Pythagoras' theorem via equilateral triangles

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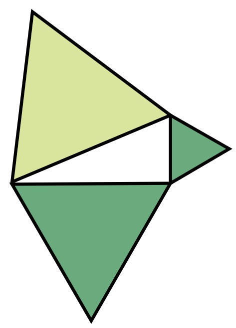
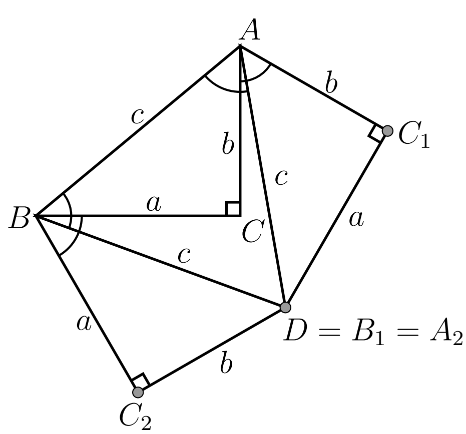
Different proofs of Pythagoras' theorem are the subject of many works along history. Here we propose still another proof which is nonstandard in that we do not use neither squares nor similarity of polygons. To proceed, consider the right triangle in the figure below, with equilateral triangles drawn on each of its sides.

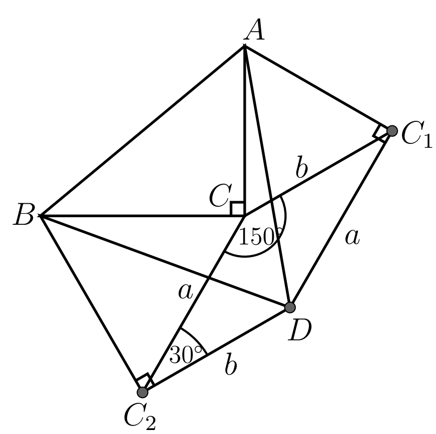


We claim that:

The areas of the small equilateral triangles sum up to the area of the larger one.

We prove this by a classical method of rearrangement. Denote A, B, C the vertices of the triangle and a, b, c the lengths of the corresponding opposite sides. Rotate the triangle ABC counter-clockwise in 60^{o} at the vertex A, and clockwise also in 60^{o} at B. Call C_1, B_1 and C_2, A_2 the image of the vertices, as shown in the picture on the left below. Notice that $BA_2 = c = AB_1$ and $\angle ABA_2 = \angle BAB_1 = 60^{o}$. Hence, A_2 and B_1 coincide. Denoting this point by D, we have that the vertices A, B and D determine an equilateral triangle of side length c.





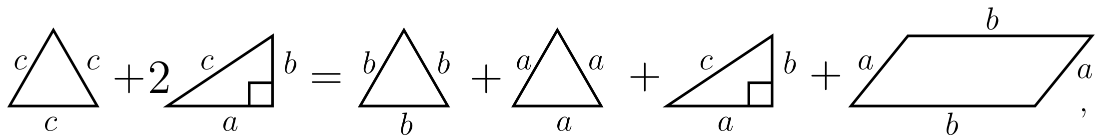
*Funded by Anillo Project 1415 "Geometry at the Frontier".

Also, notice that BCC_2 and ACC_1 are equilateral triangles of side length a and b, respectively. Moreover, triangles BC_2D and AC_1D are both congruent to triangle ABC, as shown on the right above.

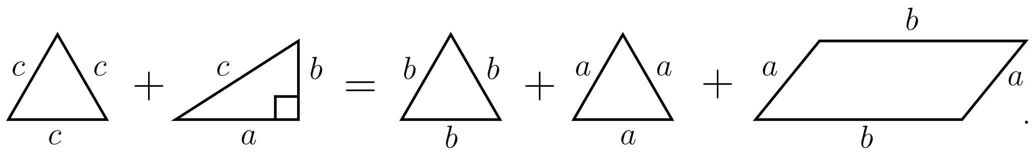
Now, referring to the areas, we have

$$ABC_2DC_1 = ABD + BC_2D + AC_1D = ACC_1 + BCC_2 + ABC + C_2DC_1C.$$

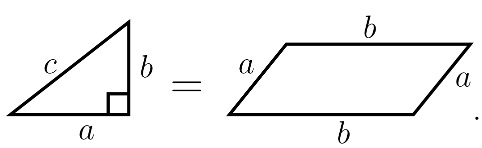
In pictures,



which yields



We are hence left to prove that

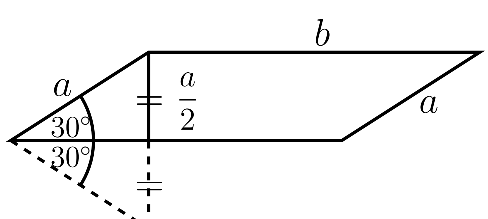


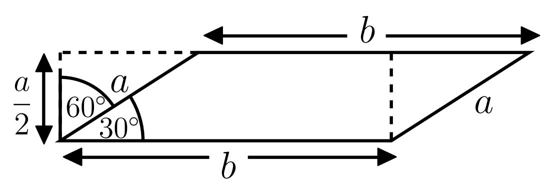
To do this, notice that C_2DC_1C is a parallelogram of side lenghts a and b. Besides, as $\angle BCA = 90^o$ and $\angle ACC_1 = \angle BCC_2 = 60^o$, we have that $\angle C_1CC_2$ must equal 150^o. Moreover, $\angle CC_2D = \angle BC_2D - \angle BC_2C = 30^o$, and similarly $\angle DC_1C = 30^o$. Thus,

$$C_1CC_2D = ab\sin(30^o) = \frac{1}{2}ab = ABC,$$

as desired.

In the last step above, one can certainly avoid the use of trigonometry just by looking at the pictures below. Indeed, it becomes clear that the height of a paralelogram of angles 30° and 150° equals half of the length of the side of the paralelogram that is not being considered as the basis. This is simply because a triangle of angles 30°, 60° and 90° is the half of an equilateral triangle, hence the length of its smallest side is half of the length of the largest one.





Why this implies Pythagoras' theorem ?

Since the area of an equilateral triangle of side length ℓ equals $\ell^2\sqrt{3}/4$, the equality proven above may be rewritten as

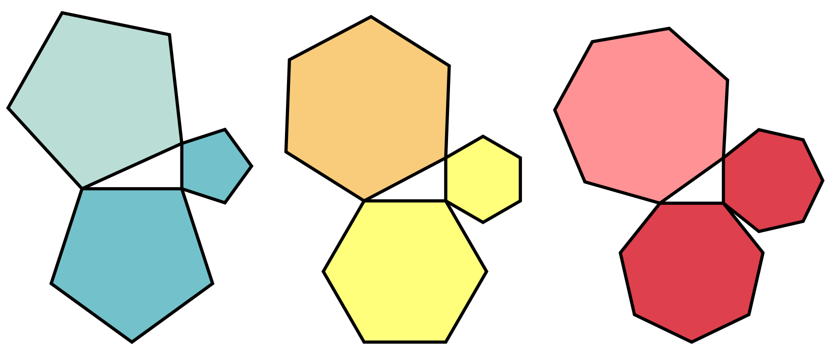
$$\frac{a^2\sqrt{3}}{4} + \frac{b^2\sqrt{3}}{4} = \frac{c^2\sqrt{3}}{4}.$$

This implies the Pythagorean equality $a^2 + b^2 = c^2$ just by cancelling the common factor $\sqrt{3}/4$.

Actually, if we start with the Pithagorean identity and multiply each term by $\sqrt{3}/4$, we obtain the equality between the area of the equilateral triangle built above the hypothenuse and the sum of those built on the catheti. The two claims are hence equivalent.

Building regular polygons on the sides

The Pythagorean equality $a^2 + b^2 = c^2$ also implies that the area of a regular n-gon of side length c is the sum of the areas of two regular n-gons of side lengths a and b, respectively. To see this, as above, it suffices to multiply each term of this equality by an appropriate constant (namely, the area of the regular n-gon of side length 1). Actually, this statement for any given $n \geq 3$ also implies (hence it is equivalent to) Pythagoras' theorem just by reversing this argument.



Certainly, a more geometric proof for regular polygons different from equilateral triangles or squares would be desirable. Of course, in this framework, Wallace-Bolyai-Gerwien's decomposition theorem applies, hence there should be a decomposition of the two small n-gons into polygonal pieces that, after rearrangement, yield the larger one. Nevertheless, the number of required pieces seems to be quite huge in general.

Acknowledgments. I wish to thank Nicolé Geyssel for drawing the pictures of this note.

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