Groups, Orders, and Dynamics
Fourth Lecture

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YGGT meeting
HAIFA, February 2013

References

- *Probabilistic and dynamical aspects of (left) orderable groups*.
  A. Navas; see
  http://imerl.fing.edu.uy/coloquio2/materiales/Curso_Navas.pdf

- *Orderable Groups*.
  R. Botto-Mura, A. Rhemtulla.

- *Right-ordered Groups*.
  V. Kopytov, V. Medvedev.

Braid groups

$$B_n = \langle \sigma_1, \ldots, \sigma_{n-1} : [\sigma_i, \sigma_j] = 1 \text{ if } |i-j| \geq 2, \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \rangle$$

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Theorem (Patrick Dehornoy)

B_n is left-orderable.

The Dehornoy ordering

Positive words

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is positive if the σ_j of minimal index j only appears with positive exponents

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Finite generation of the positive cone

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Proof [N-Wiest]. For $n = 3$: the conjugates $(\preceq_D)_{h_j}$ accummulate $\preceq_D$, where $h_j = \sigma_1^{-1}\sigma_2^j$.

Accummulating the Dehornoy order

- If w is a 1-positive word: $w = \sigma_2^{k_1}\sigma_1\sigma_2^{k_2}\sigma_1 \dots \sigma_2^{k_{\ell-1}}\sigma_1\sigma_2^{k_\ell}$, then

$$\begin{aligned}\sigma_1^{-1}\sigma_2^{j}w\sigma_2^{-j}\sigma_1 &= \underline{\sigma_1^{-1}\sigma_2^{j}(\sigma_2^{k_1}}\sigma_1\sigma_2^{k_2}\sigma_1 \dots \sigma_2^{k_{\ell-1}}\sigma_1\sigma_2^{k_\ell})\sigma_2^{-j}\sigma_1 \\ &= \underline{\sigma_2\sigma_1^{j+k_1}\sigma_2^{-1}}\sigma_2^{k_2}\sigma_1 \dots \sigma_2^{k_{\ell-1}}\sigma_1\sigma_2^{k_\ell}\sigma_2^{-j}\sigma_1.\end{aligned}$$

Thus $\sigma_1\sigma_2^{-j}w\sigma_2^{j}\sigma_1$ is 1-positive for sufficiently large j (namely for $j > -k_1$).

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Thus $\sigma_1\sigma_2^{-j} w \sigma_2^{j}\sigma_1$ is 1-positive for sufficiently large j (namely for $j > -k_1$).

• If w is 2-positive: $w = \sigma_2^{k}$ ($k > 0$), then...

• Finally, $(\prec_D)_j$ is different from $\prec_D$ for all positive integers j, since its smallest positive element is the conjugate of σ_2 by $\sigma_2^{-j}\sigma_1$, and this is different from σ_2 (the smallest element of $\preceq_D$).

Nielsen-Thurston orders (Short-Wiest)

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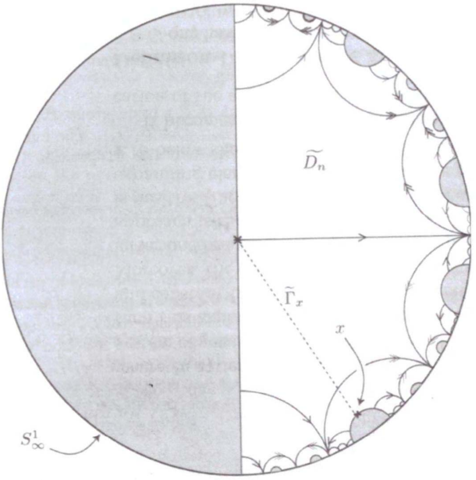
For many geodesics Γ, this gives a total (left-invariant) ordering (se so-called Nielsen-Thurston ordering associated to Γ).

Nielsen-Thurston orders



No isolated Nielsen-Thurston order

Theorem (N-Wiest)

No Nielsen-Thurston order is isolated in the space of braid orders.

Bi-orders on pure braid groups

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- If a left-order restricted to a finite-index subgroup is Conradian, then it is Conradian.
- For $n \geq 5$, the commutator subgroup B_n' is (finitely-generated and) perfect (hence B_n is not locally indicable).

Stability of the Conrad property

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