Normal forms for birthdate magic squares and their symmetries: the panmagic and other permutation groups

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Abstract. We relate normal forms of different types of magic squares to their symmetries. In particular, we define the panmagic groups, and we determine these groups for the case of squares of orders 4 and 5. Some other magic configurations are considered from this point of view. only this ?

Since the Antiquity, much literature about magic squares has been produced. A general panorama on the mathematical aspects of this is provided in the classical textbook [2]. Remind that a *magic square* is an $n \times n$ square of numbers for which the sums along rows, columns, and the main diagonals, are the same (this sum is the *magic number* of the square). In case the sum along all diagonals (including the "broken" ones) is also the magic number, the square is said to be *panmagic*. In Part I of this work, we discuss general formulas (normal forms) for the entries of 4×4 and 5×5 squares of numbers satisfying these magic properties and related ones. The formulas, which are in the spirit of those given by Lucas for 3×3 magic squares, are related to the ones that appear in Ramanujan's notebooks and the classical work of Euler [6]. In particular, they are shown to be the most general expressions for birthdate magic squares.

In Part II of this work, we study the group of symmetries of normal forms of panmagic squares. More precisely, we consider the *panmagic groups*, which are the subgroups of the corresponding group of permutations that send every panmagic square into a panmagic one. For 4×4 squares, this group has order 384, and can be realized as the group of isometries of a certain discrete torus which, in its turn, is isometric to the set of vertices of the hypercube graph. This is closely related to the well-known fact that the number of distinct perfect magic squares made of consecutive numbers is precisely 384. A similar analysis for 5×5 panmagic squares is carried out: the corresponding panmagic group has order 28 800 and, thought is not realized as an isometry group, it has a quite similar algebraic structure.

In Part 3, some generalizations of the results above are given for magic squares with supplementary properties, as well as for other type of magic configurations (hexagonal, circular, etc) that are well known and spread along the literature. More importantly, a general question concerning permutation groups of coordinates of affine varieties is stated and discussed. really ?

Part I: On the normal forms of magic squares

Lucas obtained the next general formula (normal form) for 3×3 magic squares.

$s-m$	$s+(m+n)$	$s-n$
$s+m-n$	s	$s-m+n$
$s+n$	$s-(m+n)$	$s+m$

Notice that the corresponding magic number is $3s$. Therefore, 3×3 magic squares of integer numbers exist only for magic numbers that are multiples of 3.

In the formula above, letting $s = 5$, $m = 1$ and $n = 3$, one obtains the famous Lo Shu square reproduced below. In there, all consecutive numbers $1, 2, \dots, 9$ do appear. It is a little exercise to show that this is the only 3×3 magic square with this property, except for those obtained either by rotation or reflexion (horizontal, vertical, and along diagonals). In other words, Lo Shu square is unique up to the action of the dihedral group D_4.

4	9	2
3	5	7
8	1	6

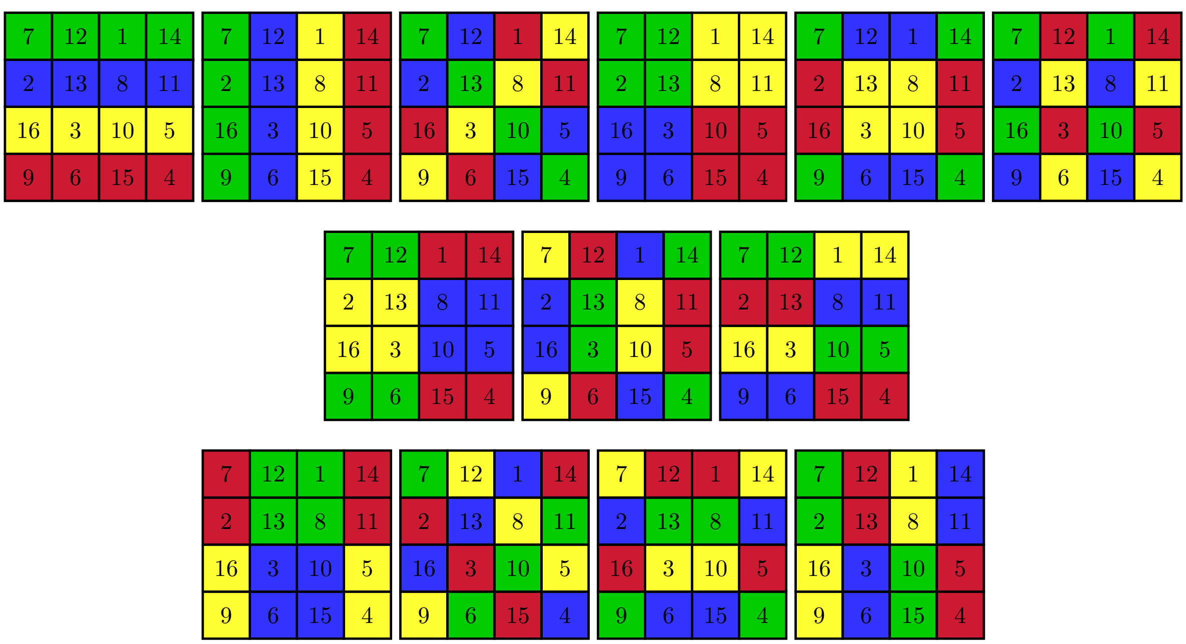
The number of magic squares of consecutive integer numbers in higher orders is quite large. In particular, the uniqueness fail: up to the action of D_4, there are 880 magic squares of order 4 and 275 305 224 of order 5 (the precise number for order 6 is unknown). However, when imposing some extra conditions (as the panmagic ones), simple normal forms for magic squares leading to some kind of uniqueness arise. Although most of these formulas are known (and several equivalent forms have been obtained by many people along the history), they are hard to find in the literature expressed in a concise way. Hence, we start with a systematic search of normal forms inspired by two remarkable magic squares: the Chautisa Yantra and Ramnanujan's birthdate magic squares.

Statement of results for 4×4 squares and some comments

With no doubt, the most beautiful 4×4 magic square of consecutive numbers is the Chautisa Yantra square below.

7	12	1	14
2	13	8	11
16	3	10	5
9	6	15	4

Besides rows, columns and main diagonals, there are other 42 symmetric combinations of four entries of this square leading to the same sum (34), as depicted below.



Theorem A. A general 4×4 magic square satisfies all the 52 conditions on the sums of the Chautisa Yantra square if and only if it has the form below, where $s = (a+b+c+d)/2$.

a	b	c	d
$s-(a-k)$	$s-(b+k)$	$s-(c-k)$	$s-(d+k)$
$s-c$	$s-d$	$s-a$	$s-b$
$c-k$	$d+k$	$a-k$	$b+k$

We will say that a 4×4 magic square is *perfect* if it satisfies the conditions of Theorem A. There is another famous magic square, reproduced below, that is not perfect but satisfies the first 36 conditions on the sums that are illustrated by the first 9 colored squares above.

22	12	18	87
88	17	9	25
10	24	89	16
19	86	23	11

This square was produced by Ramanujan. Therein, the numbers in the first row are those of his birthdate (22/December/1887).

Theorem B. A general 4×4 magic square satisfies all the 36 conditions on the sums of the Ramanujan birthdate square if and only if it has the form below.

a	b	c	d
$d+m+n$	$c-m-n$	$b-m+n$	$a+m-n$
$b-m$	$a+m$	$d+m$	$c-m$
$c-n$	$d+n$	$a-n$	$b+n$

The 24 sums illustrated by the first 6 colored squares above are interesting by themselves, and lead to the next theorem.

Theorem C. A general 4×4 magic square satisfies all the 24 conditions on the sums represented by the first 6 colored squares above if and only if it has the form below.

a	b	c	d
$d+p$	$c-p$	$b-q$	$a+q$
r	$a-b+r+p+q$	$b+d-r$	$b+c-r-p-q$
$b+c-r-p$	$b+d-r-q$	$a-b+r+q$	$p+r$

The conditions involved in the previous theorem are stronger that those of a magic square. Indeed, it is not hard to check that the magic properties only imply four extra conditions, so that there are 14 sums which are involved.

Theorem D. A general 4×4 magic square satisfies the four extra conditions that are represented by the fourth colored square depicted above, and must have the form below.

a	b	c	d
e	$b+c+2d-e-f-h$	$a+e+f-d-g$	$g+h-e$
f	g	h	$a+b+c+d-f-g-h$
$b+c+d-e-f$	$a+e+f+h-b-d-g$	$b+2d+g-e-f-h$	$e+f-d$

In the four theorems above, letting vary a, b, c, d, one can produce birthdate magic squares associated to any date. Notice, however, that the magic number of the square in Theorem A is $a+b+c+d = 2s$, which is an even number in case of integer entries. Thus, only people having born on a date leading to an even magic number can have a perfect birthdate magic square. Ramanujan was not among them, since $22+10+18+87 = 139$ is an odd number; this explains why he didn't produce a perfect birthdate square for himself. Below we give some perfect magic birthdate squares that are particularly nice in that they are made by non-negative different numbers (in most cases, entries are forced to be either equal and/or negative).

16	**5**	**17**	**18**
25	13	21	0
11	10	12	23
7	28	3	15

Maria Agnesi ($S = 56$)

25	**10**	**18**	**11**
5	4	12	3
9	16	2	17
15	14	22	13

Évariste Galois ($S = 54$)

13	**6**	**19**	**26**
21	26	15	4
14	5	20	27
18	29	12	7

John Nash ($S = 62$)

Since normal forms of magic squares involve 4 extra free parameters besides those of the first row, one may wonder whether it is possible to incorporate two birthdates in the same magic square. Unfortunately, the nature of the four extra conditions that appear in Theorem D imply that this cannot be done in a symmetric way. However, it directly follows from the same theorem that one can incorporate two different birthdates leading to the same magic number into a single magic square by placing one in the first row and the other one in the third row. Although there is an extra free parameter e in the formula, no particular choice can lead to more magic sums than the 14 sums involved in the theorem. Below, we have merged in single magic squares the birthdates of Leibniz and Riemann (with $e = 29$), Poincaré and Thurston (with $e = 28$), and Hilbert and Mirzakhani (with $e = 52$). Again, the (non)appearance of negative and/or repeated entries is just a matter of (bad) luck.

Godfrey Leibniz

1	**7**	**16**	**46**
29	49	−6	−2
17	**9**	**18**	**26**
23	5	42	0

Bernhard Riemann

Henri Poincaré

29	**4**	**18**	**54**
28	53	23	1
30	**10**	**19**	**46**
18	34	45	4

William Thurston

David Hilbert

23	**1**	**18**	**62**
52	79	11	−36
3	**5**	**19**	**77**
26	20	56	2

Maryam Mirzakhani

Proof of Theorems A, B, C and D

We start proving Theorem D. First, we establish that the entries of every magic square satisfy the four conditions illustrated by the fourth colored square. To do this, let us consider a general magic square of the form below, and let us denote $S = a + b + c + d$ its magic number.

a	b	c	d
x_1	x_2	x_3	x_4
y_1	y_2	y_3	y_4
z_1	z_2	z_3	z_4

If, from the sum of the expressions involved in the equalities

$$a + x_2 + y_3 + z_4 = S \qquad \text{and} \qquad d + x_3 + y_2 + z_1 = S$$

we substract the sum of the expressions involved in

$$b + x_2 + y_2 + z_2 = S \qquad \text{and} \qquad c + x_3 + y_3 + z_3 = S,$$

we obtain

$$a + d + z_1 + z_4 = b + c + z_2 + z_3.$$

If, instead, we substract

$$x_1 + x_2 + x_3 + x_4 = S \qquad \text{and} \qquad y_1 + y_2 + y_3 + y_4 = S,$$

we obtain

$$a + d + z_1 + z_4 = x_1 + x_4 + y_1 + y_4.$$

Finally, if we substract

$$a + x_1 + y_1 + z_1 = S \qquad \text{and} \qquad d + x_4 + y_4 + z_4 = S,$$

we obtain

$$x_1 + x_4 + y_1 + y_4 = x_2 + x_3 + y_2 + y_3.$$

Since the sum of all entries of the square is $4S$, the previous equalities obviously imply

$$a + d + z_1 + z_4 = x_2 + x_3 + y_2 + y_3 = b + c + z_2 + z_3 = x_1 + x_4 + y_1 + y_4 = S,$$

as announced.

Next, to obtain the formal norm, let us begin with the square below.

a	b	c	d
e			
f	g	h	

The entry in the fourth position of the first column and, after this, the third position in the second row, can be easily computed using the magic properties. Using the fact that $x_2 + x_3 + y_2 + y_3 = S$ proved above, one can compute the second entry of the second row. Computing the rest of the entries then becomes straightforward using the magic properties. We leave the details to the reader.

Conversely, every square in the normal form given in Theorem D satisfies the magic properties. This closes the proof of the theorem.

Let us now proceed to the proof of Theorem C. If we consider the new relation

$$a + c + f + h = S = a + b + c + d,$$

we obtain

$$h = b + d - f.$$

Introducing this value in the formulae of the normal form given by Theorem D, we obtain the square below.

a	b	c	d
e	$c + d - e$	$a + e + f - d - g$	$b + d + g - e - f$
f	g	$b + d - f$	$a + c - g$
$b + c + d - e - f$	$a + e - g$	$d + g - e$	$e + f - d$

Let us introduce the new variable $k = b + d + g - e - f$, so that $g = k + e + f - b - d$. Then the square above takes the form below.

a	b	c	d
e	$c + d - e$	$a + b - k$	k
f	$k + e + f - b - d$	$b + d - f$	$a + b + c + d - k - e - f$
$b + c + d - e - f$	$a + b + d - k - f$	$k + f - b$	$e + f - d$

Finally, letting $r = f$, $q = k - a$ (so that $k = a + q$) and $p = e - d$ (so that $e = d + p$), the square takes the very same form as that announced in the theorem.

Conversely, one easily checks that a square as in Theorem C satisfies the required conditions on the 24 sums, thus closing the proof of the theorem.

Proving Theorem B is now easy. Indeed, the equality $a + b + z_1 + z_2 = S$ yields

$$a + b + (b + c - r - p) + (b + d - r - q) = a + b + c + d,$$

from which one easily computes

$$r = b - \frac{p+q}{2}.$$

Introducing this value in the square above we obtain the following one.

a	b	c	d
$d + p$	$c - p$	$b - q$	$a + q$
$b - \frac{p+q}{2}$	$a + \frac{p+q}{2}$	$d + \frac{p+q}{2}$	$c - \frac{p+q}{2}$
$c - \frac{p-q}{2}$	$d + \frac{p-q}{2}$	$a + \frac{q-p}{2}$	$b - \frac{q-p}{2}$

Finally, letting

$$m = \frac{p+q}{2} \qquad \text{and} \qquad n = \frac{p-q}{2},$$

we have $p = m + n$ and $q = m - n$, and the entries of the square coincide with those appearing in the theorem.

Conversely, one easily checks that a square of this form satisfies the required condition on the 36 sums, thus closing the proof of Theorem B.

To prove Theorem A, we just need to use one extra condition. Indeed, from the equality $b + c + x_2 + x_3 = S$ it follows

$$S = b + c + (c - m - n) + b - m + n = 2\,b + 2\,c - 2\,m,$$

that is,

$$2\,m = 2\,b + 2\,c - (a + b + c + d) = b + c - a - d.$$

Replacing this value of m in the square of Theorem B and letting $k = n$ we obtain the square of Theorem A. Conversely, one easily checks that every square of this form satisfies the required condition on the 52 sums, thus closing the proof.

Remark 1. Imposing certain extra conditions on symmetric sums to the 52 above yields to restrictions on a, b, c, d that are never satisfied for perfect magic squares of consecutive numbers. For example, letting $y_1 + z_2 + z_3 + y_4 = S$ gives

$$S = (s - c) + (d + e) + (a - e) + (s - b) = 2s + a + d - b - c,$$

hence $a+d=b+c$. Now, since $a+b+c+d=2s$, the condition $a+d=b+c$ implies that, actually, $d=s-a$ and $a=s-d$ and, similarly, $c=s-b$ and $b=s-c$. However, according to Theorem A, the entries $s-a$, $s-b$, $s-c$ and $s-d$ also appear in the third row of the square.

In the normal form for perfect magic squares, we have stressed the role of the first row. This way, it becomes evident that horizontal translations (mod. 4) preserve perfect squares, and an extra computation leads to invariance under vertical translations (mod. 4) as well.

However, in order to study more symmetries of perfect magic squares, having in mind other normal forms gives useful insight. For example, an appropriate change of variables transforms the normal form of Theorem A into that below (referred to as the *central normal form*), from which the invariance under rotations becomes slightly more transparent.

A	$D-\tilde{n}$	$C+\tilde{n}$	B
$s-A+\tilde{n}$	$s-D$	$s-C$	$s-B-\tilde{n}$
$s-C-\tilde{n}$	$s-B$	$s-A$	$s-D+\tilde{n}$
C	$B+\tilde{n}$	$A-\tilde{n}$	D

Indeed, a counterclockwise 90° rotation of this square yields the one below.

B	$s-B-\tilde{n}$	$s-D+\tilde{n}$	D
$C+\tilde{n}$	$s-C$	$s-A$	$A-\tilde{n}$
$D-\tilde{n}$	$s-D$	$s-B$	$B+\tilde{n}$
A	$s-A+\tilde{n}$	$s-C-\tilde{n}$	C

Then, letting $N=B+C-s+\tilde{n}$, this results into the next square in central normal form.

B	$C-N$	$A+N$	D
$s-B+N$	$s-C$	$s-A$	$s-D-N$
$s-A-N$	$s-D$	$s-B$	$s-C+N$
A	$D+N$	$B-N$	C

On the Ramanujan formulae for magic squares

As we already mentioned, equivalents forms of Theorems A, B, C and D are spread in the literature, and are well-known to specialists on magic squares. In this regard, it is quite remarkable that, during his childhood, Ramanujan proposed many constructions of magic squares in the spirit of those given by de La Hire (in the XVII century) and Leonhard Euler [6], and some of the remarks contained in his notebooks [3] are closely related to these theorems.

On 3×3 magic squares. Ramanujan explicitly proposed a construction leading to 3×3 magic squares. Provided that A,B,C are in arithmetic progression, as well as P,Q,R, this is summarized in the square below.

$C+Q$	$A+P$	$B+R$
$A+R$	$B+Q$	$C+P$
$B+P$	$C+R$	$A+Q$

This is obtained as the sum of the two Latin squares below.

C	A	B
A	B	C
B	C	A

Q	P	R
R	Q	P
P	R	Q

The condition on the sum along rows and columns is obvious, and that on the diagonals follows from the arithmetic progression assumption.

What is remarkable is that this form is equivalent to that of Lucas. Indeed, letting

$$m = B - C = A - B \qquad \text{and} \qquad n = Q - R = P - Q,$$

Ramanujan's 3×3 square transforms into those below, where the last change arises by letting $s = B + Q$.

$B-m+Q$	$B+m+Q+n$	$B+Q-n$		$s-m$	$s+(m+n)$	$s-n$
$B+m+Q-n$	$B+Q$	$B-m+Q-n$	$=$	$s+m-n$	s	$s-m+n$
$B+Q+n$	$B-m-C-n$	$B+m+Q$		$s+n$	$s-(m+n)$	$s+m$

On 4×4 **magic squares.** Ramanujan proposed the following 4×4 magic square, which is the sum of two Latin squares as well.

$A+P$	$D+S$	$C+Q$	$B+R$
$C+R$	$B+Q$	$A+S$	$D+P$
$B+S$	$C+P$	$D+R$	$A+Q$
$D+Q$	$A+R$	$B+P$	$C+S$

He also included the conditions $A + D = B + C$ and $P + R = Q + S$ with a somewhat confusing discussion. What is remarkable here is that:

– this square satisfies the properties on the 24 sums of Theorem C,

– under the condition $A + D = B + C$, the properties on the 36 sums of Theorem B are satisfied,

– under the additional condition $P + R = Q + S$, the properties on the 52 sums of Theorem A are satisfied.

We leave to the reader the task of finding appropriate changes of variables that transform this square into the normal form provided by the corresponding theorem in each case. It is worth pointing out that this square above was first proposed by Euler [6].

Inspired by the results above, a systematic search for (normal forms of) higher-order magic squares (that may satisfy supplementary properties) appears as an interesting topic. In this direction, useful insight is provided by [7]. Therein, the sets of squares with different magic properties are analyzed as linear spaces (addition and scalar multiplications are the obvious ones), and their dimensions (over different fields) are explicitly computed. Obviously, in each case, the dimension corresponds to the number of free parameters in the normal form (yet the places where these parameters appear is not arbitrary). For concreteness, let us state one of the results of [7] as follows: The space of $n \times n$ magic squares has dimension $n^2 - 2n$, and the dimension of the space of pangmagic squares is $n^2 - 4n + \alpha(n) + 1$, where $\alpha(n) = 3$ for n odd and $\alpha(n) = 4$ for n even.

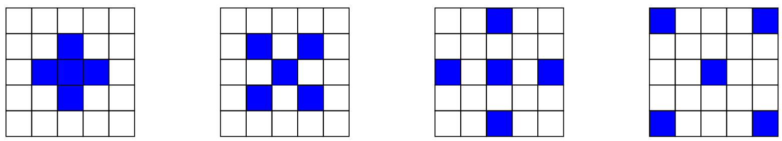
On 5×5 **magic squares.** Let us consider a very concrete example of a panmagic square taken from Ramanujan's notebooks. (Again, Ramanujan was not aware that this square appears in the work of Euler [6].)

$A+P$	$E+R$	$D+T$	$C+Q$	$B+S$
$C+T$	$B+Q$	$A+S$	$E+P$	$D+R$
$E+S$	$D+P$	$C+R$	$B+T$	$A+Q$
$B+R$	$A+T$	$E+Q$	$D+S$	$C+P$
$D+Q$	$C+S$	$B+P$	$A+R$	$E+T$

It is worth pointing out that 4×4 panmagic squares are necessarily perfect (this is well known and follows, for instance, by an easy computation using the Euler-Ramanujan normal form for 4×4 magic squares). According to the result from [7] previously stated, the dimension of the linear space of 5×5 panmagic squares equals 9, and this dimension decreases to 8 for the simplex obtained by prescribing the value of the magic sum. In the expression above, there are 10 parameters. However, adding a constant m to A, B, C, D, E and, simultaneously, substracting the same constant m to P, Q, R, S, T yields the same square. Therefore, the genuine number of parameters is 9, hence the Euler-Ramanujan square is a normal form. Notice that imposing the value of the magic sum (which makes the number of parameters decreasing to 8) corresponds to imposing a condition for

$$A + B + C + D + E + P + Q + R + S + T.$$

The normal form above shows that there are other configurations along which the entries yield the magic sum. These correspond to four types of quincuxes, whose representatives centered at the center of the square are depicted below. Notice that this yields to 120 magic sums: $4 \times 25 = 100$ of them are associated to these quincunxes, 5 to files, 5 to columns, and 10 to diagonals.



There are many other normal forms for 5×5 panmagic squares; two of them are given below.

a	b	c	d	e
p	$c+d+u-p-q$	$a+e+p+q-c-u-v$	$b+c+v-p-q$	q
$b+e-u$	$c+v-p$	$p+q-c$	$c+u-q$	$a+d-v$
u	$a+p-v$	$b+c+d-p-q$	$e+q-u$	v
$c+d-p$	$e+p+q-c-u$	$c+u+v-p-q$	$a+p+q-c-v$	$b+c-q$

$s+n-\mu$	k	$p+q-\mu$	ℓ	$r+m-\mu$
p	$\ell+r-\mu$	$m+n-\mu$	$k+s-\mu$	q
$k+m-\mu$	$q+s-\mu$	μ	$p+r-\mu$	$\ell+n-\mu$
r	$p+n-\mu$	$k+\ell-\mu$	$q+m-\mu$	s
$\ell+q-\mu$	m	$r+s-\mu$	n	$k+p-\mu$

Notice that the value of the magic sum for the last square is

$$k + \ell + m + n + p + q + r + s - 3\mu.$$

What is nice with this normal form is that it can serve for merging in a single 5×5 panmagic square two birthdates that cannot be merged in a 4×4 magic square. Moreover, using the parameter μ, one can make vary the magic number of the square. Below we illustrate this fact with two concluding examples having magic numbers 111 and 151, respectively.

Georg Cantor

20	**03**	43	**18**	27
03	41	18	04	**45**
02	46	05	26	32
28	17	16	44	**06**
58	**04**	29	**19**	01

Kurt Gödel

Sofia Kovalevskaya

53	**01**	63	**18**	16
15	28	23	35	**50**
05	84	02	25	35
12	32	17	54	**36**
66	**06**	46	**19**	14

Vladimir Arnold

Part II: On the symmetries of perfect and panmagic squares

Many works along history have dealt with the difficult (and still widely open) problem of enumeration of squares of consecutive numbers with different magic properties. In cases where this has been solved, a common strategy has consisted in considering squares of a certain type, and then looking at their images under rotations, reflexions and translations. However, it turns out that, in most cases, there are other transformations that preserve the magic properties. This naturally leads to considering the general group of permutations that preserve these properties. We deal with these groups in this part of the work. In particular, we do explicit computations for 4×4 and 5×5 panmagic squares. MAS ???

Again, after proving most of the results of the following sections, we came to the end of the hard task of looking for references for these issues (see [8], [9] and [11]). Actually, it came as a surprise to us that results of this type were known since longtime ago, yet the corresponding references (notably [9]) seem to have been neglected in the subsequent study of magic squares. Because of this, a somewhat detailed exposition follows for the case of 4×4 squares.

Statements of the results for 4×4 perfect squares

We view the symmetric group S_{16} as a group of permutations of the boxes of a 4×4 square. We call *fantastic square* a perfect magic square filled up with the numbers $1, 2, \ldots, 16$ (each being included once).

Theorem E. The subgroup of S_{16} formed by the elements that transform every perfect magic square into another one has order $2^7 \times 3 = 384$, and acts freely and transitively on the set of fantastic squares.

Part of the content of Theorem E is that there are no permutations that preserve fantastic squares but send a perfect square into a non-perfect one. In other words, the Chautisa Yantra square does not admit any special symmetry that is not shared by all perfect magic squares.

Before passing to the proof, we notice that there is a natural distance between boxes in a 4×4 square. More precisely, the distance between two positions is the minimum number of borders of squares that are crossed by a path (made up of horizontal and vertical segments) going from one to the other in the space (torus) obtained by identifying the leftmost edge with the rightmost one and, simultaneously, the bottom edge with the top one. The maximal possible distance between two positions is 4; actually, given a position, there is a unique one at distance 4 from it. Positions at distance 4 one from the other will be said to be *opposite*.

When we speak about isometries of the 4×4 square, this will refer to the permutations of the 16 positions that preserve this distance. The next result complements Theorem E above. It relates fantastic squares to the isometries of the 4×4 square.

Theorem F. The subgroup of S_{16} formed by the elements that transform every perfect 4×4 magic square into another one coincides with the group of isometries of the 4×4 square.

We will refer to the group involved in Theorems E and F as the Chautisa Yantra group, and we will denote it using the yantra symbol .

Along the proofs, many intermediate results of interest by themselves will be given. In particular, we will see that the discrete torus defined above is isometric to the set of vertices of the hypercube graph Q_4, hence  is nothing but the group of automorphisms of Q_4. Using this fact, we will give a detailed description of the structure of .

On the number of fantastic squares

The next result is very well known, though it is not easy to find a concise proof in the literature. For the reader's convenience, we give a short argument of our own that will be helpful to do explicit computations for the Chautisa Yantra group.

Proposition 1. There exist 384 fantastic squares.

Proof. The starting point is that, according to the normal form of Theorem A, in every perfect magic 4×4 square, opposite positions must be filled by numbers that add s (a half of the magic number). In particular, for fantastic squares, these numbers must add 17.

Next, a key point is the following:

- if two positions are at distance 1, then the entries are simultaneously involved in 4 of the 52 sums of a perfect magic square;
- if the positions are at distance 2, then they are simultaneously involved in 2 of these sums;
- if the positions are at distance 3, then they are simultaneously involved only in 1 sum;
- if the positions are at distance 4, then they are simultaneously involved in 5 sums.

All of this can be checked by inspection on the 13 colored squares depicted just before Theorem A. Using this remark, we will show that the entries 15, 14 and 12 must be placed at distance 3 from 16. To see this for 15, notice that there is only one form to complement $16 + 15 = 31$ with two different summands to give 34, namely $1 + 2$. However, if 16 and 15 were placed at distance $\neq 3$, then they would need to be complemented in at least two different ways. A similar argument applies to 14, since $16 + 14 = 30$ can be complemented only as $1 + 3$. Finally, for 12, the sum $16 + 12 = 28$ can be complemented only as $4 + 2$ and $1 + 5$. Hence, if the distance between the positions of 16 and 12 was not 3, then it would necessary equal 2. Let us consider the case below, all the other being analogous (actually, they are all equivalent under the action of the group H introduced below, hence it suffices to consider this case).

16			
	12		
		1	
			5

The entries 2 and 4 must occur down and to the right of 16. Both cases are analogous, hence let us suppose that 4 appears down to 16 and 2 to the right, as illustrated below. (Again, one case follows from the other by a reflexion along the diagonal.)

16	2		
4	12		
		1	
			5

This forces 13 and 15 to appear in the corresponding opposite positions, as illustrated below.

16	2		
4	12		
		1	15
		13	5

Since we know that 14 must appear at a position at distance 3 from that of 16, the two cases below are the only possible ones. However, in both cases, the row with three entries must be completed with a 4. Since 4 was already in the square, it is impossible to produce a fantastic square this way.

16	2		
4	12	14	
		1	15
		13	5

16	2		
4	12		
	14	1	15
		13	5

Assume now that 16 appears in the leftmost top position. Then 1 must appear at the opposite position and, as proved above, 12, 14 and 15 must appear in the 4 neighbor positions to this one. One readily checks that for any such choice of positions, a unique fantastic square arises. An example is shown below.

16			
		12	
	14	1	15

$\rightsquigarrow$

16	2	13	3
9	7	12	6
4	14	1	15
5	11	8	10

Since there are $4 \times 3 \times 2 = 24$ ways of arranging these entries in the 4 available positions, this leads to 24 distinct fantastic squares with an entry 16 in the leftmost top position. Finally, by translation along the 16 different positions (remind that translations preserve the set of perfect magic squares), this yields to a total number of $16 \times 24 = 384$ fantastic squares. □

Remark 2. Again, the action of the group G defined below allows reducing to check that for a single choice of positions of $12, 14$ and 15 as above, a unique fantastic square arises. It is worth pointing out that uniqueness also follows by looking at adapted normal forms (as the one below) for perfect magic squares. (We leave the verification as an exercise.) Notice that the entries $u = 14$, $v = 12$ and $w = 15$ force, for instance,

$$3p + q - u - v - w = 48 + 1 - 14 - 12 - 15 = 8,$$

hence 8 is the other entry surrounding 1 (that is, an entry at distance 1 from it).

p	$p+q-w$	$u+w-p$	$p+q-u$
$u+v+w-2p$	$2p+q-u-v$	v	$2p+q-v-w$
$2p+q-u-w$	u	q	w
$p+q-v$	$v+w-p$	$3p+q-u-v-w$	$u+v-p$

Remark 3. With a little extra effort, uniqueness can be proved in the most general context. In precise terms, given any perfect square filled up with 16 different entries, the total number of perfect magic squares that can be filled up with these 16 entries is 384. (Compare [1, Problem 2.1].) Actually, each of them can be obtained from the original one letting act the group G defined below. To check this, we may assume that the smallest entry of the square is 0, hence the largest one is s, the half of the magic number. (Substracting the minimum entry to all entries of the square reduces the general case to this one.) Modulo the action of G, we may arrange s and 0 to be in the positions illustrated in the left bellow. If $m_1 < m_2$ are the smallest nonzero entries of the square, then $s - m_1$ and $s - m_2$ must be placed in neighboring boxes to that of 0. Otherwise, s and $s - m_1$ (resp. s and $s - m_2$) would appear in positions involved in at least two magic sums, hence the complement of their sum, namely m_1 (resp. m_2), should be written in at least two different ways as sum of the entries; however, the only possible such writing is $0 + m_1$ (resp. $0 + m_2$; notice that $m_1 + m_1$ is not allowed since it uses m_1 twice).

s			
		0	

s	m_1		
m_2	$s - m_1 - m_2$		
		0	$s - m_1$
		$s - m_2$	$m_1 + m_2$

Again, modulo the action of the group G, we may suppose that the positions of m_1 and m_2 are those showed in the right above. The values of the rest of the entries that appear therein follow from the magic properties (and the fact that entries in opposite positions add up to s). Now, letting m be any other number that appears in the square, modulo the action of G we may place it in one of the four positions that remain empty in the first two rows. This leads to four different squares that are non-equivalent under the action of G, which are shown below.

s	m_1	m	$s-m-m_1$
m_2	$s-m_1-m_2$	$s+m_2-m$	$m+m_1-m_2$
$s-m$	$m+m_1$	0	$s-m_1$
$m-m_2$	$s+m_2-m-m_1$	$s-m_2$	m_1+m_2

s	m_1	$s-m-m_1$	m
m_2	$s-m_1-m_2$	$m+m_1+m_2$	$s-m-m_2$
$m+m_1$	$s-m$	0	$s-m_1$
$s-m-m_1-m_2$	$m+m_2$	$s-m_2$	m_1+m_2

s	m_1	$s+m_2-m$	$m-m_1-m_2$
m_2	$s-m_1-m_2$	m	$s+m_1-m$
$m-m_2$	$s+m_1+m_2-m$	0	$s-m_1$
$s-m$	$m-m_1$	$s-m_2$	m_1+m_2

s	m_1	$m+m_2-m_1$	$s-m-m_2$
m_2	$s-m_1-m_2$	$s+m_1-m$	m
$s+m_1-m-m_2$	$m+m_2$	0	$s-m_1$
$m-m_1$	$s-m$	$s-m_2$	m_1+m_2

Nevertheless, a lengthly case-by-case analysis shows that the new entries are different from one square to another unless some of the entries of the original square were equal. Hence, only one family of squares (with its 384 members obtained by the action of G) can appear. We leave the details to the reader.

It is worth analyzing the 52 sums involved in the magic properties of a perfect magic square in terms of distance. Notice that the corresponding 4-uples can divided in three types:

– Type 1-1-2: starting at any position, the other positions involved are at distance 1, 1 and 2 from it. There are 24 such configurations, and these correspond to the 24 sums that appear in the colored squares 1, 2, 4, 5, 7 and 10 before Theorem A.

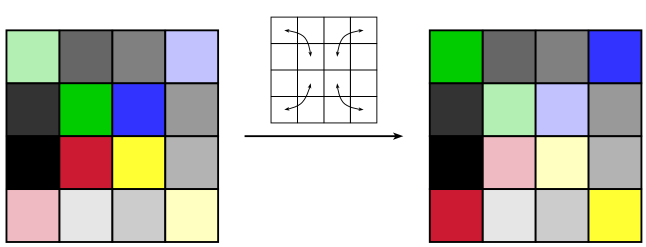
– Type 2-2-4: starting at any position, the other positions involved are at distance 2, 2 and 4 from it. There are 12 such configurations, which correspond to the 12 sums that appear in the colored squares 3, 6 and 11 before Theorem A.

– Type 1-3-4: starting at any position, the other positions involved are at distance 1, 3 and 4 from it. There are 16 such configurations, which correspond to the 16 sums that appear in the colored squares 8, 9, 12 and 13 before Theorem A.

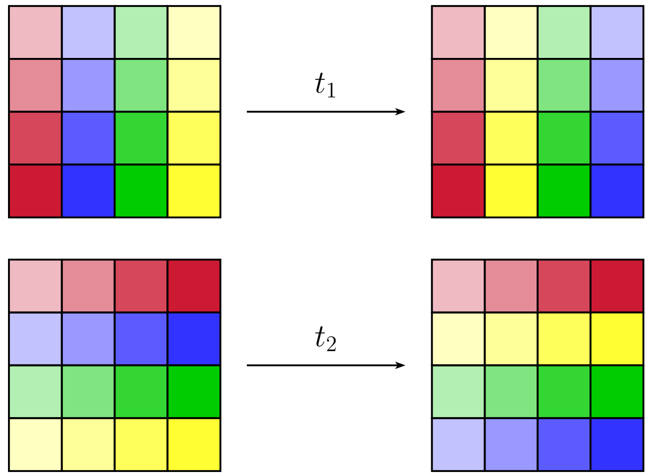
This completes the enumeration of the 52 sums. In particular, no other type of configuration is involved.

The group of symmetries of 4×4 perfect magic squares

We consider the subgroup G of S_{16} that contains all translations (mod 4), rotations by 90^o, 180^o and 270^o centered at the center of the 4×4 square, reflexions along the main diagonals, the middle vertical and the middle horizontal lines, and the quite remarkable involution (to be denoted s) depicted below. (As we will see, this group can be generated by four elements.) Notice for the moment that elements in G send contiguous boxes (*i.e.* boxes at distance 1) into contiguous ones.



Using these transformations, one can readily construct many others, as for instance those denoted by t_1 and t_2 illustrated below (these are nothing but conjugates of the vertical and horizontal reflexions by appropriate translations).

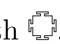


The following two important claims can be readily checked by inspection:

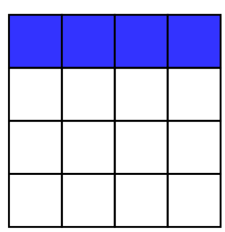
- The group G is contained in ; in other words, elements in G transform magic squares into magic squares;

- The group G acts transitively on the set of 4-uples of type 1-2-2. (The same holds for types 2-2-4 and 1-3-4, but we will not need these later facts.)

Obviously, the (extended) second claim implies the first one.

Proposition 2. The group G has order 384 and coincides with .

Proof. One readily checks that the action of G on the set of 4-uples of type 1-1-2 is transitive. Since there are 24 such 4-uples, we need to show that the stabilizer of each of them has order 16. Denote by G_1 the stabilizer of the 4-uple depicted below.



Obviously, the subgroup G_2 of G_1 made by the elements that fix the leftmost top position has index 4 in G_1. Since elements in G send contiguous boxes into contiguous ones, elements in G_2 necessarily fix the third position of the 4-uple above. An index-2 subgroup G_3 of G_2 is made by all the elements that fix the four positions. (The nontrivial class in G_1/G_2 is that of the element t_1 that permutes the second and the fourth columns and fixes all other positions).

We claim that G_3 is a group of two elements, the nontrivial one being the isometry t_2 that permutes the second and the fourth rows. Indeed, given a position in the first row and that of the third row from the same column, one readily checks that the other two positions of this column are the unique ones that complete with them a 4-uple of type 1-1-2. Since elements in G preserve the set of 4-uples of type 1-1-2, an element of G_2 must either fix these positions or permute them.

However, it is easy to see (again using the invariance of 4-uples of type 1-1-2) that if an element fixes a pair of such positions, then it is forced to fix all of them. Similarly, if it permutes one of them, it is forced to permute all of them.

We have hence proved that the order-2 group G_3 has order 2 in G_2, which in its turn has order 4 in G_1. Therefore, G_1 has order 16, which (as explained above) implies that G has order 384.

Since the action of [symbol] on the set of fantastic squares is obviously free, it follows from Proposition 1 that the order of [symbol] is smaller than or equal to 384. Since G is an order-384 subgroup of [symbol], it must necessarily coincide with the full group. This closes the proof of the proposition and, moreover, establishes the transitivity of the action of [symbol] on the set of fantastic squares. □

The previous results become more clear after realizing the following important fact, which can be easily checked:

– The group G (hence [symbol]) is contained in the group of isometries of the 4×4 square.

To prove Theorem E, we need to show the converse inclusion to that above, namely, that every isometry of the 4×4 square belongs to [symbol]. However, this follows from the fact that the 52 sums involved in the magic properties are in correspondence to all sets of 4-uples of types 1-2-2, 2-2-4 and 1-3-4, and each of these sets is obviously preserved by isometries.

On the structure of the Chautisa Yantra group

Up to isomorphism, there are 20 169 groups of order 384. Therefore, knowing the order of the Chautisa Yantra group doesn't provide much information on its structure. Below we give some insight leading to partial results on this via the action of [symbol] on the set of fantastic squares.

How to build group elements. Let σ be any element of S_4, which will be considered as a permutation of the "symbols" 8, 12, 14 and 15. From the preceding results and discussion, it becomes obvious that associated to σ there is a unique element g_σ in [symbol] satisfying:

– It fixes the box in which 1 appears (hence also that of 16);

– It takes the Chautisa Yantra square into the (unique) fantastic square for which the entries $\sigma(8)$, $\sigma(12)$, $\sigma(14)$ and $\sigma(15)$ appear in the previous positions of $8, 12, 14$ and 15, respectively.

Indeed, as it was mentioned, the partially filled square with entries 1, $\sigma(8)$, $\sigma(12)$, $\sigma(14)$ and $\sigma(15)$ can be completed in a unique way into a fantastic square. The element g_σ is hence the one that maps the box in which the entry k appears in the Chautisa Yantra square into the box where the same number k appears in the new fantastic square. As an example, for the permutation

$$\sigma = \begin{pmatrix} 8 & 12 & 14 & 15 \\ 12 & 14 & 8 & 15 \end{pmatrix}$$

we obtain the fantastic square in the right below.

7	12	1	14
2	13	8	11
16	3	10	5
9	6	15	4

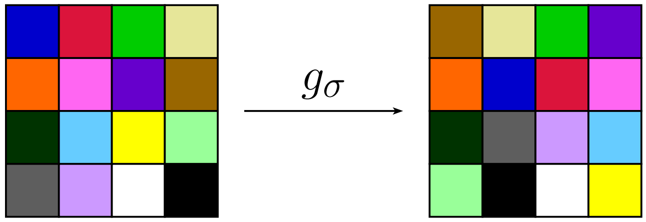
$\xrightarrow{g_\sigma}$

	14	1	8
		12	
		15	

$\rightsquigarrow$

11	14	1	8
2	7	12	13
16	9	6	3
5	4	15	10

Keeping in mind the Chautisa Yantra square in the left above, this yields the order-3 element g_σ in [symbol] depicted below.



It becomes obvious from the discussion above that contains a copy of S_4. Moreover, every element in can be written in a unique way as a product of a permutation (an element of S_4) and a translation (an element of $(\mathbb{Z}/4\mathbb{Z})^2$). Indeed, given g in , let T_g be the (unique) translation that sends the box of 1 (*i.e.* the third in the first row) into that in which 1 appears in image of the Chautisa Yantra square under g. Then $g\,T_g^{-1}$ fixes the box of 1, and is therefore completely determined by the new distribution of $8, 12, 14$ and 15 in the boxes surrounding that box; the later information is completely encoded by an element of S_4.

As a consequence of the claim above, it follows that can be generated by four elements: the horizontal and vertical translations of displacement 1, and any two generators of the symmetric group S_4.

Some examples. The canonical form above allows explicitly computing elements in . For example, for the permutation

$$\sigma = \begin{pmatrix} 8 & 12 & 14 & 15 \\ 12 & 15 & 14 & 8 \end{pmatrix}$$

and the translation T that moves boxes 2 spaces downwards and 2 spaces to the left, the associated element $g = T\sigma$ in arises from the diagram below.

7	12	1	14
2	13	8	11
16	3	10	5
9	6	15	4

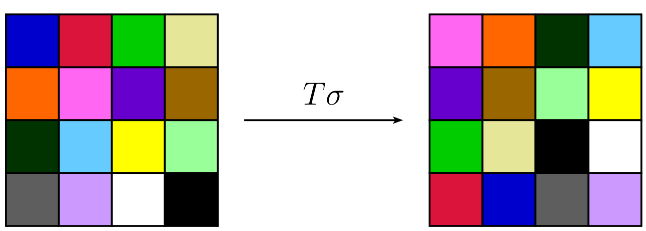
$\xrightarrow{g = T\sigma}$

8			
1	14		15
12			

$\rightsquigarrow$

13	2	16	3
8	11	5	10
1	14	4	15
12	7	9	6

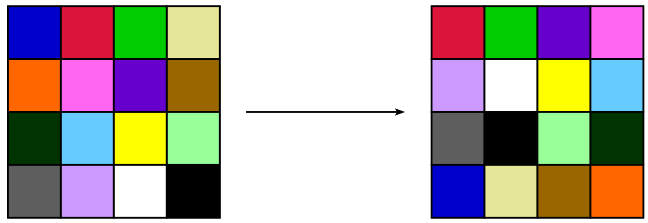
Hence, g acts as illustrated next.



It is not hard to check the somewhat surprising fact that the order of this element $g = T\sigma$ is 6. Another surprising element is the one depicted below, which arises from the translation of one space to the left and the permutation

$$\sigma = \begin{pmatrix} 8 & 12 & 14 & 15 \\ 15 & 12 & 8 & 14 \end{pmatrix}$$

We leave to the reader the task of checking that this element has order 8.



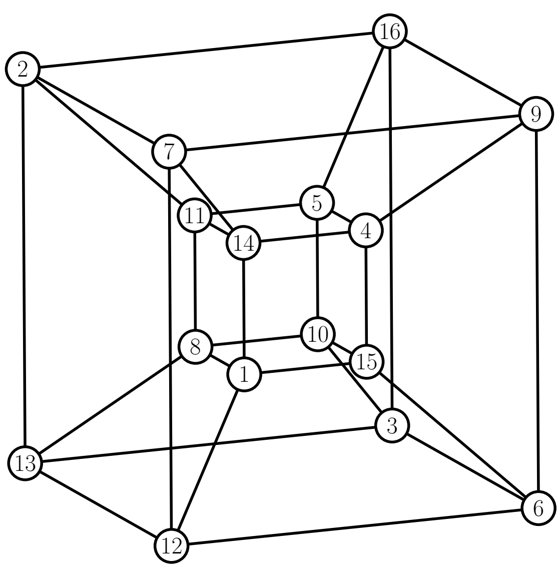
The Chautisa Yantra hypercube

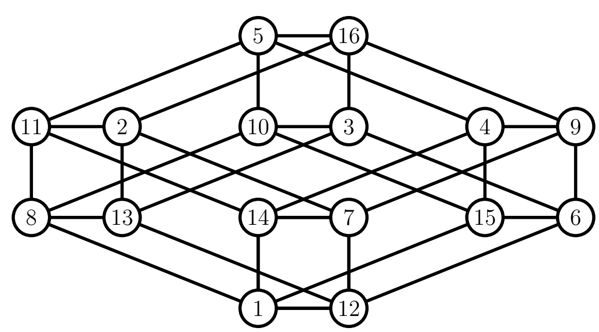
It seems very hard to completely understand the group structure of ◌ via the realization above. However, as it was suggested to us by G. Luchini, our discrete torus is nothing but the set of vertices of the hypercube graph $\mathcal{H}$. Therefore, ◌ coincides with the group of automorphisms of $\mathcal{H}$, which is a very well known object. Needed to say, the first explicit mention of the hypercube in relation to perfect magic squares in the literature seems to have been made by Coxeter in regard to the work of Rosser and Walker [9] (see also [10]).

The hypercube has vertex set $\{0,1\}^4$. The identification with the discrete torus is explicit in the labelling below.

0000	0001	0011	0010
0100	0101	0111	0110
1100	1101	1111	1110
1000	1001	1011	1010

Thus, the Chautisa Yantra square can be presented as an hypercube in which every vertex is labelled with a number from 1 to 16. Two different models are exhibited below.





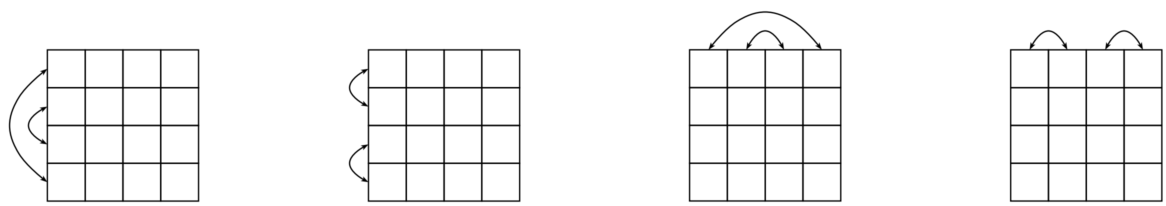
Remark 4. It may seem hard to fill the entries of the fantastic hypercubes because it is not obvious how to recognize 4-uples of types 1-3-4 and 2-2-4 in $\mathcal{H}$. However, using those of type 1-1-2 (which correspond to the 2-faces of the hypercube) is almost sufficient. To better explain this, it is worth mentioning that 4×4 squares for which all 4-uples of type 1-1-2 lead to the same sum (these are usually called *semimagic squares*) have the normal form below. We leave the verification of the details to the reader.

a	b	c	d
$s-(a-k)$	$s-(b+k)$	$s-(c-k)$	$s-(d+k)$
ℓ	$a+b-\ell$	$c+\ell-a$	$a+d-\ell$
$s-k-\ell$	$c+d+k+\ell-s$	$s+a-c-k-\ell$	$s-a-d+k+\ell$

Notice that, automatically, all 4-uples of type 1-3-4 lead to the same sum. However, imposing any single extra condition coming from a 4-uple of type 2-2-4 (as, for instance, any diagonal) leads to a perfect magic square. This is equivalent to imposing the condition that one of the entries of the square plus the entry in the opposite box add up to s.

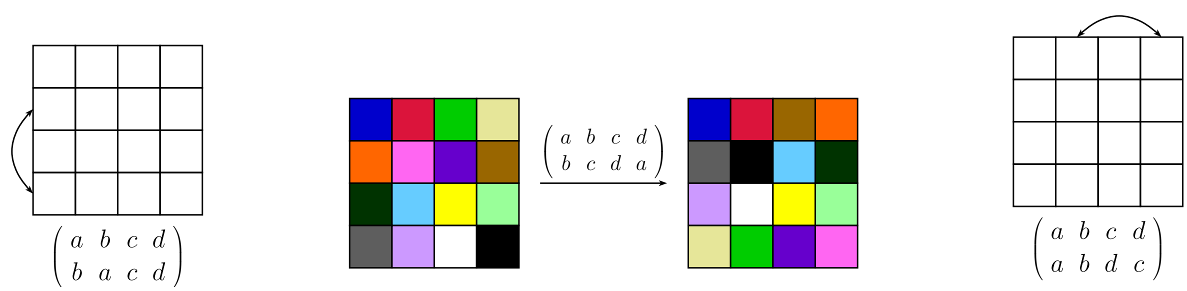
Therefore, in order to fill up a fantastic hypercube, it suffices to choose a vertex and labelling it 1, to label 8, 12, 14 and 15 the neighboring vertices (any choice is possible), and then labeling all vertices so that the sum of the labels along all faces equals 34 and the sum of the labels of opposite vertices equals 17.

There are two distinguished types of automorphisms of $\mathcal{H}$: dyadic translations and permutations of entries. The dyadic translations form a subgroup $\mathcal{D}$ of $\mathcal{H}$ with a preferred system of generators, namely, those that consist in adding 1 (mod. 2) to a previously chosen entry; see the pictures below. Notice that the dyadic translations correspond to order-2 elements, and $\mathcal{D}$ is an order 16 subgroup of ✤.



$(a,b,c,d) \to (a+1,b,c,d)$ $(a,b,c,d) \to (a,b+1,c,d)$ $(a,b,c,d) \to (a,b,c+1,d)$ $(a,b,c,d) \to (a,b,c,d+1)$

The action of some other permutations is shown below. Notice that permutations easily provide elements of order 2, 3 or 4 in ✤.



$\begin{pmatrix} a & b & c & d \\ b & a & c & d \end{pmatrix}$ $\begin{pmatrix} a & b & c & d \\ b & c & d & a \end{pmatrix}$ $\begin{pmatrix} a & b & c & d \\ a & b & d & c \end{pmatrix}$

Using dyadic translations and permutations, we may produce all group elements. For instance, in the hypercube model, the torus translation of displacement one to the right corresponds to the composition of the transposition $(c\,d)$ and the dyadic translation $(a,\, b,\, c,\, d) \to (a,\, b,\, c,\, d+1)$. As a another example, the counterclockwise rotation of angle 90^o is the product of the two transpositions $(a\,c)$ and $(b\,d)$ together with the dyadic translation $(a,\, b,\, c,\, d) \to (a+1,\, b,\, c,\, d)$.

It is not very hard to check that $\mathcal{D}$ is a normal subgroup in ✤. The corresponding quotient ✤$/\mathcal{D}$ is isomorphic to the permutation group S_4. Thus, ✤ is the semidirect product $S_4 \ltimes \mathcal{D}$, which can be also presented as the wreath product $S_4 \wr (\mathbb{Z}/2\mathbb{Z})$.

Using the isomorphisms above, it becomes evident that an element in ✤ can have order 2, 3, 4, 6 or 8. Indeed, its projection in ✤$/\mathcal{D}$ can have order 2, 3 or 4, and since nontrivial elements in $\mathcal{D}$ have order 2, this leads to the possibilities listed above.

On the 5-panmagic group

For $n \geq 1$, the *n-panmagic group* is the group of permutations of the entries of the $n \times n$ square that sends every $n \times n$ panmagic group into a panmagic one. For $n = 1, 2, 3$, this is the corresponding full permutation group. This is due to the well-known fact that, in each case, the entries of a panmagic square are necessarily equal. The 4-panmagic group is nothing than the Chautisa Yantra group. Our aim now is to explicitly determine the 5-panmagic group.

As in the case of 4×4 squares, we will say that an $n \times n$ square is fantastic if it is panmagic and is filled up with all the entries $1, 2, \ldots, n^2$. It is widely known that all 5×5 fantastic squares are obtained via the Euler-Ramanujan formal norm below for any choice of the parameters such that either $\{A, B, C, D, E\} = \{0, 5, 10, 15, 20\}$ and $\{P, Q, R, S, T\} = \{1, 2, 3, 4, 5\}$, or $\{A, B, C, D, E\} = \{1, 2, 3, 4, 5\}$ and $\{P, Q, R, S, T\} = \{0, 5, 10, 15, 20\}$ (see for instance [5]).

$A+P$	$E+R$	$D+T$	$C+Q$	$B+S$
$C+T$	$B+Q$	$A+S$	$E+P$	$D+R$
$E+S$	$D+P$	$C+R$	$B+T$	$A+Q$
$B+R$	$A+T$	$E+Q$	$D+S$	$C+P$
$D+Q$	$C+S$	$B+P$	$A+R$	$E+T$

The procedure above gives a total number of $2 \times (5!)^2 = 28\,800$ fantastic squares of order 5. One example is the square below, which has the supplementary property of being *symmetric*, which means that every entry and the entry in the opposite position with respect to the center sum up to 26.

1	15	22	18	9
23	19	6	5	12
10	2	13	24	16
14	21	20	7	3
17	8	4	11	25

Theorem G. The 5-panmagic group is isomorphic to $S_2 \ltimes (S_5 \times S_5)$, and acts transitively on the set of 5×5 fantastic squares.

Proof. The group $S_2 \ltimes (S_5 \times S_5)$ naturally acts on the set of 5×5 panmagic squares: one factor S_5 refers to the parameters $\{A, B, C, D, E\}$, the other one to the parameters $\{P, Q, R, S, T\}$, and the S_2 factor to the possibility of intertwining these parameters. This group freely acts on the set of 5×5 fantastic squares. Since the group has order $2 \times (5!)^2$, which equals the number of 5×5 fantastic squares, it must coincide with the whole 5-panmagic group. $\square$

It is worth stressing that, unlike the Chautisa Yantra group, the 5-panmagic group is not a group of isometries of the corresponding discrete torus. Indeed, group elements may send rows to very different configurations. In this regard, it is nice to realize that the family of configurations made of rows, columns, diagonals and quincunxes is invariant under the group action.

Notice that the reflexion along the main descendent diagonal arises from the S_2 factor, that is, from changing A, B, C, D, E into P, Q, R, S, T, respectively. Certain group elements that have no analog in other dimensions deserve particular attention. For example, the permutations

$$\begin{pmatrix} A & B & C & D & E \\ A & C & E & B & D \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} P & Q & R & S & T \\ P & R & T & Q & S \end{pmatrix}$$

yield the transformation

$A+P$	$E+R$	$D+T$	$C+Q$	$B+S$
$C+T$	$B+Q$	$A+S$	$E+P$	$D+R$
$E+S$	$D+P$	$C+R$	$B+T$	$A+Q$
$B+R$	$A+T$	$E+Q$	$D+S$	$C+P$
$D+Q$	$C+S$	$B+P$	$A+R$	$E+T$

$\longrightarrow$

$A+P$	$D+T$	$B+S$	$E+R$	$C+Q$
$E+S$	$C+R$	$A+Q$	$D+P$	$B+T$
$D+Q$	$B+P$	$E+T$	$C+S$	$A+R$
$C+T$	$A+S$	$D+R$	$B+Q$	$E+P$
$B+R$	$E+Q$	$C+P$	$A+T$	$D+S$

Geometrically, this corresponds to permuting the columns 1, 2, 3, 4, 5 into 1, 3, 5, 2, 4 and, right after this, permuting the columns 1, 2, 3, 4, 5 into 1, 3, 5, 2, 4.

Another interesting transformation arises from

$$\begin{pmatrix} A & B & C & D & E \\ A & E & D & C & B \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} P & Q & R & S & T \\ P & S & Q & T & R \end{pmatrix},$$

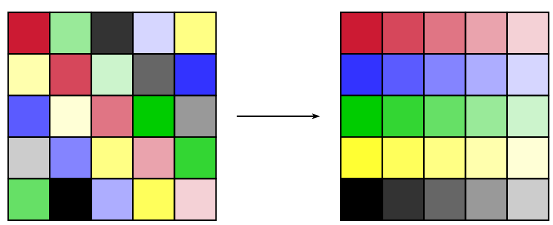
which yield

$A+P$	$E+R$	$D+T$	$C+Q$	$B+S$
$C+T$	$B+Q$	$A+S$	$E+P$	$D+R$
$E+S$	$D+P$	$C+R$	$B+T$	$A+Q$
$B+R$	$A+T$	$E+Q$	$D+S$	$C+P$
$D+Q$	$C+S$	$B+P$	$A+R$	$E+T$

$\longrightarrow$

$A+P$	$B+Q$	$C+R$	$D+S$	$E+T$
$D+R$	$E+S$	$A+T$	$B+P$	$C+Q$
$B+T$	$C+P$	$D+Q$	$E+R$	$A+S$
$E+Q$	$A+R$	$B+S$	$C+T$	$D+P$
$C+S$	$D+T$	$E+P$	$A+Q$	$B+R$

Geometrically, this corresponds to transforming diagonal into rows, as depicted below.



On the 6-panmagic group

As is well known (and not very hard to establish), there is no $n \times n$ fantastic square for any integer n of the form $4m+2$. In particular, no 6×6 fanstastic square exists. However, the 6-panmagic group is well defined.

6	33	36	48	19	8
29	41	5	15	13	47
40	1	34	12	43	20
2	31	42	44	17	14
35	37	3	21	9	45
38	7	30	10	49	16

On the 7-panmagic group

NORMAL FORM dimension 25
18 parameters, 25 genuine.

A_1	A_2	A_3	A_4	A_5	A_6	A_7
A_6	A_7	A_1	A_2	A_3	A_4	A_5
A_4	A_5	A_6	A_7	A_1	A_2	A_3
A_2	A_3	A_4	A_5	A_6	A_7	A_1
A_7	A_1	A_2	A_3	A_4	A_5	A_6
A_5	A_6	A_7	A_1	A_2	A_3	A_4
A_3	A_4	A_5	A_6	A_7	A_1	A_2

$+$

B_1	B_2	B_3	B_4	B_5	B_6	B_7
B_5	B_6	B_7	B_1	B_2	B_3	B_4
B_2	B_3	B_4	B_5	B_6	B_7	B_1
B_6	B_7	B_1	B_2	B_3	B_4	B_5
B_3	B_4	B_5	B_6	B_7	B_1	B_2
B_7	B_1	B_2	B_3	B_4	B_5	B_6
B_4	B_5	B_6	B_7	B_1	B_2	B_3

$+$

C_1	C_2	C_3	C_4	C_5	C_6	C_7
C_4	C_5	C_6	C_7	C_1	C_2	C_3
C_7	C_1	C_2	C_3	C_4	C_5	C_6
C_3	C_4	C_5	C_6	C_7	C_1	C_2
C_6	C_7	C_1	C_2	C_3	C_4	C_5
C_2	C_3	C_4	C_5	C_6	C_7	C_1
C_5	C_6	C_7	C_1	C_2	C_3	C_4

$+$

D_1	D_2	D_3	D_4	D_5	D_6	D_7
D_3	D_4	D_5	D_6	D_7	D_1	D_2
D_5	D_6	D_7	D_1	D_2	D_3	D_4
D_7	D_1	D_2	D_3	D_4	D_5	D_6
D_2	D_3	D_4	D_5	D_6	D_7	D_1
D_4	D_5	D_6	D_7	D_1	D_2	D_3
D_6	D_7	D_1	D_2	D_3	D_4	D_5

Geometric generalizations of magic squares in higher dimension

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