An example concerning the Theory of Levels for codimension-one foliations

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An important aspect of foliations concerns the existence of local minimal sets. Recall that a foliated manifold has the LMS property if, for every open, saturated set W and every leaf $L \subset W$, the relative closure $\bar{L} \cap W$ contains a minimal set of $F|_W$. A fundamental result (due to Cantwell-Conlon [2] and Duminy-Hector [5]) establishes the LMS property for codimension-one foliations that are transversely of class $C^{1+\mathrm{Lipschitz}}$. This is the basic tool of the so-called *Theory of Levels*.

A classical example due to Hector (which corresponds to the suspension of a group action on the interval) shows that the LMS property is no longer true for codimension-one foliations which transversely are only continuous (see [1, Example 8.1.13]). Despite of this, in recent years, the possibility of extending some of the results of the Theory of Levels to smoothness smaller than $C^{1+\mathrm{Lipschitz}}$ has been naturally addressed [3, 4]. In this Note we will show that, however, analogues of Hector's example appear in class C^1 (and actually in class $C^{1+\alpha}$ for small values of α).

1 A General Construction

Let $(a_n)_{n\in\mathbb{Z}}$ be a sequence such that $a_{n+1} < a_n$ for all $n \in \mathbb{Z}$, $a_n \to 0$ as $n \to \infty$, and $a_n \to 1$ as $n \to -\infty$. Let (n_k) be a strictly increasing sequence of positive integers, and let $f\colon [0,1] \to [0,1]$ be a homeomorphism such that $f(a_{n+1}) = a_n$ for all $n \in \mathbb{Z}$. For each $k > 0$, we let u_k, v_k, b_k, c_k be such that $a_{n_k+1} < b_k < u_k < v_k < c_k < a_{n_k}$. For each $i \in \{0, \ldots, n_{k+1} - n_k\}$, we set $u_k^i := f^i(u_k)$ and $v_k^i := f^i(v_k)$. Notice that

$$f^i([u_{k+1}^0, v_{k+1}^0]) = [u_{k+1}^i, v_{k+1}^i] \subset f^i([a_{1+n_{k+1}}, a_{n_{k+1}}]) = [a_{n_{k+1}-i+1}, a_{n_{k+1}-i}].$$

Now, we let $g\colon [0,1] \to [0,1]$ be a homeomorphism such that:
– $g = Id$ on $[a_{n+1}, a_n]$ for each $n < 0$, as well as each $n > 0$ such that $n \neq n_k$ for every k;
– $g = Id$ on $[a_{1+n_k}, b_k] \cup [c_k, a_{n_k}]$, $g(u_k^0) = v_k^0$, and g has no fixed point on $]b_k, c_k[$.

Main assumption: In order that f, g generate a group of homeomorphisms of $[0,1]$ whose associated suspension does not have the LMS property, we assume that (see Figure 1)

$$u_{k+1}^{n_{k+1}-n_k} = b_k \qquad \text{and} \qquad v_{k+1}^{n_{k+1}-n_k} = c_k.$$

With these general notations, Hector's example corresponds to the choice $n_k = k$. We will show that, by taking $n_k = 2^k$, one may perform this construction in such a way the

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resulting maps f and g are diffeomorphisms of class C^1 (actually, of class $C^{1+\alpha}$ for any $\alpha < (\sqrt{5}-1)/2$). It is quite possible that slightly improving our method, one can smooth the action up to the class $C^{2-\delta}$ for any $\delta > 0$. (Compare [7], where for a similar construction, T. Tsuboi deals with the $C^{3/2-\delta}$ case before the $C^{2-\delta}$ case due to technical difficulties.)

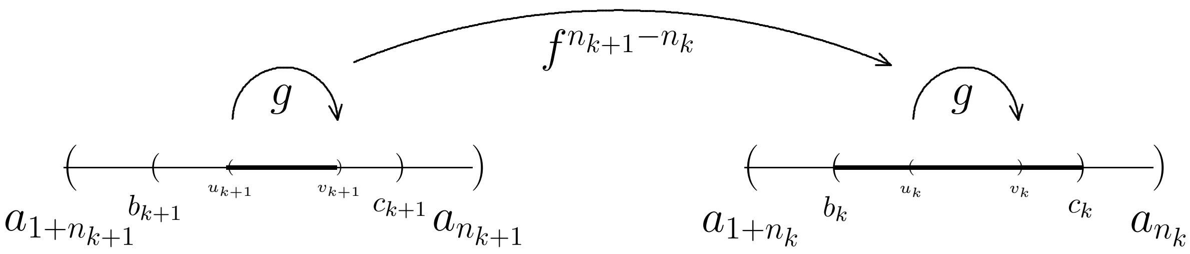

Figure 1

2 The length of the intervals and some estimates

We let $|[u^i_{k+1}, v^i_{k+1}]| := \lambda^i_k |[u_{k+1}, v_{k+1}]|$, where the constant $\lambda_k > 1$ satisfies the compatibility relation

$$\lambda_k^{2^k} = \frac{|[u^{2^k}_{k+1}, v^{2^k}_{k+1}]|}{|[u_{k+1}, v_{k+1}]|} = \frac{|[b_k, c_k]|}{|[u_{k+1}, v_{k+1}]|}. \tag{1}$$

Let $\varepsilon > 0$ be very small (to be fixed in a while). We set:

– $|[a_{n+1}, a_n]| := \frac{c_\varepsilon}{(1+|n|)^{1+\varepsilon}}$, where c_ε is chosen so that $\sum_{n\in\mathbb{Z}} |[a_{n+1}, a_n]| = 1$;

– $|[b_k, c_k]| := \frac{1}{2}|[a_{2^k+1}, a_{2^k}]| = \frac{c_\varepsilon}{2(1+2^k)^{1+\varepsilon}}$, where $k > 0$;

– $|[u_k, v_k]| := |[b_k, c_k]|^{1+\theta}$.

We assume that the center of $[a_{2^k+1}, a_{2^k}]$ coincides with the center of $[b_k, c_k]$ and with that of $[u_k, v_k]$. Furthermore, we assume that for each $i \in \{0, \ldots, 2^k\}$, the centers of $[u^i_{k+1}, v^i_{k+1}]$ and $[a_{2^{k+1}-i+1}, a_{2^{k+1}-i}]$ coincide.

For the estimates concerning regularity, we will strongly use the following lemma from [6].

Technical Lemma. *Let $\omega : [0, \eta] \to [0, \omega(\eta)]$ be a function (modulus of continuity) such that $s \mapsto s/\omega(s)$ is non-increasing. If I, J are closed non-degenerate intervals such that $1/2 \leq |I|/|J| \leq 2$ and*

$$\left|\frac{|J|}{|I|} - 1\right| \frac{1}{\omega(|I|)} \leq M,$$

then there exists a $C^{1+\omega}$ diffeomorphism $f : I \to J$ that is tangent to the identity at the endpoints and whose derivative has ω-norm bounded from above by $6\pi M$.

Actually, for $I := [a, b]$ and $J := [a', b']$, one may take $f = \varphi^{-1}_{a',b'} \circ \varphi_{a,b}$, where $\varphi_{a,b}$ is defined by (a similar definition stands for $\varphi_{a',b'}$)

$$\varphi_{a,b}(x) = -\frac{1}{(b-a)}\mathrm{ctg}\left(\pi\Big(\frac{x-a}{b-a}\Big)\right).$$

The condition on the derivative at the endpoints allows us to fit together the maps in order to create a diffeomorphism of a larger interval. Actually, if all of the involved sub-intervals of type I, J satisfy the hypothesis of the lemma above for the same constant M, then the ω-norm of the derivative of the induced diffeomorphism is bounded from above by $12\pi M$.

In what follows, we will deal with the modulus of continuity $\omega(s) = s^\alpha$ for the derivative, where $\alpha > 0$. A constant depending on the three parameters $\alpha, \theta, \varepsilon$, and whose value is irrelevant for our purposes, will be generically denoted by M.

Estimates for f: The diffeomorphism f is constructed by fitting together the maps provided by the Technical Lemma sending (see Figure 2):

(i) $[u^i_{k+1}, v^i_{k+1}]$ into $[u^{i+1}_{k+1}, v^{i+1}_{k+1}]$,

(ii) $[a_{2^{k+1}-i}, u^i_{k+1}]$ into $[a_{2^{k+1}-i-1}, u^{i+1}_{k+1}]$,

(iii) $[v^i_{k+1}, a_{2^{k+1}-i-1}]$ into $[v^{i+1}_{k+1}, a_{2^{k+1}-i-2}]$.

For (i), we have

$$\left|\frac{|[u^{i+1}_{k+1}, v^{i+1}_{k+1}]|}{|[u^i_{k+1}, v^i_{k+1}]|} - 1\right| \frac{1}{|[u^i_{k+1}, v^i_{k+1}]|^\alpha} = |\lambda_k - 1| \frac{1}{(\lambda_k^i |[u^0_{k+1}, v^0_{k+1}]|)^\alpha} \leq |\lambda_k - 1| \frac{1}{|[b_{k+1}, c_{k+1}]|^{(1+\theta)\alpha}}.$$

Now from (1) one obtains

$$\lambda_k^{2^k} = \frac{\frac{c_\varepsilon}{2(1+2^k)^{1+\varepsilon}}}{\big(\frac{c_\varepsilon}{2(1+2^{k+1})^{1+\varepsilon}}\big)^{1+\theta}} \leq M\Big(\frac{(1+2^{k+1})^{1+\theta}}{1+2^k}\Big)^{1+\varepsilon} \leq M2^{k\theta(1+\varepsilon)}.$$

From the inequality $|2^\alpha - 1| \leq \alpha$ (which holds for α positive and small) one concludes that

$$|\lambda_k - 1| \leq M\frac{k}{2^k}.$$

On the other hand,

$$\frac{1}{|[b_{k+1}, c_{k+1}]|} \leq M(1+2^{k+1})^{1+\varepsilon} \leq M2^{k(1+\varepsilon)}.$$

Therefore,

$$\left|\frac{|[u^{i+1}_{k+1}, v^{i+1}_{k+1}]|}{|[u^i_{k+1}, v^i_{k+1}]|} - 1\right| \frac{1}{|[u^i_{k+1}, v^i_{k+1}]|^\alpha} \leq M\frac{k}{2^k}2^{k(1+\varepsilon)(1+\theta)\alpha}. \quad (2)$$

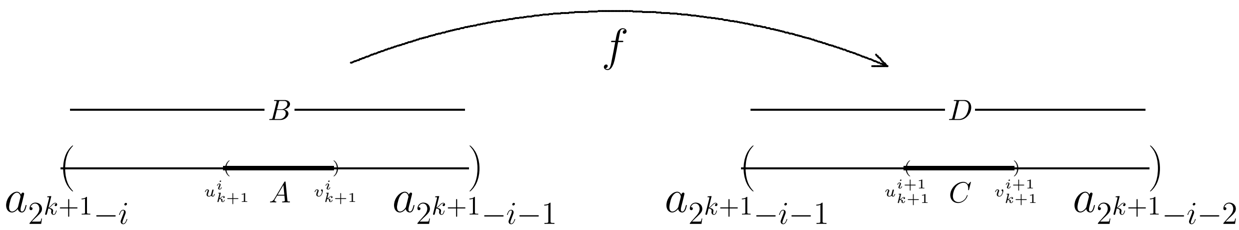

Figure 2

Now, for (ii), set $A := |[u^i_{k+1}, v^i_{k+1}]|$, $B := |[a_{2^{k+1}-i}, a_{2^{k+1}-i-1}]|$, $C := |[u^{i+1}_{k+1}, v^{i+1}_{k+1}]|$, and $D := |[a_{2^{k+1}-i-1}, a_{2^{k+1}-i-2}]|$. Then

$$\left| \frac{|[a_{2^{k+1}-i-1}, u^{i+1}_{k+1}]|}{|[a_{2^{k+1}-i}, u^i_{k+1}]|} - 1 \right| \frac{1}{|[a_{2^{k+1}-i}, u^i_{k+1}]|^\alpha} = \left| \frac{D-C}{B-A} - 1 \right| \frac{2^\alpha}{(B-A)^\alpha}.$$

Moreover, since $A \leq B/2$ and $C = \lambda_k A$,

$$\begin{aligned}
\left| \frac{D-C}{B-A} - 1 \right| &\leq \left| \frac{D-B}{B-A} \right| + \left| \frac{C-A}{B-A} \right| \leq 2 \left| \frac{D-B}{B} \right| + |\lambda_k - 1| \\
&= \frac{M}{B} \left[\frac{1}{(2^{k+1}-i-2)^{1+\varepsilon}} - \frac{1}{(2^{k+1}-i-1)^{1+\varepsilon}} \right] + M\frac{k}{2^k} \\
&\leq MB \left[(2^{k+1}-i-1)^{1+\varepsilon} - (2^{k+1}-i-2)^{1+\varepsilon} \right] + M\frac{k}{2^k} \\
&\leq \frac{M}{2^{k(1+\varepsilon)}} 2^{k\varepsilon} + M\frac{k}{2^k} \\
&\leq M\frac{k}{2^k}.
\end{aligned}$$

Therefore,

$$\left| \frac{D-C}{B-A} - 1 \right| \frac{2^\alpha}{(B-A)^\alpha} \leq M\frac{k}{2^k} 2^{k(1+\varepsilon)\alpha},$$

hence

$$\left| \frac{|[a_{2^{k+1}-i-1}, u^{i+1}_{k+1}]|}{|[a_{2^{k+1}-i}, u^i_{k+1}]|} - 1 \right| \frac{1}{|[a_{2^{k+1}-i}, u^i_{k+1}]|^\alpha} \leq M\frac{k}{2^{k(1-(1+\varepsilon)\alpha)}}. \tag{3}$$

Finally, notice that by construction, the estimates for (iii) are the same as those for (ii).

Estimates for g: The diffeomorphism g is obtained by fitting together the maps provided by the Technical Lemma sending:

(i) $[b_k, u^0_k]$ into $[b_k, v^0_k]$,

(ii) $[u^0_k, c_k]$ into $[v^0_k, c_k]$,

(iii) $[a_{2^k+1}, b_k]$ and $[c_k, a_{2^k}]$ into themselves as the identity.

For (i), notice that

$$\left| \frac{|[b_k, v^0_k]|}{[b_k, u^0_k]} - 1 \right| \frac{1}{|[b_k, u^0_k]|^\alpha} = \frac{|[u^0_k, v^0_k]|}{|[b_k, u^0_k]|^{1+\alpha}} \leq \frac{2^{1+\alpha}|[u^0_k, v^0_k]|}{(|[b_k, c_k]| - |[u^0_k, v^0_k]|)^{1+\alpha}} = \frac{2^{1+\alpha}|[b_k, c_k]|^{1+\theta}}{(|[b_k, c_k]| - |[b_k, c_k]|^{1+\theta})^{1+\alpha}},$$

thus

$$\left| \frac{|[b_k, v^0_k]|}{[b_k, u^0_k]} - 1 \right| \frac{1}{|[b_k, u^0_k]|^\alpha} \leq M|[b_k, c_k]|^{\theta-\alpha}. \tag{4}$$

The estimates for (ii) are similar to those for (i) and we leave them to the reader.

The choice of the parameters: According to our Technical Lemma, and due to (2), (3), and (4), sufficient conditions for the $C^{1+\alpha}$ smoothness of f, g are:

– $(1+\varepsilon)(1+\theta)\alpha < 1$,

– $\frac{1}{1+\varepsilon} > \alpha$

– $\theta > \alpha$.

Now, for $0 < \alpha < (\sqrt{5}-1)/2$, one easily checks that these conditions are satisfied for $\theta := \alpha + \varepsilon$, where $\varepsilon > 0$ is small enough so that $(1+\varepsilon)(1+\alpha+\varepsilon)\alpha < 1$.

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