Research Proposal

The research program that follows is a continuation of the research carried out by the Main Researcher (MR, for short) over the last two years. It is not focused on a single mathematical problem, but rather inspired by several topics in dynamics and group actions on which the MR is familiar and has concrete ideas to make significant progress leading to publications in reputed mathematical journals in the next period of two years. More ambitious problems/programs for which results are less difficult to ensure within this period (*e.g. the* existence of left-orderable / circularly orderable infinite Kazhdan groups or the existence of a regular stationary measure for groups of diffeomorphisms of 1-manifolds acting minimally) are not explicitly stated (discussed) as part of this project.

Besides, three interdisciplinary lines of research are described, one in ethnomathematics, one in mathematics education and one in mathematics epidemiology.

Last but not least, concrete actions for dissemination of mathematics are proposed.

I. RESEARCH ON FUNDAMENTAL MATHEMATICS

1) Group actions on surfaces. The first question in this regard to deal with is motivated by the well-known fact that the property of faithfully acting by homeomorphisms of a one-manifold is "closed", which is reflected by the characterization in terms of left-orderability for actions on the line [DNR16] and in terms of bounded cohomology for actions on the circle [Ghy01]. It is widely unknown whether something similar holds in higher dimension, yet conjecturally this shouldn't be the case. Here is a concrete question in this direction:

Question: Is the set of finitely generated groups that faithfully act by homeomorphisms of a given compact surface closed as a subset of the space of marked groups (endowed with the corresponding Chabauty–Gromov topology [Cha00])?

One of the main difficulties to answer this question is the absence of enough criteria to establish that certain families of groups cannot act on surfaces. For instance, no example is known of a finitely generated, torsion-free group for which non-existence of faithful actions on surfaces has been shown. The existence of such a group is a question interesting in itself.

In what concerns differentiable actions, we will concentrate on the so-called Milnor alternative for groups of diffeomorphisms. Recall that this refers to the absence in certain frameworks of *groups of intermediate growth*, that is, groups for which the number of elements that can be written as a product of no more than n generating factors grows faster than polynomially but slower than exponentially in n. This is the case, for instance, inside Lie groups, but a famous example of Grigorchuk shows that groups of intermediate growth naturally arise as groups acting on Cantor sets. Since the seminal construction in [Gri84], many generalizations have been produced.

The question of whether a group of intermediate growth can arise as a group of diffeomorphisms of a low-dimensional manifold is very interesting. It is known that, for

1-manifolds, such a group may appear as a group of C^1 diffeomorphisms, but the derivative of some group element must fail to be Hölder continuous [Nav08]. We will intend to establish a similar statement in the two-dimensional case:

Question: Does there exist a group of intermediate growth of C^2 diffeomorphisms of a compact surface?

It seems that the answer to this question is negative in general, but the case of groups of area-preserving diffeomorphisms seems to be more tractable. Indeed, if the group is solvable, then it cannot have intermediate growth due to a classical theorem of Milnor [Mil68]. By the work of Ghys [Ghy93], an extension of this result to groups satisfying weaker solvability-like properties would yield that groups of diffeomorphisms that are conjugate to groups generated by elements close to the identity also satisfy the Milnor alternative (Ghys' techniques deal with real-analytic diffeomorphisms, but some of them still apply in lower differentiability). Now, a key argument from [HKR20] implies that groups that do not satisfy this conjugacy property have elements with hyperbolic-like behavior. If this behavior yields the existence of homoclinic intersections, this would force a pair of elements for which a classical (positive) ping-pong argument would imply the existence of free semigroups inside the group, hence exponential growth (compare [HX21]).

2) On conjugates of linear cocycles. A particular class of 2-dimensional systems is given by a PSL(2,R) cocycle over a 1-dimensional dynamics (the linear action can be though of as an action on the projective line). In this framework, we are interested in a very concrete problem dealing with discontinuity of Lyapunov exponents along the closure of a single conjugacy class. Here, by "conjugacy class" we refer to cohomology: Remind that two cocycles A: $S^1 \to$ PSL(2,R) and A': $S^1 \to$ PSL(2,R) over a dynamics T: $S^1 \to S^1$ are said to be cohomologous if there exists a cocycle B: $S^1 \to$ PSL(2,R) such that, for all points x of S^1,

$$A(x) = B^{-1}(T(x))\, A'(x)\, B(x)$$

Question. Given an irrational angle α, is it possible to build a continuous cocycle A: $S^1 \to$ PSL(2,R) over the circle rotation of angle α and a sequence of cohomologous cocycles A_n that converges to a to cocycle B such that each A_n has zero Lyapunov exponents yet B has nonzero Lyapunov exponents ? What about convergence in finer (Hölder) topology ?

There exists a vast literature on (dis)continuity of Lyapunov exponents in various frameworks (see [Via2020] for a full panorama on this). Nevertheless, no known example answers the question above. This would require a careful use of existing techniques, most probably in combination with an Anosov-Katok like method [FK04].

Besides being a natural question, let us mention that our interest on this comes from the study of the asymptotic distortion of diffeomorphisms of 1-manifolds described below. Indeed, this is a kind of (global) Lyapunov exponent (at the level of second derivatives), and examples of discontinuity of its value naturally arise along the closure of a single conjugacy class; see [EN93].

3) On distorted diffeomorphisms in different regularities. The notion of *distortion* is ubiquitous in mathematics. In general, an object is said to be "distorted" in an ambient space if it doesn't take its "natural" form therein. Measuring the degree of distortion (either quantitatively or qualitatively) yields useful tools in each context, with a broad variety of potential applications. This part of the research project focuses on two concrete problems concerning general differentiable/continuous dynamics.

Following the seminal work of Gromov [Gro93], given a finitely generated group G, an infinite-order group element is *distorted* if the word-length of its powers grows sublinearly (this is independent of the finite system of generators). More generally, an element is distorted inside an arbitrary group if it is distorted inside a finitely-generated subgroup. Thus, a given element can be undistorted in a given group, though distorted in a larger group containing it. Here is a very general question on this concerning diffeomorphisms of manifolds:

Question: Given a compact manifold M and numbers $r > s$ larger than or equal to 1, does there exist a C^rdiffeomorphism of M that is distorted in the group of C^s diffeomorphisms but undistorted in the group of C^rdiffeomorphisms?

This question comes from [Nav21] (see also [DE20]), where the answer is shown to be affirmative for a 1-manifold and $s = 2$. The key tool to detect undistortion is the so-called "asymptotic distortion", introduced in [Nav18] and later developed in [EN19] and [EN20]. This corresponds to the stable version of the variation of the logarithm of the derivative of the diffeomorphism; more precisely,

$$\mathrm{dist}_\infty(f) = \lim_{n\to\infty} \frac{\mathrm{var}(\log(Df^n))}{n}$$

(the existence of the limit above follows from the subadditivity property of the variation of the logarithm of the derivative). Detecting a similar object in higher dimensions seems to be a fundamental step to make some progress on the question above, and is interesting in itself. Nevertheless, it may happen that no extension exists, which would also be very interesting; compare [Sev98].

Part of this research will be carried out by the PhD student Leonardo Dinamarca under the orientation of the MR.

4) On distortion and conjugacy.

Asymptotic distortion was introduced as a tool to deal with the conjugacy problem. In concrete terms, it is proved in [Nav18] that a C^{1+bv} diffeomorphism of a (compact) 1-manifold has vanishing asymptotic distortion if and only if it admits a sequence of conjugates that converges to an isometry in the C^{1+bv} topology. (More generally, the asymptotic distortion of such a diffeomorphism is the infimum value of the total variation of the logarithm of the derivative of conjugates of the diffeomorphism.) Whether or not this result has a version in the C^2 topology is an interesting question. A partial result for the case of the interval appears in [EN19], but the case of the circle is widely open.

The proof of [Nav18] uses an explicit sequence of conjugates that comes from looking at the affine derivative as a cocycle with values in $L^1(S^1)$ and performing a Césaro mean procedure on it. Proceeding this way, the natural conjugating maps h_n are those satisfying

$$Dh_n = \frac{\sqrt[n]{Df(x)\, Df^2(x) \cdots Df^{n-1}(x)}}{\int \sqrt[n]{Df(s)\, Df^2(s) \cdots Df^{n-1}(s)}\, ds}$$

Using an Anosov-Katok like method, we plan to build a C^2 circle diffeomorphism of irrational rotation number (for which the asymptotic distortion always vanishes, by another result of [Nav18]) so that the sequence of conjugates by h_n does not converge to a rotation in the C^2 topology. This would give some evidence on that no analog of the conjugacy theorem exists in C^2 regularity.

This part of the research will be carried out in collaboration with Mario Ponce (who will participate in this project as an associate researcher).

In a broader context, we will study whether distorted homeomorphisms always admit sequences of conjugates that converge to "well-behaved" homeomorphisms (*e.g.* maps lying in a compact maximal subgroup, whenever this makes sense), and viceversa. This seems to be the case for linear groups except for particular cases (certain solvable groups). Some of the groups to deal with here are general groups of homeomorphisms of the Cantor set (whose infinite-order elements are always distorted [Cor20]), the subgroups of computable and regular homeomorphisms, groups acting by isometries of nonpositively curved spaces, Cremona groups, etc. Relations with the most general notion of distortion (for non locally compact groups) introduced by Rosendal [Ros17] seems plausible (and promising) here.

Rather than oriented towards a precise question, this is mostly a research program. Part of this research will be carried out by the PhD student Lautaro Vásquez under the orientation of the MR.

5) On discrete sets of the plane. Passing now to the context of metric spaces, distortion arises when looking at embeddings of one space into another one, and refers to the change of the original metric under the map. The study of distortion for maps defined from (families of) discrete —even finite— metric spaces is particularly interesting and has broad applications not only in geometry but also in crystallography (in relation to the mathematical study of quasicrystals) and theoretical computer science (for the quite relevant problem of packing data with low cost). One of the milestones is the celebrated Bourgain's theorem [Bou85], according to which the distortion of embeddings of finite metric spaces of cardinality $n$into Hilbert spaces can be promoted to be of order no larger than $\log(n)$.

In the remarkable recent work [DKK18], Dymond, Kaluža and Kopecká managed to deal with Lipschitz equivalence —with no hypothesis of lower Lipschitz constants for the maps— using ideas coming from harmonic analysis (mainly, the notion of Lipschitz regular maps). In particular, they proved the following rather surprising result:

“For each n, there exists a subset of $\mathbb{Z}^2$ of cardinality n^2 such that, if we denote by $L(n)$ the infimum among all Lipschitz constants for bijections from the set into $\{1,...,n\}^2$, then $L(n)$ diverges as n goes to infinity”.

This is only an existence result, and no control on neither the sets in the statement nor the constants $L(n)$ can be obtained from the proof. In this direction, it is quite plausible that, by developing ideas and techniques from discrete analysis, one may tackle the following:

Goal: Make the result above both explicit and quantitative, thus obtaining a lower bound for $L(n)$ that goes to infinity as n does.

In a related direction, we will pursue prior work (see [CN16] and [Nav16]) to deal with the longstanding question of the existence of a cut-and-project Delone set of the plane that is not bi-Lipschitz equivalent to the standard lattice (this important problem comes from the seminal work of Burago and Kleiner; see [BK98] and [BK02]). Notice that, in case these sets exist (which is seemingly the case), the ideas and techniques from [DKK18] should also yield the existence of such sets with the reinforced property that no Lipschitz equivalence to the standard lattice exists.

6) **On “elementary” geometry.** Here we propose to focus on the Morley's theorem [Mo00, Mo24] from a dynamical point of view. Remind that this surprising and beautiful result states that for every (Euclidean) triangle, the three intersection points determined by the angle trisectors of a triangle and that are next to the sides are the vertices of an equilateral triangle. A good reading on this is the remarkable survey by Baker and Oakley [OaBa78], where almost 200 relevant references can be found, and this only up to 1978. Since then, there have been as many relevant contributions, one of the most famous ones being the one by Alain Connes [Co98].

Let us point out that Morley’s theorem does no longer hold in non-Euclidean geometries; for instance, in hyperbolic geometry, Svrtan and Veljan [SvVe17] do explicit computations of the lengths of the sides of the resulting triangle, thus explicitly showing that it is not necessarily equilateral. Also, for many constructions similar to Morley’s where one replaces angle trisectors by different types of sets, the result also fails. We propose to use differentiable and dynamical tools to accurately estimate the deviation from an equilateral triangle of constructions of this type that have the following general scheme below.

First, given two sets A, B of the plane, two curves T_{AB} and T_{BA} are attached to them according to an specific rule. We describe some examples:

- If A and B are the edges of an angle in the plane, we can consider T_{AB}, T_{BA} as the two trisector rays of the angle formed by A and B.

- If A and B are points in the plane, we can consider T_{AB} and T_{BA} as the perpendicular lines to the segment AB that pass through the points that divide this segment into 3 equal parts.

- If A and B are points in the plane, we can consider T_{AB} and T_{BA} as the two trisector curves defined in [AsMaTo07a]: The curve T_{AB} borders the region of the plane that contains A and whose points are closer to A than to points in the region containing B, and similarly for T_{BA}.

- If A and B are the edges of an angle in the plane and n> 3, we can consider T_{AB} and T_{BA} as the two rays that form equal angles with A and B of amplitude <AB/n [Le40].

Next, given three points A, B, C in the plane, we consider the three intersection points:
- A' as the intersection of T_{AB} with T_{AC} ;
- B' as the intersection of T_{BA} with T_{BC} ;
- C' as the intersection of T_{CB} with T_{CA} .

We then consider the dynamics Γ that sends the triangle (A, B, C) into (A', B', C') up to a suitable renormalization whenever this makes sense (say, we impose the resulting triangle to have the same area in case of Euclidean geometry). For example, in the first construction introduced above, Γ is trivial, since the map immediately sends any triangle into the equilateral triangle. In the second construction, it is not hard to see that the triangle (A', B', C') is a similar to (A, B, C), hence Γ is nothing but the identity.

A reasonable (say, symmetric) dynamics Γ of this type should have the equilateral triangle as a fixed point. We plan to compute and/or understand the behavior of the derivative $D\Gamma$ in a neighborhood of the equilateral triangle. We hope being able to establish that, in most of the situations, the equilateral triangle is a (global) attractor for the iterations of Γ, and to estimate the exponential speed of convergence. In the context of the third construction above, this would advance the geometric understanding of the configuration of neutral zones [AsMaTo07b].

7) On magic squares. Though magic squares of numbers (and, in general, geometric arrangements of numbers satisfying "magic" equal-sum type properties) have been seriously considered for the study of history of mathematics [Ses19], they are usually presented as mere mathematical curiosities. However, there are still many interesting open questions regarding them, as those regarding explicit counting of the number of distinct magic squares of consecutive numbers (starting at 1). In the spirit of this research program, here we will deal with group actions on magic configurations. This is nothing but the group of permutations of the positions that transform any magic configuration of numbers into another one. The main motivation comes from the recent reconstruction by the MR of a beautiful though forgotten (and not fully understood) theorem that claims that the 4-panmagic group is naturally isomorphic to the group of 384 isometries of the hypercube (see [Nav20] and references therein). (Remind that a panmagic square is a numerical square for which the sums along rows, columns and diagonals -including the broken ones- are the same.)

Goal. Describe the panmagic groups for higher-order squares.

For order-5 squares, this group is also known: it is naturally isomorphic to the semidirect product of $A_5 \times A_5$ with Z/2Z (the order-2 element corresponds to interchanging the first two factors). However, the proof of this result and that for order-4 panmagic squares above require knowing in advance the total number of panmagic squares of consecutive numbers starting at 1, which is unavailable in higher order. We expect, however, that developing direct arguments would lead to explicit descriptions of all panmagic squares.

II. RESEARCH ON ETHNOMATHEMATICS: patters and culture

Starting from the 20th century, ornaments of ancient cultures have been studied and classified using mathematical methods. Although this study has focused mainly on the group structure of geometric patterns (see [WC91] and [WC04]), it can be further extended to include for instance symmetries of magic squares (see [Nav20]), configurations spaces of patterns exhibiting some degrees of freedom (see [Chi21]), patterns with more complex (fractal) structure, etc. Here we will focus on the last of the aforementioned topics.

In concrete terms, we will study in detail the patterns of the Shipibo culture from the Amazonian Perú. The analysis will be done in collaboration with the archaeologist Paola González, who is a specialist on this culture and its connections to Chilean cultures [Gon16]. The methods to be pursued here are related to dynamical systems. Indeed, Shipibo ornaments are rather special because they look like fractals; see for instance https://www.perunorth.com/news/2016/3/30/art-of-the-shipibo. These are the mathematical objects that usually appear in dynamical systems of geometric origin, and were largely popularized in the second half of the 20th century by B. Mandelbrot (who actually created the word "fractal"; see [Man93]). The research to be carried out here will focus on two different aspects:

- The search for an explanation for the production of fractal-like iconography: Our thesis here goes back to an ancient Shipibo legend, according to which the sacred serpent Ronin has all possible designs in his skin (see [Geb85] and [Roe82]). Mathematics enters into the play because of a classical theorem of Anderson [And58] and Betsvina [Bet88], which claims that there is a unique homogeneous curve that contains a (topological) copy of all curves, namely the Menger curve, which is a fractal object. Thus, in a certain way, Shipibo cultures have been producing fractal-like patterns because one of their original myths forces them to respect a specific mathematical structure.

- A systematic study of the variations in the patterns from one Shipibo village to another: Shipibo culture is still very active in producing iconography, but the patterns vary from place to place. There have been many explanations of this based on cultural issues. Here we propose to add our mathematical viewpoint in relation to the emergence of fractals and the logic rules of construction. This will certainly require visiting the places where production of Shipibo art follows the most ancient tradition.

III. RESEARCH ON MATHEMATICS EDUCATION: didactic transposition of dynamical concepts

Interdisciplinary research has acquired great relevance in didactic of mathematics and, in general, in didactic of science; see for instance [Ara16] and [WR19]. We propose to start carrying out a very ambitious program in this direction which is innovative even at an international level, namely to establish the basis for the introduction of concepts as nonlinearity, complexity, sensitiveness to initial conditions, randomness, entropy and chaos, in a school context. This is certainly a necessary task, since these notions govern several global phenomena of our 21^{st} century world.

In concrete terms, our research in this domain will lead us to elaborate a guide text introducing the concepts above that could be used by school teachers in their initial and permanent careers. In this way, they will be able to learn about these notions during their university studies not only from a classical pure mathematical perspective, but also in a contextualized form. Thanks to this, they will proceed to the transfer of knowledge in a responsible manner along the exercise of their profession, by respecting the principles of cognitive and epistemological vigilance. In summary, this will allow them to continuously apply the didactic transposition of these very elaborate (and quite important) mathematical concepts [Che84].

This proposal becomes even more pertinent in the Chilean context because of the recent changes in the program of school mathematics, particularly that of the later years. The recent introduction of differential calculus and probability, with an emphasis in the resolution of problems and mathematical modelling, are among the main changes that are currently being implemented. In this direction, and following the lines of [KB15], a special effort in both our research and the outcome text will be devoted to showing how the dynamical notions above appear in the mathematical modelling of the evolution of a pandemic as well as the climate change phenomenon.

IV. RESEARCH ON MATHEMATICS EPIDEMIOLOGY: models incorporating unreported cases

The COVID19 pandemic has revealed the necessity of revisiting the most classical epidemiological models by incorporating some new elements that we have confronted over the last two years. One of the most relevant ones is that not all the cases are systematically reported, so that the total number of infected people is much larger than the one which is reported. Although *a-posteriori* correctionsvto this are easy to implement [G05], it is important to incorporate them in the modeling, since they create a dynamics of the pandemic that is not captured by classical SIR and SEIR models (a kind of second -and larger- Lyapunov exponent for the evolution).

This is what has been done by Liu et al. in a series of papers, each of which deals with a specific country (see for instance [LMSW20], [LMSW21], [LMW21]). However, the qualitative theory of this extended model is still incomplete. Part of this has been settled in [NV20], where general results on existence of solutions for the corresponding ODE equations and some of their properties are shown. In particular, a very simple example for which two peaks for the curve of infected people is exhibited. We plan to pursue on this research. In particular, we still do not know whether certain initial conditions can lead to more than two peaks. This is a challenging pure mathematical problem.

V. WORK ON DISSEMINATION OF MATHEMATICS

Last but not least, a great effort within this project will be made on communication of science, particularly mathematics, to society. This includes:

- The publication of a new book of dissemination ofmMathematics via Ed. Planeta, more precisely, a second volume of “Lecciones de Matemáticas para el Recreo” (see https://www.planetadelibros.cl/libro-lecciones-de-matematicas-para-el-recreo/280849);

- Administration of the Facebook fanpage “Un Viaje a las Ideas Matemáticas” (see https://www.facebook.com/unviajealasideas), that should pass from about 25,000 followers to 40,000 during the next 2 years (this would lead to about 3,5 million interactions per year);

- Collaboration with the electronic newspaper “El Mostrador” (see https://www.elmostrador.cl/autor/anavas/) by publishing at least 3 new articles per year therein;

- Collaboration with the Festival de Matemáticas as a continuation of the EXPLORA Project ED230120, with the realization of at least one version per year (see https://festivaldematematica.cl/);

- Collaboration with the Olimpiada Nacional de Matemáticas (ONM) and Campeonato Escolar de Matemáticas (CMAT);

- Maintenance of the dissemination website “Paisajes Matemáticos”, a translated (an adapted) version of the French site “Images des Mathématiques”, mainly sponsored by CNRS, which has been in activity over the last two years with great success: see https://images.math.cnrs.fr for the official website and https://www.elmostrador.cl/cultura/2019/04/16/matematico-chileno-pone-al-alcance-de-toda-hispanoamerica-un-portal-de-divulgacion-para-todo-publico/ for a press report of it.

VI. ON METHODOLOGY AND WORK PLAN

Research on fundamental mathematics is very hard to predict. It certainly involves standard activities as reading relevant bibliography, organizing seminars and workshops, working with students, collaborating (both nationally and internationally), etc. Nevertheless, trying to predict a Gantt Chart for this is totally useless, since publications depend on individual progress. Besides the associate researcher of this project, Mario Ponce (from PUC Chile; collaboration on topics I.4 and I.6), among the potential collaborators we can mention Cristóbal Rivas (USACH; collaboration on topic I.1), Jimena Royo-Letelier (USACH; III), María José Moreno (USACH; 1.7), Sebastian Barbieri (USACH; I.4 and II), Elisabeth Montoya (PUCV; III), Ximena Colipan (U. Talca; III), Paola González (U. of Chile; II), Rodolfo Viera (PUC; I.6), Cristóbal Rojas (U. Andrés Bello; I.4), Fabio Brochero (Univ. Minas Gerais; I.7), Alejandro Kocsard (U. Fed. Rio de Janeiro; I.1), María Antonieta Emparan (Freibourg Univ.; II), Hélène Eynard-Bontemps (Inst. Fourier; I.2), Bassam Fayad (U. Paris; I.3), Yves de Cornulier (Univ. Lyon 1; I.4), Gastón Vergara-Hermosilla (U. Bordeaux; IV), Kathryn Mann (Cornell. Univ.; I.4), Christian Rosendal (Univ. Illinois at Chicago; I.4), Jairo Bochi (Penn. State Univ.; 1.2), and Sebastian Hurtado (Chicago Univ.; I.1). The capacity of the MR and collaborators to implement the proposal is attested by their prior work on the different subjects, which in many cases has yield to joint publications (this has been particularly the case with the associate researcher).

The proposed research on ethnomathematics is more easily predictable since it corresponds to undergoing work. During the first year we will concentrate on the theoretical basis of it, which should lead to a publication. The second year will be mostly devoted to classify different patterns according to particular properties of the fractal-like models.

The proposed research on mathematics education is a first step of a major program, and as such a fruitful outcome would be a document (publication) describing the problem and presenting a method to deal with each specific topic. Elaborating documents on each of these topics will be a task for the next years and/or another project fully focused on this aspect.

The proposed research on mathematics epidemiology is a continuation of prior work on models incorporating unreported cases. Publishing an article on this will depend on whether we could solve one of our main problems, namely the existence of several peaks (more than 2) for the curves arising in these extended models.

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