ON GROUPS AS DYNAMICAL OJECTS

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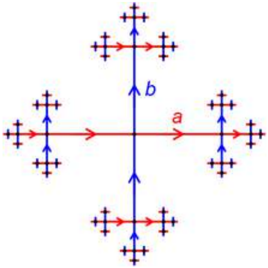
Edges: connect any two elements that differ by (right) multiplication by a generator.



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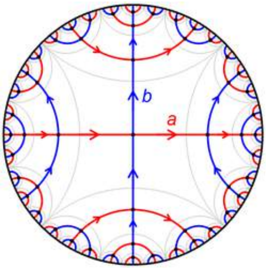

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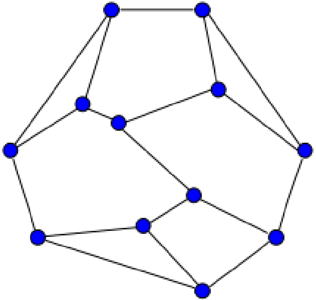

A General Principle

If a group (class of groups) acts nicely on a nice space, then the action should reveal some algebraic structure ($\rightsquigarrow$ nice theorem).

Groups acting on the Cantor set

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The group of piecewise dyadic homeomorphisms of the binary Cantor set is finitely presented and simple (Thompson's group V). It contains a copy of every finite group. Every automorphism is inner (Bleak, Lanoue, Yonah; N).

- The elements of V that respect the cyclic order form the subgroup T; this may be seen also as a group of piecewise-affine, orientation-preserving homeomorphisms of the circle.

R.Thompson's groups

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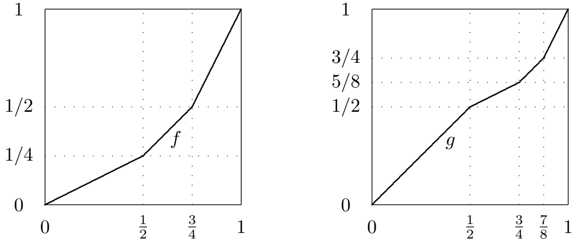
$$F = \langle f, g : \ [fg^{-1}, f^{-1}gf] = [fg^{-1}, f^{-2}gf^2] = id \rangle.$$



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Theorem (Matui, Grigorchuk-Medynets, Juschenko-Monod)

If f is minimal, then the commutator subgroup of its full topological group is finitely generated, amenable, and simple.

Recall that a group is *amenable* if all its actions by homeomorphisms of compact metric spaces preserve a probability measure. Such a group cannot contain free subgroups.

The Burnside groups

Question (Burnside)

Let Γ be a finitely-generated group in which every element has finite order (perhaps uniformly bounded). Is Γ necessarily finite ?

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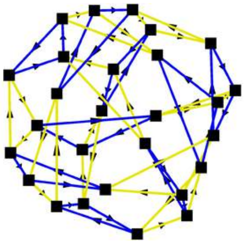

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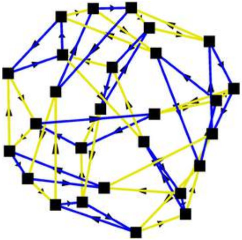

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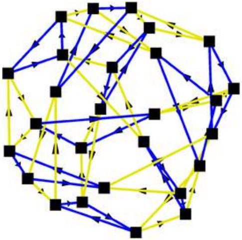

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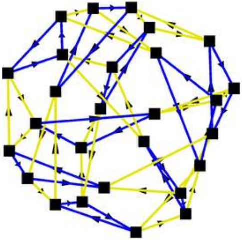

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- $B(7)$ should be infinite (hyperbolic; obvious for Gromov).
- $B(5)$ should still be infinite (Zelmanov).
- For odd $n > 666$, $B(n)$ is infinite (non-amenable; Adian-Novikov).

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• According to a theorem of Kerékjártó (based on the work of Brouwer), every finite-order homeomorphism of the sphere is conjugate to a rotation.

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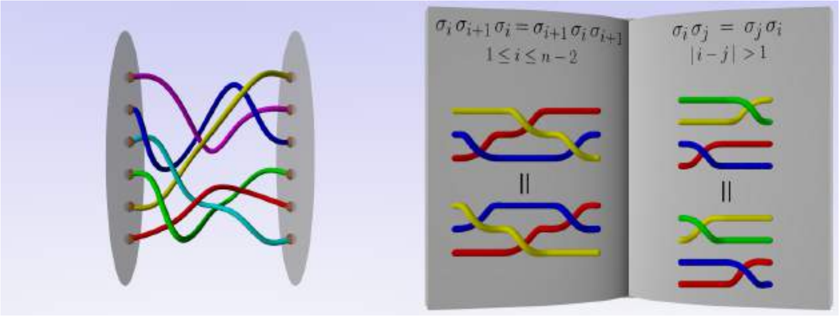
Question

Does there exist an algebraic characterization of groups that do act faithfully by homeomorphisms of a certain 2-manifold ?

Ordering braids



$B_n = \langle \sigma_1, \ldots, \sigma_{n-1} \colon \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1},\ \sigma_i\sigma_j = \sigma_j\sigma_i \text{ if } |i-j| > 1 \rangle$

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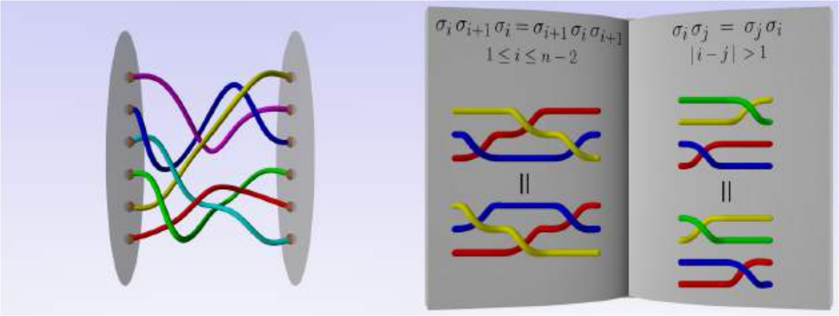

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The braid group B_n is left-orderable.

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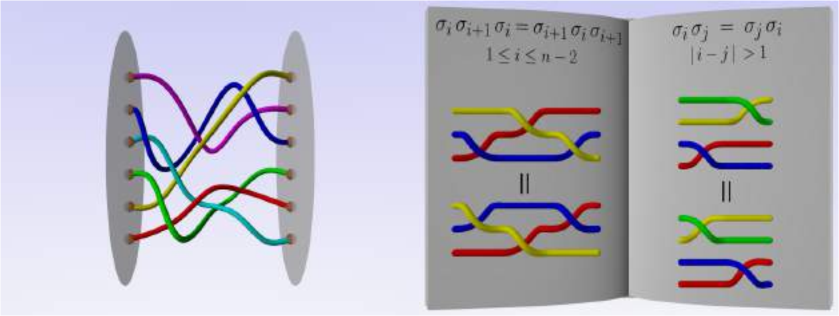

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An element is "positive" if it may be written as a word in the generators such that the generator σ_i with smallest index i that appears is raised only to positive exponents (ex: $\sigma_2\sigma_4^7\sigma_2^2\sigma_3^{-500}$).

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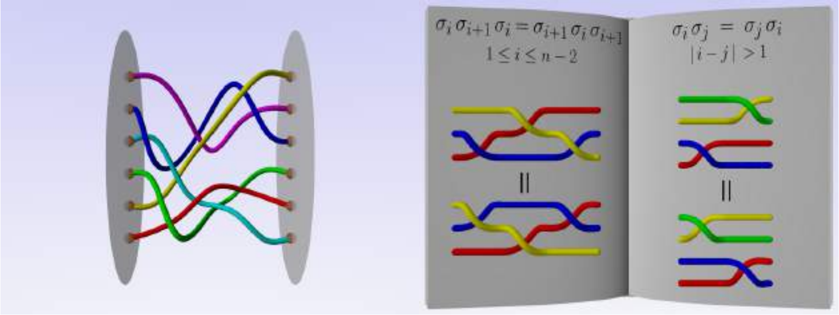

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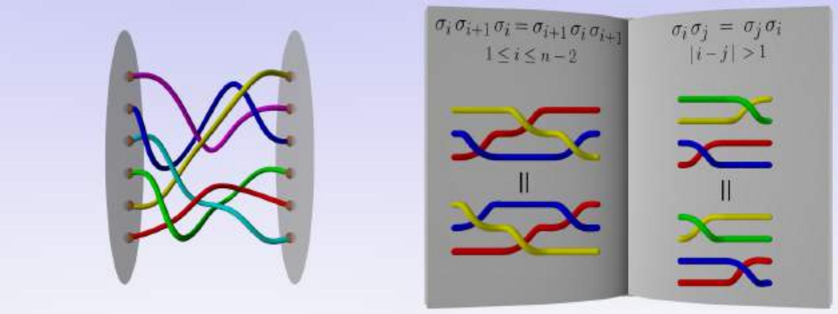

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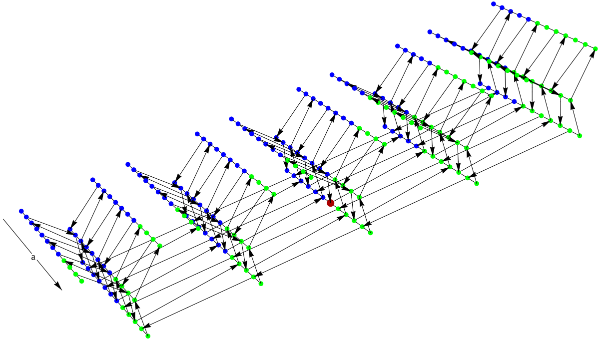

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• The Dehornoy order is not finitely determined (N; N-Wiest).

In dynamical terms

Actions coming from finitely determined orders are *structurally stable* in a very strong way:

Up to topological conjugacy, there is a unique action of B_3 on the real line without global fixed points and for which the generators $a_1 = \sigma_1 \sigma_2$ and $a_2 = \sigma_2^{-1}$ send the origin into positive real numbers.

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Question

Does there exist a finitely-generated, structurally-stable group of homeomorphisms of the sphere ?

The regularity enters into the game

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Every nontrivial finitely-generated subgroup of $\mathrm{Diff}^1_+([0,1])$ admits a nontrivial homomorphism into $\mathbb{R}$ (*i.e.* $\mathrm{Diff}^1_+([0,1])$ is *locally indicable*).

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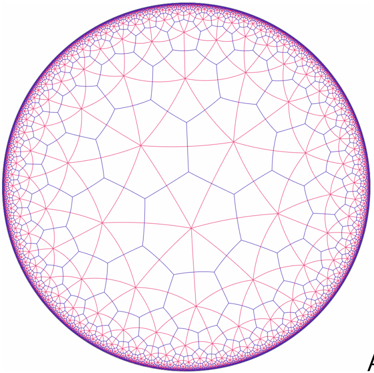
Local indicability does not hold for $\mathrm{Homeo}_+([0,1])$ (even for the group of Lipschitz homeomorphisms). An example (also due to Thurston) is the lifting to $\widetilde{\mathrm{PSL}}(2,\mathbb{R})$ of the (2,3,7)-triangle subgroup of $\mathrm{PSL}(2,\mathbb{R})$.

Local indicability



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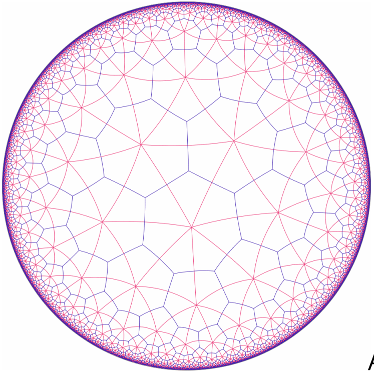

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Theorem (N)

Local indicability is not the only obstruction for embeddings into $\mathrm{Diff}_+^1([0,1])$.

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Notice that $\mathrm{Diff}^{1+\alpha}(M)$ is a group:

$$\|Df(x) - Df(y)\| \leq C|x-y|^{\alpha}$$

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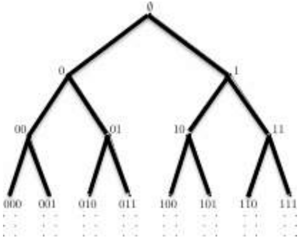
• Exponential growth means that the number of elements that may be written as products of no more than n generators grows exponentially as a function of n.

• The class of groups of polynomial growth coincides with that of almost-nilpotent ones (Bass, Guivarch; Gromov).

The Grigorchuk group: $G = \langle \bar{a}, \bar{b}, \bar{c}, \bar{d} \rangle$



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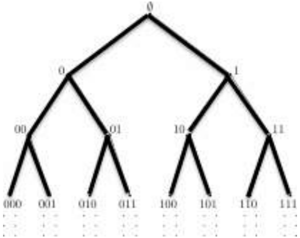

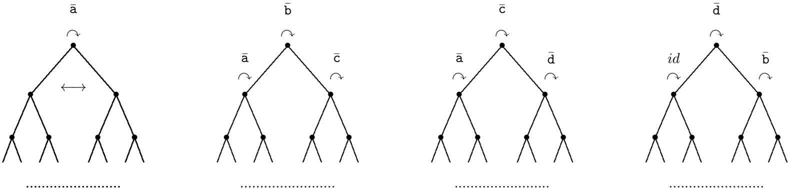

The generators of the Grigorchuk group

The Grigorchuk group: $G = \langle \bar{a}, \bar{b}, \bar{c}, \bar{d} \rangle$

Some formulae: for $l_i \in \{0, 1\}$,

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• Nonexistence of embeddings group of intermediate growth (as for example GM) in $\mathrm{Diff}_+^{1+\alpha}([0,1])$ is established by using (nontrivial extensions) of classical techniques in 1-dimensional dynamics.

Actions of "small" groups

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• Several announcements (including published papers with reviews) claiming for a proof or a disproof of amenability for F. There are even serious reasons to think that this problem may be undecidable.

Actions of "large" groups

Theorem (Witte Morris)

If Γ is a finite-index subgroup of $\mathrm{SL}(3,\mathbb{Z})$, then every action of Γ on S^1 (resp. $[0,1]$) has a finite image (resp. is trivial).

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In a certain (very precise) sense, most groups do satisfy Kazhdan’s property (T).

Actions of Kazhdan groups

Theorem (N)

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Corollary (N)

Thompson's group T does not satisfy Kazhdan's property (T).

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A map g is projective iff the following holds for all x, y:

$$\frac{Dg(x)Dg(y)}{(g(x)-g(y))^2} = \frac{1}{(x-y)^2}.$$

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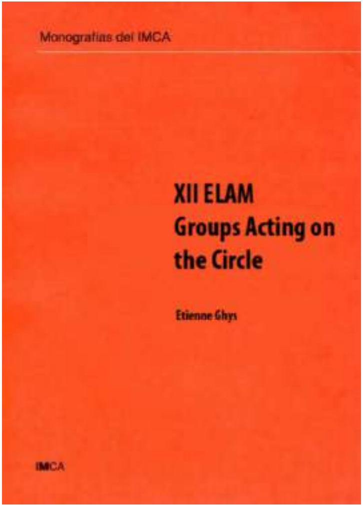
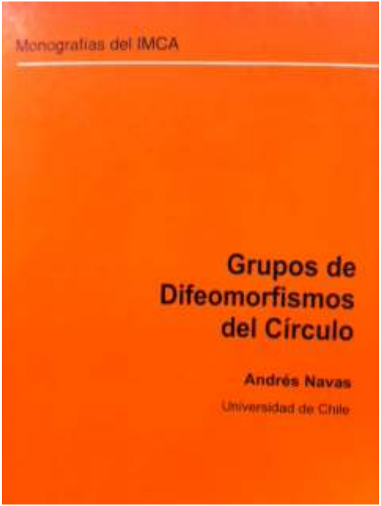
The Schwarzian derivative of g equals

$$S(g)(x) = \frac{1}{6} \lim_{y \to x} \left[\frac{Dg(x)Dg(y)}{(g(x)-g(y))^2} - \frac{1}{(x-y)^2} \right].$$

Some references





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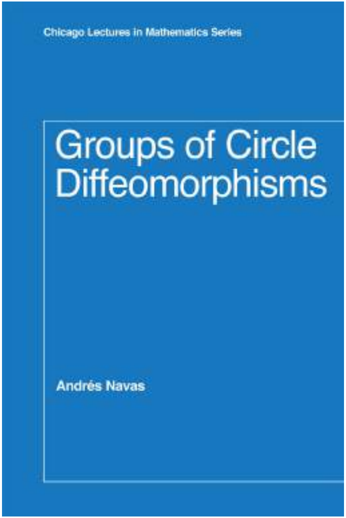

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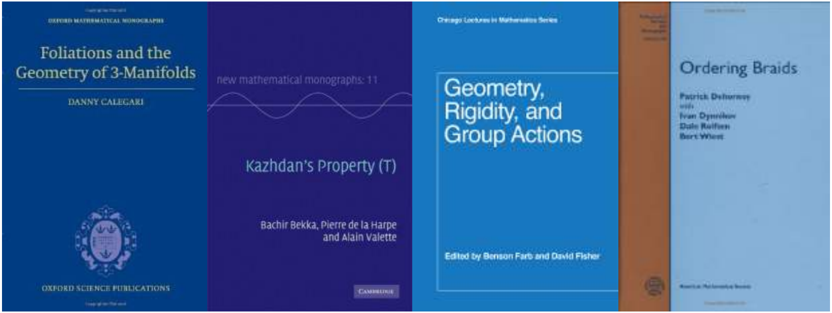

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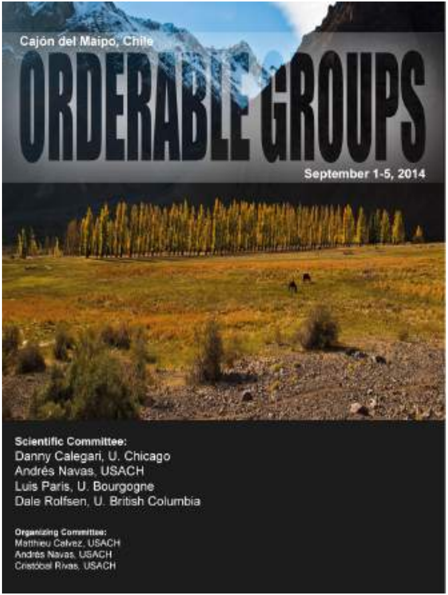
Cajón del Maipo, Chile
ORDERABLE GROUPS
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Andrés Navas, USACH
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Dale Rolfsen, U. British Columbia
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MUCHAS GRACIAS

MANY THANKS

MUITO OBRIGADO