On invariant distributions of circle diffeomorphisms and a equidistribution theorem for smooth potentials

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Main references

This is essentially a joint work with Michele Triestino (ENS-Lyon):

[NT, 2012] On the invariant distributions of C^2 circle diffeomorphisms with irrational rotation number.

Remarks and related results by/with S. Crovisier and V. Kleptsyn will also be discussed.

Invariant distributions

• A probability measure is a point in the dual of the space of continuous functions. (Duality is realized by integration.)

• Given a manifold M, a k-distribution on M is a point in the dual space of $C^k(M)$.

• A k-distribution L is invariant under a $C^{k'}$ diffeomorphism $f : M \to M$, with $k' \geq k$, if for all $\varphi \in C^k(M)$:

$$L(\varphi) = L(\varphi \circ f).$$

An example

Assume that $x_0 \in M$ (1-dimensional) is such that

$$\sum_{n \in \mathbb{Z}} Df^n(x_0) < \infty.$$

Then

$$\varphi \mapsto \sum_{n \in \mathbb{Z}} D\varphi(f^n(x_0)) \cdot Df^n(x_0)$$

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This situation arises for hyperbolic-like dynamics. Hence, it is natural to first deal with elliptic-like dynamics...

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In dual form:

Theorem

For every C^k function $\varphi : \mathrm{S}^1 \to \mathrm{S}^1$ having zero mean with respect to the f-invariant measure, there exists a sequence of C^k functions $\psi_n : \mathrm{S}^1 \to \mathrm{S}^1$ such that

$$\psi_n \circ f - \psi_n \longrightarrow \varphi$$

in the C^k topology.

Main Results

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The theorem above is sharp in what concerns regularity of f. More precisely, there are C^1 "counterexamples" f that:

– preserve an invariant Cantor set (Denjoy,...); these can be made $C^{1+\alpha}$ for all $\alpha < 1$.

– are minimal (Kodama-Matsumoto); remains unknown in class $C^{1+\alpha}$.

An Equidistribution Theorem

Theorem

If $f \in C^{1+bv}$ and φ is of class C^1, then (for $\alpha \sim \frac{p_n}{q_n}$)

$$S_{q_n}(\varphi) - q_n \int_{\mathbb{S}^1} \varphi d\mu \longrightarrow 0.$$

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$$\left| S_{q_n}(\varphi) - q_n \int_{\mathrm{S}^1} \varphi d\mu \right| \leq \mathrm{var}(\varphi), \quad \varphi \in C^{bv} \text{ (Denjoy-Koksma)}$$

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Proof. Apply the Equidistribution Theorem to $\varphi := \log(Df) \in C^1$. Since

$$\int_{\mathbb{S}^1} \log(Df)\, d\mu = 0,$$

we get

$$\log(Df^{q_n}) = S_{q_n}(\log(Df)) \longrightarrow 0.$$

Since $f^{q_n} \to id$ in the C^0 topology (Denjoy), we must have $f^{q_n} \to id$ in the C^1 topology.

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• Let ψ_m be such that $\psi_m \circ f - \psi_m \longrightarrow \varphi$ in the C^1 topology. Then:

$$\begin{aligned} S_{q_n}(\varphi) &= S_{q_n}(\varphi - [\psi_m \circ f - \psi_m]) + S_{q_n}(\psi_m \circ f - \psi_m) \\ &= S_{q_n}(\varphi - [\psi_m \circ f - \psi_m]) + \psi_m \circ f^{q_n} - \psi_m. \end{aligned}$$

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Hence,

$$\begin{aligned} |S_{q_n}(\varphi)| &\leq |S_{q_n}(\varphi - [\psi_m \circ f - \psi_m])| + |\psi_m \circ f^{q_n} - \psi_m| \\ &\leq \mathrm{var}(\varphi - [\psi_m \circ f - \psi_m]) + |\psi_m \circ f^{q_n} - \psi_m| \\ &\leq \|\varphi - [\psi_m \circ f - \psi_m]\|_{C^1} + |\psi_m \circ f^{q_n} - \psi_m|. \end{aligned}$$

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Then (remarkable identity)

$$\psi_n \circ f - \psi_n = \varphi - \frac{S_n(\varphi)}{n}$$

Hence, if φ has zero mean, then ψ_n yields the desired approximation of φ by coboundaries in the C^0 topology.

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Proof of the Main Theorem II

But: for generic φ, the derivative of $\psi_n \circ f - \psi_n$ exploses.

However, for $\varphi(x) := f(x) - x$, this works, provided $f \in C^{1+bv}$. Indeed, the previous identity becomes

$$\psi_n \circ f - \psi_n = \varphi - \frac{f^n(x) - x}{n}$$

and the derivative of the term $\frac{f^{q_m}(x)-x}{q_m}$ converges to zero as $m \to \infty$, because of Denjoy's inequality.

Proof of the Main Theorem III

A very simple idea: if we wish to C^1-approximate φ by functions of the form $\psi \circ f - \psi$, then $D\varphi$ should be C^0-approximable by functions of the form $\xi \circ f \cdot Df - \xi$. (Namely, for $\xi = D\psi$.)

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• If the functions ξ had zero mean (Leb), this would solve the problem (just by integration).

• Otherwise, just substract the integral of ξ...

Proof of the Main Theorem IV

End of the proof: let $c_n := \int \xi_n$ for ξ_n such that

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Now recall: $\psi_{q_m} \circ f - \psi_{q_m} \longrightarrow f(x) - x$ in C^1, hence

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Thus,

$$(\xi_n - c_n + c_n D\psi_{q_m}) \circ f \cdot Df - (\xi_n - c_n + c_n D\psi_{q_m}) \longrightarrow D\varphi.$$

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Theorem

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This result is the dual statement of the fact that there are no f-conformal measures other than the Lebesgue measure (which is a result due to Douady,Yoccoz/Katok).

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Definition

A measure ν is f-conformal if for every continuous function Ψ:

$$\int \Psi \circ f \cdot Df \, d\nu = \int \Psi \, d\nu.$$

Group actions

For (finitely-generated) group actions by circle diffeomorphisms without invariant measure, there should be no invariant distribution. This is well established in some cases (real-analytic, free groups), and follows (among others) from recent works with Deroin and Kleptsyn, as well as the work of Filimonov and Kleptsyn:

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- Minimal actions are ergodic with respect to the ergodic measure. (Exceptional minimal sets have zero Lebesgue measure.)

- There are no conformal measure other than Leb...

Conjugacies

The twisted cohomological equation is closely related to the next

Question

Let f be a C^2 (even C^∞) circle diffeomorphism of irrational rotation number. Can f be C^2-conjugated to diffeomorphisms arbitrarily C^2-close to the corresponding rotation ?

In class C^1, this is known and easy: just use the Cohomological Identity for $\log(Df)$.

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In class C^1, this is known and easy: just use the Cohomological Identity for $\log(Df)$. Less trivial is that such a method works for nilpotent groups:

Theorem

Every C^1 action of a nilpotent group on S^1 (resp. [0,1]) is topologically conjugated to actions arbitrarily C^1-close to actions by rotations (resp. the trivial action).

MANY THANKS