Three results on cocycles of Banach space isometries over a minimal dynamics

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Abstract. We show that skew-actions by isometries of Banach spaces over a minimal dynamics having a bounded orbit admit continuous skew-invariant sections in case of reflexive or L^1 spaces, and that this is no longer true for L^∞. This extends one of the results in [2] and [6], which establishes the same result for cocycles of Hilbert space isometries.

Let Γ be a semigroup acting minimally by homeomorphisms of a compact metric space X. Given a topological group G, we consider a cocycle I over this dynamics taking values in G, that is, a continuous map $I : \Gamma \times X \to G$ such that $I(gh, x) = I(g, hx)I(h, x)$ holds for all g, h in Γ and all $x \in X$. We are interested in the case where G is the group of (affine) isometries of a Banach space $\mathbb{B}$. It is not hard to check that, in this setting, the cocycle relation is equivalent to that if for each $g \in G$ we consider the map $(x, v) \to (g(x), I(g, x)(v))$ on $X \times \mathbb{B}$, then this defines a (skew) group action.

It was shown in [2] that if $\mathbb{B}$ is a separable Hilbert space, then the existence of a bounded orbit for the skew-action implies the existence of a continuous skew-invariant section, that is, a continuous function $\varphi : X \to \mathbb{B}$ such that $I(g, x)\varphi(x) = \varphi(gx)$ holds for all $x \in X$ and all $g \in \Gamma$. The necessity of extending this result to more general Banach spaces has been pointed out by J. Renault [6]. Let us mention that slight extensions of the arguments of [2] show that the result still holds when $\mathbb{B}$ is a separable and uniformly-convex Banach space, so that we will refer to the Main Theorem of [2] as being this generalized version. Our first result deals with the even more general case of a separable, reflexive space, and may be considered as an extension of the famous Ryll-Nardzewski theorem for cocycles over a minimal dynamics.

Theorem A. *In the setting above, if $\mathbb{B}$ is separable and reflexive, then the existence of a bounded orbit for the skew-action implies the existence of a continuous skew-invariant section $\varphi \colon X \to \mathbb{B}$.*

Our next two results concern two important cases of non-reflexive spaces, namely L^1 and L^∞ spaces, for which the situations appear to be opposite.

Theorem B. *In the setting above, if Γ is a group and $\mathbb{B}$ is the space $L^1(\Omega, \mu)$ of integrable (real or complex valued) functions over a σ-finite measured space (Ω, μ), then the existence of a bounded orbit for the skew-action implies the existence of a continuous skew-invariant section $\varphi \colon X \to L^1(\Omega, \mu)$.*

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This theorem may also be considered as a version for cocycles of the main result of [1], where it is shown that every group action by isometries of an L^1 space with a bounded orbit has a fixed point. As in the case of [1], Theorem B above still holds when replacing $L^1(\Omega, \mu)$ by a separable predual of a von Newmann algebra. We do not know whether the theorem (as well as the main result of [1]) still holds for semigroup actions.

Using dynamical arguments, it has been proved in [2] that every semigroup action by isometries of an L^∞ space having a bounded orbit must have a fixed point. (Notice that for real or complex valued functions, this also follows from the Ryll-Nardzewski theorem.) Nevertheless, the situation for cocycles of isometries of an L^∞ space over a minimal dynamics is different.

Theorem C. *There exists a compact metric space X endowed with a minimal flow T^t associated to which there is a cocycle of translations $c_t : X \to L^\infty(\Omega, \mu)$ having a bounded orbit but no continuous skew-invariant section.*

We should point out that continuity here is understood in the strong topology. Indeed, the arguments of the first part of the proof of Theorem A show that if $\mathbb{B}$ is an L^∞ space, the existence of a bounded orbit for a skew-action implies the existence of a weakly-continuous skew-invariant section.

1 Proof of Theorem A

The proof is strongly inpired by Namioka-Asplund's proof of the Ryll-Nardzewski theorem [5], and can be easily modifed to show a more general result concerning cocycles of affine noncontractions that preserve a weakly-compact subset (we leave this to the reader).

First step. Existence of a weakly-continuous skew-invariant section.

Let us consider the product $X \times \mathbb{B}$ endowed with the weak topology along the fibers. If there is a bounded orbit for the skew-action, then its closure is a compact skew-invariant set. The same holds for the set obtained by taking the convex-closure along the fibers. The family $\mathcal{F}$ of compact skew-invariant sets that are convex along the fibers is hence nonempty, and Zorn's lemma yields a minimal set (for the inclusion relation) in $\mathcal{F}$. Let us denote M this set. Since the action on X is minimal, the projection of M into X must coincide with the whole space. We will show that for all $x \in X$, the fiber $M_x := \{v \in \mathbb{B} : (x, v) \in M\}$ of M above x must consist of a single point. Assuming this, we have that M is the graph of a skew-invariant function φ, which must be weakly-continuous because its graph M is compact (for the weak topology on the fibers).

To show that M_x is a single point, we first recall that every separable Banach space is *dentable*. This means that given any bounded subset $B \subset \mathbb{B}$ and $\varepsilon > 0$, there exists a convex subset $C_\varepsilon \subset \mathbb{B}$ such that the diameter of $B \setminus C_\varepsilon$ is smaller than ε. Assuming that M_x has two points $v_1 \neq v_2$, we let $\varepsilon := \|v_1 - v_2\|/2$. By convexity, the point (x, v_*) belongs to M, where $v_* = (v_1 + v_2)/2$. We claim that the orbit of (x, v_*) must intersect $X \times (B \setminus C_\varepsilon)$, that is, $I(g)(v_*)$ belongs to $B \setminus C_\varepsilon$ for a certain $g \in \Gamma$. Otherwise, the convex-closure of the orbit of (x, v_*) would be an element of $\mathcal{F}$ contained in C_ε, thus contradicting the minimality of M. Now, by the definition of ε, none of the points $I(g)(v_1)$ and $I(g)(v_2)$ can be contained in

$B \setminus C_\varepsilon$. Therefore, they are both in $B \cap C_\varepsilon$. By the convexity of C_ε, their midpoint $I(g)(v_*)$ must also be contained in C_ε, which is a contradiction.

Second step. Strong continuity of weakly-continuous skew-invariant sections.

The argument of [2, §4.3.3] shows that the set S of strong-continuity points of a weakly-continuous skew-invariant section φ is closed and Γ-invariant. By the minimality of the Γ-action on X, either S is empty or it coincides with the whole space. The fact that it is not empty follows from the following general fact: if $\mathbb{B}$ is separable, then every weakly-continuous function from a compact metric space into $\mathbb{B}$ is strongly-continuous on a G_δ-set (see [7]).

2 Proof of Theorem B

Given a Banach space $\mathbb{B}$ with bidual $\mathbb{B}^{**}$, we say that $\mathbb{B}$ is complemented in $\mathbb{B}^{**}$ if there exists a subspace $\mathbb{B}_1$ of $\mathbb{B}^{**}$ such that $\mathbb{B}^{**} = \mathbb{B} \oplus \mathbb{B}_1$ and the norm in $\mathbb{B}^{**}$ is the sum of the norms in $\mathbb{B}$ and $\mathbb{B}_1$. As in [1], the main property of $\mathbb{B} = L^1(\Omega, \mu)$ that we will exploit (and which is shared by preduals of von Newmann algebras) is that it is complemented in its double dual.

To prove Theorem B, we first notice that I has a unique extension to a weakly*-continuous isometric action on $\mathbb{B}^{**}$. Namely, let $\theta(g,x)$ and $c(g,x)$ be the associated linear and translation part of $I(g,x)$, respectively, so that $I(g,x) = \theta(g,x) + c(g,x)$. Then we let

$$I^{**}(g,x) := \theta^{**}(g,x) + c(g,x),$$

where, for $\omega \in \mathbb{B}^{**}$ we define $\theta^{**}(g,x)(\omega)$ by letting

$$\theta^{**}(g,x)(\omega)(U) := \omega(U \circ \theta(g,x)) \quad \text{for all } U \in \mathbb{B}^*.$$

To check that this defines a cocycle of isometries, we first need to check that each $I^{**}(g,x)$ is an isometry of $\mathbb{B}^{**}$, which is equivalent to checking that each $\theta^{**}(g,x)$ is an isometry. To do this, notice that, since Γ is assumed to be a group, each $I(g,x)$ has inverse $I(g^{-1}, g^{-1}x)$, hence each $\theta(g,x)$ is a bijective linear isometry, with inverse $\theta(g^{-1}, g(x))^{-1}$. Then, for all $g \in \Gamma$ and $x \in X$, we compute:

$$\begin{aligned}
\|\theta^{**}(g,x)(\omega)\|_{\mathbb{B}^{**}} &= \sup_{\|U\|_{\mathbb{B}^*}=1} |\theta^{**}(g,x)(\omega)(U)| \\
&= \sup_{\|U\|_{\mathbb{B}^*}=1} |\omega(U \circ \theta(g,x))| \\
&= \sup_{\|V\|_{\mathbb{B}^*}=1} |\omega(V)| \\
&= \|\omega\|_{\mathbb{B}^{**}},
\end{aligned}$$

where, in the third equality, we have used the fact that $\theta(g,x)$ is a bijective linear isometry. Moreover, given g_1, g_2 in Γ, letting $\theta_1 := \theta(g_1, g_2(x))$ and $\theta_2 := \theta(g_2, x)$, we have

$$\begin{aligned}
\theta^{**}(g_1 g_2, x)(\omega)(U) &= \omega(U \circ \theta(g_1 g_2, x)) \\
&= \omega(U \circ (\theta_1 \theta_2))
\end{aligned}$$

$$
\begin{aligned}
&= \omega((U \circ \theta_1) \circ \theta_2) \\
&= \theta^{**}(g_2, x)(\omega)(U \circ \theta_1) \\
&= \theta^{**}(g_1, g_2(x))\theta^{**}(g_2, x)(\omega)(U),
\end{aligned}
$$

which shows that θ^{**} defines a cocycle of linear isometries. Finally, to see that I^{**} is weakly* continuous, we need to check that θ^{**} is weakly continuous, that is, if x_n converges to x in X, then for each $\omega \in \mathbb{B}^{**}$, one has $\theta^{**}(g, x_n)(\omega) \rightharpoonup \theta^{**}(g, x)(\omega)$. To see that this holds, simply notice that, for all $U \in \mathbb{B}^*$, one has the convergence $U \circ \theta(g, x_n) \to U \circ \theta(g, x)$ in the operator norm. Therefore, $\omega(U \circ \theta(g, x_n)) \to \omega(U \circ \theta(g, x))$, which yields the desired convergence.

Now, the strategy of proof of Theorem B is as follows:

– The existence of a bounded orbit plus the minimality of the action on the basis implies that the orbit $\mathcal{O}$ of the trivial section $X \times \{0\}$ is bounded;

– For each $x \in X$, we let $ctr(\mathcal{O}_x)$ be the set of Chebyshev centers of $\mathcal{O}_x$ (the fiber of $\mathcal{O}$ over x). This set, which is a priori contained in $\mathbb{B}^{**}$, is actually contained in $\mathbb{B}$.

– The set

$$
\mathcal{C} := \{(x, v) \colon x \in X, v \in ctr(\mathcal{O}_x)\}
$$

is compact (for the weak topology along the fibers) and invariant.

– We can hence consider the family of compact invariant subsets contained in $\mathcal{O}$. A minimal set therein is the graph of a continuous skew-invariant section $\varphi : X \to \mathbb{B}$.

First Step. Let (x_0, v_0) be a point of $X \times \mathbb{B}$ having bounded orbit. To be more precise, let L be such that, for all $h \in \Gamma$, one has $\|I(h, x_0)(v_0)\| \leq L$. Notice that this implies that, for all $v \in \mathbb{B}$,

$$
\|I(h, x_0)(v)\| = \|I(h, x_0)(v) - I(h, x_0)(v_0) + I(h, x_0)(v_0)\| \leq \|v - v_0\| + L \leq \|v\| + 2L.
$$

Moreover, since

$$
\|I(h, x_0)^{-1}(v_0) - v_0\| = \|v_0 - I(h, x_0)(v_0)\|,
$$

one has

$$
\|I(h, x_0)^{-1}(v_0)\| \leq 2\|v_0\| + \|I(h, x_0)(v_0)\| \leq 3L.
$$

Now, for each $x \in X$ there is a sequence $g_n \in \Gamma$ such that $g_n(x_0)$ converges to x. The previous remarks then yields

$$
\|I(g, g_n(x_0))(v_0)\| = \|I(gg_n, x_0) I(g_n, x_0)^{-1}(v_0)\| \leq \|I(g_n, x_0)^{-1}(v_0)\| + 2L \leq 5L.
$$

Since $I(g, x)(v_0) = \lim_{n \to \infty} I(g, g_n(x_0))(v_0)$, this implies that $\|I(g, x)(v_0)\| \leq 5L$, and therefore, for all $g \in \Gamma$,

$$
\|I(g, x)(0)\| = \|I(g, x)(0) - I(g, x)(v_0) + I(g, x)(v_0)\| \leq \|v_0\| + \|I(g, x)(v_0)\| \leq 6L
$$

Thus, the orbit $\mathcal{O}$ of the trivial section $X \times \{0\}$, which is nothing but the set

$$
\{(g(x), I(g, x)(0)) \colon x \in X, g \in \Gamma\},
$$

satisfies that all its fibers are made of vectors having norm $\leq 6L$. Notice that the fiber of $\mathcal{O}$ over $x \in X$ is $\mathcal{O}_x = \{I(g, g^{-1}(x))(0) \colon g \in \Gamma\}$.

Second Step. Since for each $x \in X$ the subset $\mathcal{O}_x \subset \mathbb{B}$ is bounded, we can consider the set $ctr(\mathcal{O}_x) \subset \mathbb{B}^{**}$ made of its Chebyshev centers. To be more precise, remind that given a bounded subset C of a Banach space $\mathbb{B}$, its *radius* is defined as

$$\rho(A) := \inf\{r > 0 \colon A \subset B(\omega, r) \text{ for a certain } \omega \in \mathbb{B}^{**}\}.$$

The set $ctr(C)$ of *Chebyshev centers* of C is the set of $\omega \in \mathbb{B}^{**}$ such that $C \subset B(\omega, \rho(C) + \varepsilon)$ for all $\varepsilon > 0$. It is easy to see that this set is convex. More importantly, using the fact that the norm is lower-semicontinuous with respect to the weak topology, one readily checks that $ctr(C)$ is weakly* compact.

A key argument for the proof in [1] is that, if $\mathbb{B}$ is complemented in its double dual, then the Chebyshev centers of every bounded subset of $\mathbb{B}$ are actually contained in $\mathbb{B}$. In particular, in our framework, each set $ctr(x)$ is contained in $\mathbb{B}$, hence $\mathcal{C}$ defined above is contained in $X \times \mathbb{B}$.

Third Step. This is the main new ingredient. The situation is quite delicate, as it was shown in [2] that taking centers along the fibers of the closure of an orbit may fail to give a continuous skew-invariant section even in the framework of Hilbert spaces (it is, however, continuous for finite dimensional vector spaces). The key point here is that we will not consider the orbit of a single point but that of a section. We start with a general lemma.

Lemma 1. *If C is a bounded subset of a Banach space $\mathbb{B}$ then, for each $\varepsilon > 0$, there exists a finite subset $\{c_1, \ldots, c_k\}$ of C such that $\rho(\{c_1, \ldots, c_k\}) \geq \rho(C) - \varepsilon$.*

Proof. Assume otherwise, and let $\varepsilon > 0$ be such that for each finite subset $F = \{c_1, \ldots, c_k\}$ of C, the set $C_{\varepsilon,F} := \{\omega \in \mathbb{B}^{**} \colon F \subset B(\omega, \rho(C)) - \varepsilon\}$ is nonempty. The semicontinuity of the norm with respect to the weak topology implies that $C_{\varepsilon,F}$ is weakly* compact (moreover, it is convex). The family $\mathcal{F}$ made by all these sets $C_{\varepsilon,F}$ (with ε fixed and F ranging all along the finite subsets of C) satisfies the finite intersection property. Indeed, if $F_1, \ldots, F_n$ are finitely many finite subsets of C, then

$$C_{\varepsilon,F_1} \cap \ldots \cap C_{\varepsilon,F_n}$$

contains $C_{\varepsilon,F_1 \cup \ldots \cup F_n}$ which, by hypothesis, is nonempty. By compactness, the whole intersection

$$\bigcap_{F \text{ finite}, F \subset C} C_{\varepsilon,F}$$

is nonempty. By definition, every point ω in this intersection satisfies

$$C \subset B(\omega, \rho(C) - \varepsilon).$$

However, this contradicts the definition of $\rho(C)$, thus proving the lemma.

A preliminary step to prove that $\mathcal{C}$ is compact (for the weak topology) is given by the next lemma.

Lemma 2. *The function that assigns to each x the radius $\rho(\mathcal{O}_x)$ of the fiber of $\mathcal{O}$ above x is constant.*

Proof. Let ρ be the supremum of of the radii of the fibers $\mathcal{O}_x$. Given $\varepsilon > 0$, let $x \in X$ be such that $\rho(\mathcal{O}_x) > \rho - \varepsilon$. We first show that there is a neighborhood V of x such that, for all $y \in V$, one still has $\rho(\mathcal{O}_y) > \rho - 3\varepsilon$. To show this, notice that Lemma 1 yields a finite subset $\{u_1, \ldots, u_k\}$ of $\mathcal{O}_x$ such that $\rho(\{u_1, \ldots, u_k\}) > \rho - 2\varepsilon$. Each v_i is of the form $I(g_i, g_i^{-1}(x))(0)$ for a certain $g_i \in \Gamma$. Choose V so that, for each $1 \leq i \leq k$,

$$\|I(g_i, g_i^{-1}(x))(0) - I(g_i, g_i^{-1}(y))(0)\| < \varepsilon. \tag{1}$$

The points $v_i := I(g_i, g_i^{-1}(y))$ all lie in $\mathcal{O}_y$. In particular, $\rho(\mathcal{O}_y) \geq \rho(\{v_1, \ldots, v_k\})$. Finally, it is clear that $\rho(\{v_1, \ldots, v_k\}) > \rho - 3\varepsilon$. Otherwise, we would have $\{v_1, \ldots, v_k\} \subset B(\omega, \rho - 3\varepsilon)$ for a certain $\omega \in \mathbb{B}^{**}$, which together with (1) implies $\{u_1, \ldots, u_k\} \subset B(\omega, \rho - 2\varepsilon)$, thus leading to a contradiction.

To close the proof of the lemma notice that, since $\mathcal{O}$ is invariant, the set $\mathcal{C}$ is also invariant. Moreover, the function $z \to \rho(\mathcal{O}_z)$ is contant along the orbits. Since the action on the basis is minimal, every orbit intersects V. Therefore, one has $\rho(\mathcal{O}_z) > \rho - 3\varepsilon$ for all $z \in X$. Since this holds for all $\varepsilon > 0$, we conclude that $\rho(\mathcal{O}_z) = \rho$ for all z, as desired.

Finally, we come up with the compactness of $\mathcal{C}$.

Lemma 3. *The set $\mathcal{C}$ is compact for the weak topology along the fibers.*

Proof. We need to prove that if x_n converges to $x \in X$ and $\omega_n \in ctr(\mathcal{O}_{x_n})$ weakly converges to ω, then ω lies in $ctr(\mathcal{O}_x)$. Assume otherwise. Then $B(\omega, \rho + \varepsilon)$ does not cover $\mathcal{O}_x$ for a certain positive ε. In other words, there exists $v \in \mathcal{O}_x$ such that $\|v - \omega\| > \rho$. This vector v may be written in the form $v = I(g, g^{-1}(x))(0)$ for a certain $g \in \Gamma$. Now consider the vector $v_n := I(g, g^{-1}(x_n)) \in \mathcal{O}_{x_n}$. We have $v_n \to v$ (in the strong topology). Therefore, $\|v - \omega\| \leq \liminf_{n \to \infty} \|v_n - \omega_n\| \leq \rho$, which is a contradiction.

Fourth Step. The rest of the proof is as in Theorem A. Knowing that $\mathcal{C} \subset X \times \mathbb{B}$ is invariant and compact for the weak topology, we can find (using Zorn's lemma) a nonempty, minimal invariant compact set M. The Namioka-Asplund argument using dentability shows that each fiber M_x is a single point, hence M is the graph of a skew-invariant, weakly-continuous, almost-invariant section. Strong continuity of this section follows as before using the Baire cathegory argument.

3 Proof of Theorem C

Without any doubt, it should be possible to give a more explicit example than the one provided below. However, after recalling the well-known link between skew-flows and the classical Favard theory in the context of almost-periodic solutions to certain ODE (see [7]), the example we provide will arise naturally.

Let X be a compact metric space endowed with a (continuous) minimal flow T^t. Given a Banach space $\mathbb{B}$ and a continuous function $\xi \colon X \to \mathbb{B}$, let us fix a point $x_0 \in X$, and let us consider the function $\psi \colon \mathbb{R} \to \mathbb{B}$ defined by $\psi(t) := \xi(T^t x_0)$. It is then easy to see that ψ is *almost-periodic*, in the sense that the set $\{\psi_t \colon t \geq 0\}$ of translated functions $\psi_t(\cdot) := \psi(t + \cdot)$ is precompact with respect to the uniform topology.

Assume conversely that $\psi \colon \mathbb{R} \to \mathbb{B}$ is almost-periodic. Let X be the closure of $\{\psi_t \colon t \geq 0\}$. Then X admits a flow, namely, the one induced by $T^t(\psi) = \psi_t$. It is not difficult to show that this flow is minimal. Moreover, letting $\xi \colon X \to \mathbb{B}$ be the evaluation-at-the-origin function (that is, $\xi(\psi) := \psi(0)$), we may recover our original function ψ by the procedure above by noticing that $\psi(t) = \psi_t(0) = \xi(T^t \psi)$.

Let us now recall a result in this framework due to Amerio, that extends a classical theorem of Bohr: if $\mathbb{B}$ is uniformly convex and the integral values $\Psi(t) := \int_0^t \psi(s) ds$ are uniformly bounded, then the function Ψ is also almost-periodic. Actually, this theorem still holds for every reflexive $\mathbb{B}$, as it can be deduced from Theorem A. (Compare [4, Theorem D].) Indeed, associated to ψ, let us consider the compact metric space X and the corresponding flow T^t on it. By hypothesis, the orbits of the skew-action on $X \times \mathbb{B}$,

$$(x, s) \mapsto (T^t x, s + \Psi(t)),$$

are bounded. By Theorem A, there exists a continuous function $\varphi \colon X \to \mathbb{B}$ such that, for all $t \geq 0$ and all $x \in X$,

$$\varphi(T^t x) - \varphi(x) = \Psi(t).$$

By the discussion above, the function $t \to \varphi(T^t x)$ is almost-periodic, and this obviously implies that Ψ is almost-periodic as well.

It follows from the proof above that if $\mathbb{B}$ is a Banach space such that there exists an almost-periodic function $\psi : \mathbb{R} \to \mathbb{B}$ having uniformly-bounded integrals Ψ in such a way that Ψ is not almost-periodic, then there is a cocycle of translations of $\mathbb{B}$ over a minimal flow on a compact metric space for which there is a bounded orbit but no continuous skew-invariant section. To show Theorem C, it is thus enough to exhibit a space $\mathbb{B} = L^\infty(\Omega, \mu)$ with such a function. The first example of this was given by Johnson in [3]. Another –more general– example (taken from [7]) works as follows: let ψ be the function in $\mathbb{B} = c_0$ defined by

$$(\psi(t))_n = \frac{1}{2^n} \sin\left(\frac{t}{2^n}\right).$$

Then ψ satisfies the desired properties. The proof is closed by noticing that c_0 embeds into $L^\infty(\Omega, \mu)$ for every space (Ω, μ) such that μ is not concentrated on finitely many sets.

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