{2})$	$(\hat{3}\hat{1}\underline{1})$	$(\hat{3}\hat{3}1)$	$(\hat{1}\hat{1}\underline{1})$	$(\hat{1}\hat{1}1)$	$(\hat{1}\hat{3}\underline{1})$	$(\hat{1}\hat{3}1)$	(200)	(000)	(222)	$(02\underline{2})$	(220)	(020)
$(\underline{1}\hat{2}\hat{0})$	$(\underline{1}\hat{2}\underline{\hat{2}})$	$(\underline{1}\hat{2}\hat{2})$	$(\hat{1}\hat{1}\underline{1})$	$(\hat{1}\hat{3}1)$	$(\underline{\hat{1}}\hat{1}\underline{1})$	$(\underline{\hat{1}}\hat{1}1)$	$(\underline{\hat{1}}\hat{3}\underline{1})$	$(\underline{\hat{1}}\hat{3}1)$	(000)	$(\underline{2}00)$	(022)	$(\underline{2}2\underline{2})$	(020)	$(\underline{2}20)$
$(1\hat{0}\underline{\hat{2}})$	$(1\hat{0}\underline{\hat{4}})$	$(1\hat{0}\hat{0})$	$(\hat{3}\underline{\hat{1}}\underline{3})$	$(\hat{3}\hat{1}\underline{1})$	$(\hat{1}\underline{\hat{1}}\underline{3})$	$(\hat{1}\underline{\hat{1}}\underline{1})$	$(\hat{1}\hat{1}\underline{3})$	$(\hat{1}\hat{1}\underline{1})$	$(2\underline{2}\underline{2})$	$(0\underline{2}\underline{2})$	(200)	$(00\underline{4})$	$(20\underline{2})$	$(00\underline{2})$
$(\underline{1}\hat{0}\hat{2})$	$(\underline{1}\hat{0}\hat{0})$	$(\underline{1}\hat{0}\hat{4})$	$(\hat{1}\underline{\hat{1}}1)$	$(\hat{1}\hat{1}3)$	$(\underline{\hat{1}}\underline{\hat{1}}1)$	**$(\underline{\hat{1}}\underline{\hat{1}}3)$**	$(\underline{\hat{1}}\hat{1}1)$	$(\underline{\hat{1}}\hat{1}3)$	$(0\underline{2}2)$	$(\underline{2}\underline{2}2)$	(004)	$(\underline{2}00)$	(002)	$(\underline{2}02)$
$(1\hat{0}\hat{0})$	$(1\hat{0}\underline{\hat{2}})$	$(1\hat{0}\hat{2})$	$(\hat{3}\underline{\hat{1}}\underline{1})$	$(\hat{3}\hat{1}1)$	$(\hat{1}\underline{\hat{1}}\underline{1})$	$(\hat{1}\underline{\hat{1}}1)$	$(\hat{1}\hat{1}\underline{1})$	$(\hat{1}\hat{1}1)$	$(2\underline{2}0)$	$(0\underline{2}0)$	(202)	$(00\underline{2})$	(200)	(000)
$(\underline{1}\hat{0}\hat{0})$	$(\underline{1}\hat{0}\underline{\hat{2}})$	$(\underline{1}\hat{0}\hat{2})$	$(\hat{1}\underline{\hat{1}}\underline{1})$	$(\hat{1}\hat{1}1)$	$(\underline{\hat{1}}\underline{\hat{1}}\underline{1})$	$(\underline{\hat{1}}\underline{\hat{1}}1)$	$(\underline{\hat{1}}\hat{1}\underline{1})$	$(\underline{\hat{1}}\hat{1}1)$	$(0\underline{2}0)$	$(\underline{2}\underline{2}0)$	(002)	$(\underline{2}0\underline{2})$	(000)	$(\underline{2}00)$

The Unique Product Property (UPP) vs orderability

	(002)	$(00\underline{2})$	$(\hat{2}1\hat{1})$	$(\hat{2}\underline{1}\underline{\hat{1}})$	$(\hat{0}1\hat{1})$	$(\hat{0}1\underline{\hat{1}})$	$(\hat{0}\underline{1}\hat{1})$	$(\hat{0}\underline{1}\underline{\hat{1}})$	$(1\hat{2}\hat{0})$	$(\underline{1}\hat{2}\hat{0})$	$(1\hat{0}\underline{\hat{2}})$	$(\underline{1}\hat{0}\hat{2})$	$(1\hat{0}\hat{0})$	$(\underline{1}\hat{0}\hat{0})$
(002)	(004)	(000)	$(\hat{2}1\hat{3})$	$(\hat{2}\underline{1}\hat{1})$	$(\hat{0}1\hat{3})$	$(\hat{0}1\hat{1})$	$(\hat{0}\underline{1}\hat{3})$	$(\hat{0}\underline{1}\hat{1})$	$(1\hat{2}\hat{2})$	$(\underline{1}\hat{2}\hat{2})$	$(1\hat{0}\hat{0})$	$(\underline{1}\hat{0}\hat{4})$	$(1\hat{0}\hat{2})$	$(\underline{1}\hat{0}\hat{2})$
$(00\underline{2})$	(000)	$(00\underline{4})$	$(\hat{2}1\underline{\hat{1}})$	$(\hat{2}\underline{1}\underline{\hat{3}})$	$(\hat{0}1\underline{\hat{1}})$	$(\hat{0}1\underline{\hat{3}})$	$(\hat{0}\underline{1}\underline{\hat{1}})$	$(\hat{0}\underline{1}\underline{\hat{3}})$	$(1\hat{2}\underline{\hat{2}})$	$(\underline{1}\hat{2}\underline{\hat{2}})$	$(1\hat{0}\underline{\hat{4}})$	$(\underline{1}\hat{0}\hat{0})$	$(1\hat{0}\underline{\hat{2}})$	$(\underline{1}\hat{0}\underline{\hat{2}})$
$(\hat{2}1\hat{1})$	$(\hat{2}1\underline{\hat{1}})$	$(\hat{2}1\hat{3})$	(020)	(002)	(220)	(222)	(200)	(202)	$(\hat{1}\hat{3}1)$	$(\hat{3}\hat{3}1)$	$(\hat{1}\hat{1}3)$	$(\hat{3}\hat{1}\underline{1})$	$(\hat{1}\hat{1}1)$	$(\hat{3}\hat{1}1)$
$(\hat{2}\underline{1}\underline{\hat{1}})$	$(\hat{2}\underline{1}\underline{\hat{3}})$	$(\hat{2}\underline{1}\hat{1})$	$(00\underline{2})$	$(0\underline{2}0)$	$(20\underline{2})$	(200)	$(2\underline{2}\underline{2})$	$(2\underline{2}0)$	$(\hat{1}\hat{1}\underline{1})$	$(\hat{3}\hat{1}\underline{1})$	$(\hat{1}\underline{\hat{1}}1)$	$(\hat{3}\underline{\hat{1}}\underline{3})$	$(\hat{1}\underline{\hat{1}}\underline{1})$	$(\hat{3}\underline{\hat{1}}\underline{1})$
$(\hat{0}1\hat{1})$	$(\hat{0}1\underline{\hat{1}})$	$(\hat{0}1\hat{3})$	$(\underline{2}20)$	$(\underline{2}02)$	(020)	(022)	(000)	(002)	$(\underline{\hat{1}}\hat{3}1)$	$(\hat{1}\hat{3}1)$	$(\underline{\hat{1}}\hat{1}3)$	$(\hat{1}\hat{1}\underline{1})$	$(\underline{\hat{1}}\hat{1}1)$	$(\hat{1}\hat{1}1)$
$(\hat{0}1\underline{\hat{1}})$	$(\hat{0}1\underline{\hat{3}})$	$(\hat{0}1\hat{1})$	$(\underline{2}2\underline{2})$	$(\underline{2}00)$	$(02\underline{2})$	**(020)**	$(00\underline{2})$	(000)	$(\underline{\hat{1}}\hat{3}\underline{1})$	$(\hat{1}\hat{3}\underline{1})$	$(\underline{\hat{1}}\hat{1}1)$	$(\hat{1}\hat{1}\underline{3})$	$(\underline{\hat{1}}\hat{1}\underline{1})$	$(\hat{1}\hat{1}\underline{1})$
$(\hat{0}\underline{1}\hat{1})$	$(\hat{0}\underline{1}\underline{\hat{1}})$	$(\hat{0}\underline{1}\hat{3})$	$(\underline{2}00)$	$(\underline{2}\underline{2}\underline{2})$	(000)	(002)	$(0\underline{2}0)$	$(0\underline{2}2)$	$(\underline{\hat{1}}\hat{1}1)$	$(\hat{1}\hat{1}1)$	$(\underline{\hat{1}}\underline{\hat{1}}3)$	$(\hat{1}\underline{\hat{1}}\underline{1})$	$(\underline{\hat{1}}\underline{\hat{1}}1)$	$(\hat{1}\underline{\hat{1}}1)$
$(\hat{0}\underline{1}\underline{\hat{1}})$	$(\hat{0}\underline{1}\underline{\hat{3}})$	$(\hat{0}\underline{1}\hat{1})$	$(\underline{2}0\underline{2})$	$(\underline{2}\underline{2}0)$	$(00\underline{2})$	(000)	$(0\underline{2}\underline{2})$	$(0\underline{2}0)$	$(\underline{\hat{1}}\hat{1}\underline{1})$	$(\hat{1}\hat{1}\underline{1})$	$(\underline{\hat{1}}\underline{\hat{1}}1)$	$(\hat{1}\underline{\hat{1}}\underline{3})$	$(\underline{\hat{1}}\underline{\hat{1}}\underline{1})$	$(\hat{1}\underline{\hat{1}}\underline{1})$
$(1\hat{2}\hat{0})$	$(1\hat{2}\underline{\hat{2}})$	$(1\hat{2}\hat{2})$	$(\hat{3}\hat{1}\underline{1})$	$(\hat{3}\hat{3}1)$	$(\hat{1}\hat{1}\underline{1})$	$(\hat{1}\hat{1}1)$	$(\hat{1}\hat{3}\underline{1})$	$(\hat{1}\hat{3}1)$	(200)	(000)	(222)	$(02\underline{2})$	(220)	(020)
$(\underline{1}\hat{2}\hat{0})$	$(\underline{1}\hat{2}\underline{\hat{2}})$	$(\underline{1}\hat{2}\hat{2})$	$(\hat{1}\hat{1}\underline{1})$	$(\hat{1}\hat{3}1)$	$(\underline{\hat{1}}\hat{1}\underline{1})$	$(\underline{\hat{1}}\hat{1}1)$	$(\underline{\hat{1}}\hat{3}\underline{1})$	$(\underline{\hat{1}}\hat{3}1)$	(000)	$(\underline{2}00)$	(022)	$(\underline{2}2\underline{2})$	(020)	$(\underline{2}20)$
$(1\hat{0}\underline{\hat{2}})$	$(1\hat{0}\underline{\hat{4}})$	$(1\hat{0}\hat{0})$	$(\hat{3}\underline{\hat{1}}\underline{3})$	$(\hat{3}\hat{1}\underline{1})$	$(\hat{1}\underline{\hat{1}}\underline{3})$	$(\hat{1}\underline{\hat{1}}\underline{1})$	$(\hat{1}\hat{1}\underline{3})$	$(\hat{1}\hat{1}\underline{1})$	$(2\underline{2}\underline{2})$	$(0\underline{2}\underline{2})$	(200)	$(00\underline{4})$	$(20\underline{2})$	$(00\underline{2})$
$(\underline{1}\hat{0}\hat{2})$	$(\underline{1}\hat{0}\hat{0})$	$(\underline{1}\hat{0}\hat{4})$	$(\hat{1}\underline{\hat{1}}1)$	$(\hat{1}\hat{1}3)$	$(\underline{\hat{1}}\underline{\hat{1}}1)$	$(\underline{\hat{1}}\underline{\hat{1}}3)$	$(\underline{\hat{1}}\hat{1}1)$	$(\underline{\hat{1}}\hat{1}3)$	$(0\underline{2}2)$	$(\underline{2}\underline{2}2)$	(004)	$(\underline{2}00)$	(002)	$(\underline{2}02)$
$(1\hat{0}\hat{0})$	$(1\hat{0}\underline{\hat{2}})$	$(1\hat{0}\hat{2})$	$(\hat{3}\underline{\hat{1}}\underline{1})$	$(\hat{3}\hat{1}1)$	$(\hat{1}\underline{\hat{1}}\underline{1})$	$(\hat{1}\underline{\hat{1}}1)$	$(\hat{1}\hat{1}\underline{1})$	$(\hat{1}\hat{1}1)$	$(2\underline{2}0)$	$(0\underline{2}0)$	(202)	$(00\underline{2})$	(200)	(000)
$(\underline{1}\hat{0}\hat{0})$	$(\underline{1}\hat{0}\underline{\hat{2}})$	$(\underline{1}\hat{0}\hat{2})$	$(\hat{1}\underline{\hat{1}}\underline{1})$	$(\hat{1}\hat{1}1)$	$(\underline{\hat{1}}\underline{\hat{1}}\underline{1})$	$(\underline{\hat{1}}\underline{\hat{1}}1)$	$(\underline{\hat{1}}\hat{1}\underline{1})$	$(\underline{\hat{1}}\hat{1}1)$	$(0\underline{2}0)$	$(\underline{2}\underline{2}0)$	(002)	$(\underline{2}0\underline{2})$	(000)	$(\underline{2}00)$

The Unique Product Property (UPP) vs orderability

	(002)	$(00\underline{2})$	$(\hat{2}1\hat{1})$	$(\hat{2}\underline{1}\underline{\hat{1}})$	$(\hat{0}1\hat{1})$	$(\hat{0}1\underline{\hat{1}})$	$(\hat{0}\underline{1}\hat{1})$	$(\hat{0}\underline{1}\underline{\hat{1}})$	$(1\hat{2}\hat{0})$	$(\underline{1}\hat{2}\hat{0})$	$(1\hat{0}\underline{\hat{2}})$	$(\underline{1}\hat{0}\underline{\hat{2}})$	$(1\hat{0}\hat{0})$	$(\underline{1}\hat{0}\hat{0})$
(002)	(004)	(000)	$(\hat{2}1\hat{3})$	$(\hat{2}\underline{1}\hat{1})$	$(\hat{0}1\hat{3})$	$(\hat{0}1\hat{1})$	$(\hat{0}\underline{1}\hat{3})$	$(\hat{0}\underline{1}\hat{1})$	$(1\hat{2}\hat{2})$	$(\underline{1}\hat{2}\hat{2})$	$(1\hat{0}\hat{0})$	$(\underline{1}\hat{0}\hat{4})$	$(1\hat{0}\hat{2})$	$(\underline{1}\hat{0}\hat{2})$
$(00\underline{2})$	(000)	$(00\underline{4})$	$(\hat{2}1\underline{\hat{1}})$	$(\hat{2}\underline{1}\underline{\hat{3}})$	$(\hat{0}1\underline{\hat{1}})$	$(\hat{0}1\underline{\hat{3}})$	$(\hat{0}\underline{1}\underline{\hat{1}})$	$(\hat{0}\underline{1}\underline{\hat{3}})$	$(1\hat{2}\underline{\hat{2}})$	$(\underline{1}\hat{2}\underline{\hat{2}})$	$(1\hat{0}\underline{\hat{4}})$	$(\underline{1}\hat{0}\hat{0})$	$(1\hat{0}\underline{\hat{2}})$	$(\underline{1}\hat{0}\underline{\hat{2}})$
$(\hat{2}1\hat{1})$	$(\hat{2}1\underline{\hat{1}})$	$(\hat{2}1\hat{3})$	**(020)**	(002)	(220)	(222)	(200)	(202)	$(\hat{1}\hat{3}1)$	$(\hat{3}\hat{3}1)$	$(\hat{1}\hat{1}3)$	$(\hat{3}\hat{1}\underline{1})$	$(\hat{1}\hat{1}1)$	$(\hat{3}\hat{1}1)$
$(\hat{2}\underline{1}\underline{\hat{1}})$	$(\hat{2}\underline{1}\underline{\hat{3}})$	$(\hat{2}\underline{1}\hat{1})$	$(00\underline{2})$	$(0\underline{2}0)$	$(20\underline{2})$	(200)	$(2\underline{2}\underline{2})$	$(2\underline{2}0)$	$(\hat{1}\hat{1}\underline{1})$	$(\hat{3}\hat{1}\underline{1})$	$(\hat{1}\underline{\hat{1}}1)$	$(\hat{3}\underline{\hat{1}}\underline{3})$	$(\hat{1}\underline{\hat{1}}\underline{1})$	$(\hat{3}\underline{\hat{1}}\underline{1})$
$(\hat{0}1\hat{1})$	$(\hat{0}1\underline{\hat{1}})$	$(\hat{0}1\hat{3})$	$(\underline{2}20)$	$(\underline{2}02)$	**(020)**	(022)	(000)	(002)	$(\underline{\hat{1}}\hat{3}1)$	$(\hat{1}\hat{3}1)$	$(\underline{\hat{1}}\hat{1}3)$	$(\hat{1}\hat{1}\underline{1})$	$(\underline{\hat{1}}\hat{1}1)$	$(\hat{1}\hat{1}1)$
$(\hat{0}1\underline{\hat{1}})$	$(\hat{0}1\underline{\hat{3}})$	$(\hat{0}1\hat{1})$	$(\underline{2}2\underline{2})$	$(\underline{2}00)$	$(02\underline{2})$	**(020)**	$(00\underline{2})$	(000)	$(\underline{\hat{1}}\hat{3}\underline{1})$	$(\hat{1}\hat{3}\underline{1})$	$(\underline{\hat{1}}\hat{1}1)$	$(\hat{1}\hat{1}\underline{3})$	$(\underline{\hat{1}}\hat{1}\underline{1})$	$(\hat{1}\hat{1}\underline{1})$
$(\hat{0}\underline{1}\hat{1})$	$(\hat{0}\underline{1}\underline{\hat{1}})$	$(\hat{0}\underline{1}\hat{3})$	$(\underline{2}00)$	$(\underline{2}\underline{2}2)$	(000)	(002)	$(0\underline{2}0)$	$(0\underline{2}2)$	$(\underline{\hat{1}}\hat{1}1)$	$(\hat{1}\hat{1}1)$	$(\underline{\hat{1}}\underline{\hat{1}}3)$	$(\hat{1}\underline{\hat{1}}\underline{1})$	$(\underline{\hat{1}}\underline{\hat{1}}1)$	$(\hat{1}\underline{\hat{1}}1)$
$(\hat{0}\underline{1}\underline{\hat{1}})$	$(\hat{0}\underline{1}\underline{\hat{3}})$	$(\hat{0}\underline{1}\hat{1})$	$(\underline{2}0\underline{2})$	$(\underline{2}\underline{2}0)$	$(00\underline{2})$	(000)	$(0\underline{2}\underline{2})$	$(0\underline{2}0)$	$(\underline{\hat{1}}\hat{1}\underline{1})$	$(\hat{1}\hat{1}\underline{1})$	$(\underline{\hat{1}}\underline{\hat{1}}1)$	$(\hat{1}\underline{\hat{1}}\underline{3})$	$(\underline{\hat{1}}\underline{\hat{1}}\underline{1})$	$(\hat{1}\underline{\hat{1}}\underline{1})$
$(1\hat{2}\hat{0})$	$(1\hat{2}\underline{\hat{2}})$	$(1\hat{2}\hat{2})$	$(\hat{3}\hat{1}\underline{1})$	$(\hat{3}\hat{3}1)$	$(\hat{1}\hat{1}\underline{1})$	$(\hat{1}\hat{1}1)$	$(\hat{1}\hat{3}\underline{1})$	$(\hat{1}\hat{3}1)$	(200)	(000)	(222)	$(02\underline{2})$	(220)	**(020)**
$(\underline{1}\hat{2}\hat{0})$	$(\underline{1}\hat{2}\underline{\hat{2}})$	$(\underline{1}\hat{2}\hat{2})$	$(\hat{1}\hat{1}\underline{1})$	$(\hat{1}\hat{3}1)$	$(\underline{\hat{1}}\hat{1}\underline{1})$	$(\underline{\hat{1}}\hat{1}1)$	$(\underline{\hat{1}}\hat{3}\underline{1})$	$(\underline{\hat{1}}\hat{3}1)$	(000)	$(\underline{2}00)$	(022)	$(\underline{2}2\underline{2})$	**(020)**	$(\underline{2}20)$
$(1\hat{0}\underline{\hat{2}})$	$(1\hat{0}\underline{\hat{4}})$	$(1\hat{0}\hat{0})$	$(\hat{3}\underline{\hat{1}}\underline{3})$	$(\hat{3}\hat{1}\underline{1})$	$(\hat{1}\underline{\hat{1}}\underline{3})$	$(\hat{1}\underline{\hat{1}}\underline{1})$	$(\hat{1}\hat{1}\underline{3})$	$(\hat{1}\hat{1}\underline{1})$	$(2\underline{2}\underline{2})$	$(0\underline{2}\underline{2})$	(200)	$(00\underline{4})$	$(20\underline{2})$	$(00\underline{2})$
$(\underline{1}\hat{0}\hat{2})$	$(\underline{1}\hat{0}\hat{0})$	$(\underline{1}\hat{0}\hat{4})$	$(\hat{1}\underline{\hat{1}}1)$	$(\hat{1}\hat{1}3)$	$(\underline{\hat{1}}\underline{\hat{1}}1)$	$(\underline{\hat{1}}\underline{\hat{1}}3)$	$(\underline{\hat{1}}\hat{1}1)$	$(\underline{\hat{1}}\hat{1}3)$	$(0\underline{2}2)$	$(\underline{2}\underline{2}2)$	(004)	$(\underline{2}00)$	(002)	$(\underline{2}02)$
$(1\hat{0}\hat{0})$	$(1\hat{0}\underline{\hat{2}})$	$(1\hat{0}\hat{2})$	$(\hat{3}\underline{\hat{1}}\underline{1})$	$(\hat{3}\hat{1}1)$	$(\hat{1}\underline{\hat{1}}\underline{1})$	$(\hat{1}\underline{\hat{1}}1)$	$(\hat{1}\hat{1}\underline{1})$	$(\hat{1}\hat{1}1)$	$(2\underline{2}0)$	$(0\underline{2}0)$	(202)	$(00\underline{2})$	(200)	(000)
$(\underline{1}\hat{0}\hat{0})$	$(\underline{1}\hat{0}\underline{\hat{2}})$	$(\underline{1}\hat{0}\hat{2})$	$(\hat{1}\underline{\hat{1}}\underline{1})$	$(\hat{1}\hat{1}1)$	$(\underline{\hat{1}}\underline{\hat{1}}\underline{1})$	$(\underline{\hat{1}}\underline{\hat{1}}1)$	$(\underline{\hat{1}}\hat{1}\underline{1})$	$(\underline{\hat{1}}\hat{1}1)$	$(0\underline{2}0)$	$(\underline{2}\underline{2}0)$	(002)	$(\underline{2}0\underline{2})$	(000)	$(\underline{2}00)$

The Unique Product Property (UPP) vs orderability

Orderable groups do satisfy the UPP.

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Question

Does there exist a UPP group which is non-orderable ?

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Theorem (Delzant, Bowditch, Chiswell)

If a hyperbolic closed manifold has a "large enough" injectivity radius, then its π_1 satisfies the UPP.

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Theorem (F.Calegari, Dunfield)

There exists a sequence of closed hyperbolic 3-manifolds with trivial first Betti numbers and injectivity radius going to infinite.

Laws and new categories of orders

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Let Γ be an orderable group satisfying a nontrivial law (identity). Is Γ necessarily solvable (or at least locally indicable) ?

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An affirmative answer to (the weak form of) this question would solve the next

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Laws and new categories of orders

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Theorem (Akhmedov, N)

If Γ is a subgroup of $\mathrm{Diff}_+^{\omega}([0,1])$ ($\mathrm{Diff}_+^{2}([0,1])$?) satisfying a nontrivial law, then Γ is solvable.

Laws and new categories of (left) orders

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Definition (Question)

Let W be a nontrivial word on two letters. Say that an order is a W-order if $W(f, g) \in P$ for all f, g in P. Does there exist W such that W-orders correspond to orders with a relevant algebraic property ?

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A link: If an orderable group satisfies a *semigroup law*, then every order on it is Conradian (Longobardi-Maj-Rhemtulla).

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A left-order $\preceq$ is *non-Conradian* iff Γ contains f, g, u, v such that

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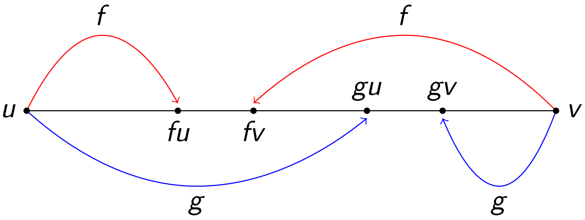

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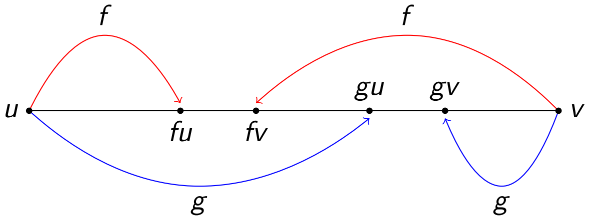

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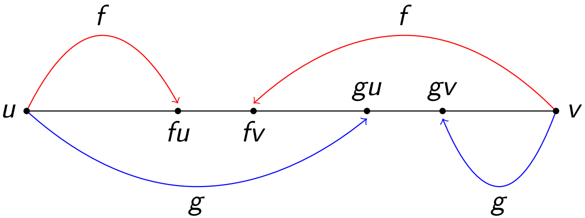

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Remark: Existence of N-crossings for all N implies that no group identity can be satisfied ("ping-pong").

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- An isolated point corresponds to an order that is determined by finitely many inequalities.

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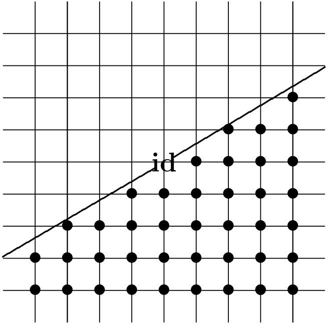

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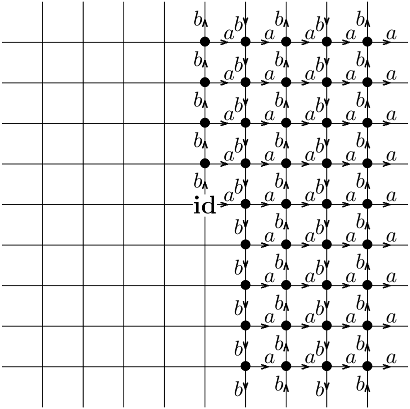

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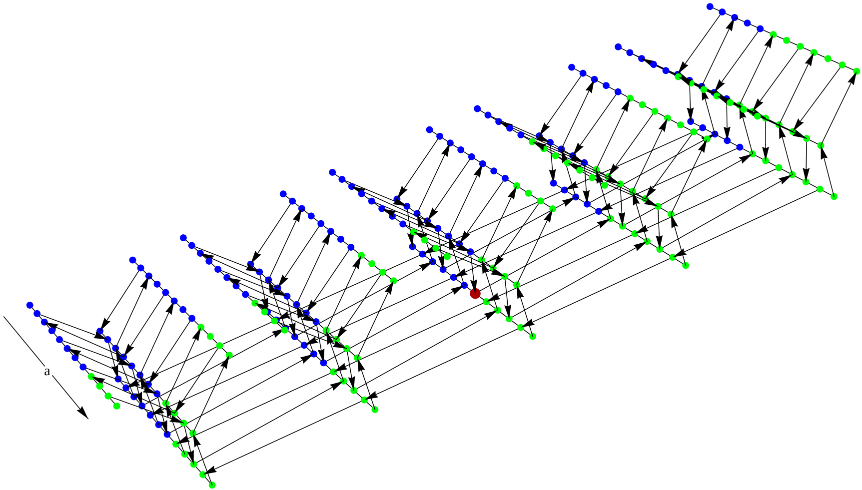

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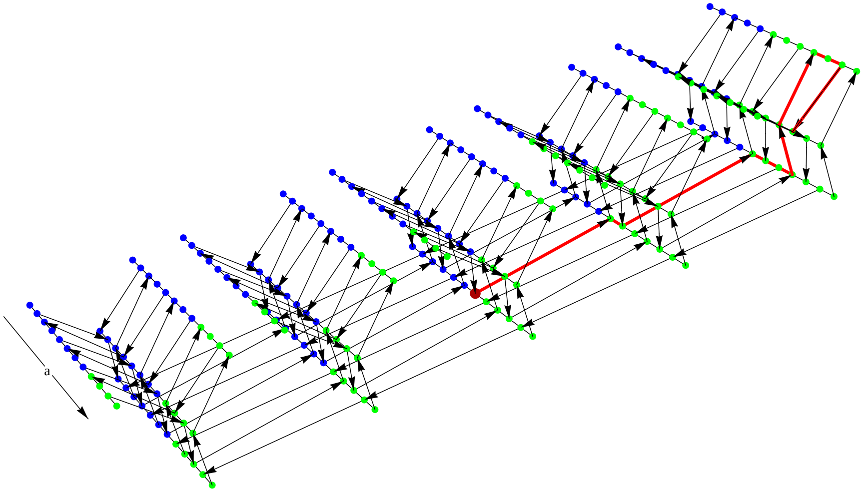

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Does the space of orders of Thompson's group F contain an isolated point ?

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Theorem (Kielak)

The free group $\mathbb{F}_2$ is not the group of fractions of any strict finitely-generated subsemigroup.

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Remark: No examples in the literature of finitely-generated *simple* orderable groups (though existence was recently announced by Bludov).

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- $\mathrm{Diff}_+^{3/2}(\mathrm{S}^1)$ contains no infinite Kazhdan subgroup (N).

Ordering large groups (?)

Some examples of non-standard torsion-free Kazhdan groups come from actions on buildings:

Example [Peyerimhoff-Vdovina]

$$\Gamma = \langle a, b \colon r_1, r_2, r_3 \rangle,$$

$$r_1 = bababab^{-3}a^{-3},$$
$$r_2 = ba^{-1}b^{-1}a^{-3}b^2a^{-1}bab,$$
$$r_3 = b^3a^{-1}baba^2b^2aba.$$