Groups, Orders, and Dynamics
Third Lecture

Andrés Navas, USACH

YGGT meeting
HAIFA, February 2013

References

- *Probabilistic and dynamical aspects of (left) orderable groups*.
  A. Navas; see
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- *Orderable Groups*.
  R. Botto-Mura, A. Rhemtulla.

- *Right-ordered Groups*.
  V. Kopytov, V. Medvedev.

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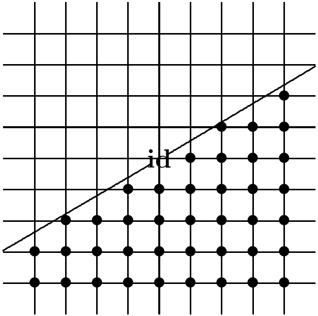

General results

Theorem (Linnell)

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Theorem (Rivas)

The space of C-orders is either finite or homeomorphic to the Cantor set.

The free group F_n, $n \geq 2$

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Theorem (Clay, Rivas)

There exist left-orders on F_n, $n \geq 2$, having dense orbits under the conjugacy action.

Detecting isolated left-orders

Lemma (Linnell)

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Proof. Assume that $g_1, \ldots, g_k$ generate $P_{\preceq}$. If $\preceq'$ is close to $\preceq$, then all these g_i are positive for $\preceq'$, hence

$$P_{\preceq} \subset P_{\preceq'}.$$

Similarly,

$$P_{\preceq}^{-1} \subset P_{\preceq'}^{-1}.$$

This forces equality.

Back to free groups

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Question

Does there exist a finitely generated semigroup S of F_2 that is disjoint from its inverse and such that every group element f can be written in the form

$$f = gh^{-1}$$

with g, h in $S \cup \{id\}$?

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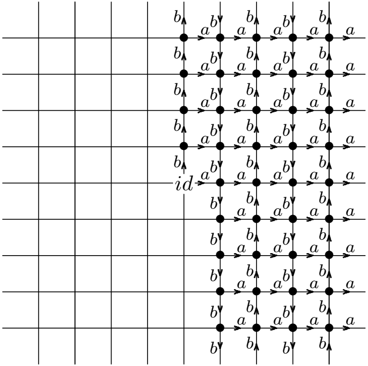

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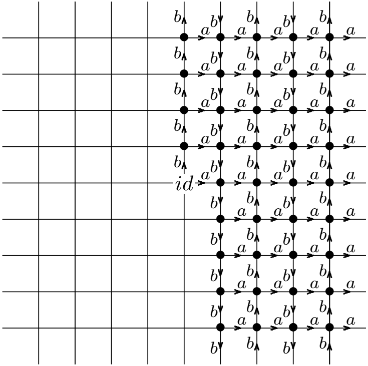

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Theorem (Dubrovina-Dubrovin)

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$$B_n = \langle c_1, \ldots, c_{n-1} \rangle^+ \sqcup \langle c_1^{-1}, \ldots, c_{n-1}^{-1} \rangle^+ \sqcup \{id\},$$

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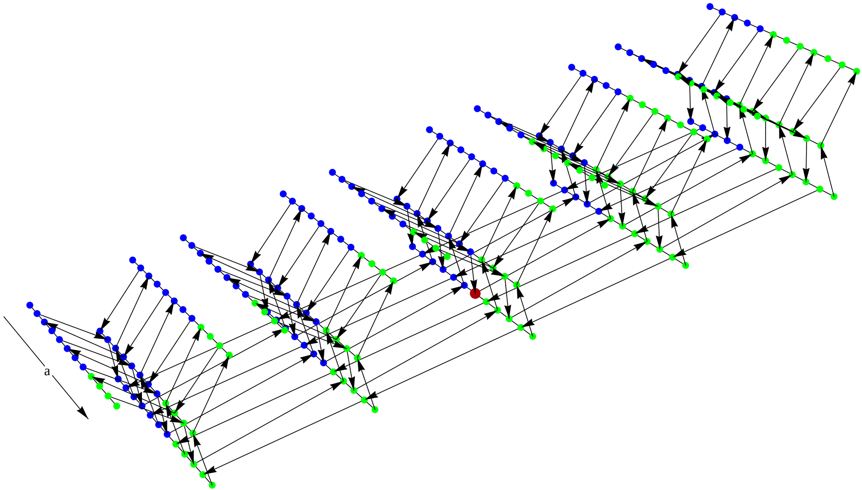

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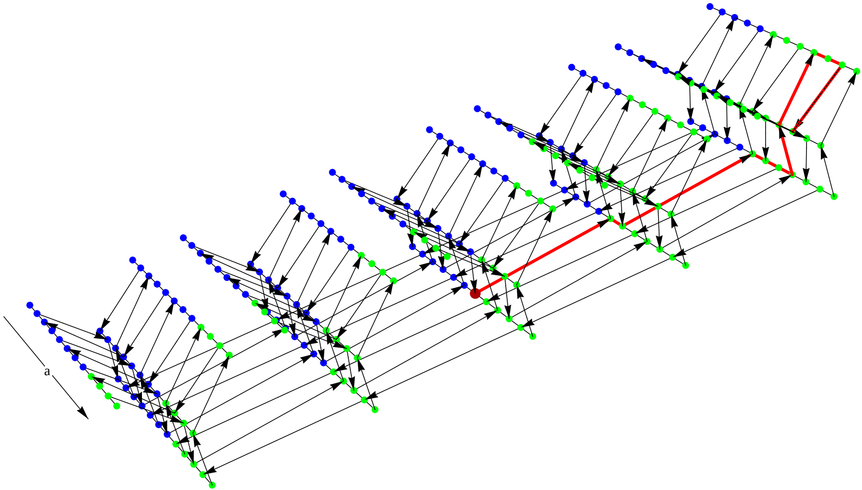

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$$\langle a_1, \ldots, a_k; a_i^{m_i+1} = a_{i+1}^{n_i+1}, i = 1, \ldots, k-1 \rangle$$

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Question

Can an amenable group with infinitely many orders have an isolated left-order?

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Question

Is the space of bi-orders of a free group homeomorphic to the Cantor set?

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- Fact: the subset $L = 1 + o(1) := \{1 + S : \; S \in o(1)\}$ is a subgroup (under multiplication) of $\mathbb{A}$. Moreover, if f, g are (free) generators of F_2, the map ϕ sending f (resp. g) to the element $1+X$ (resp. $1+Y$) in $\mathbb{A}$ extends in a unique way into an injective homomorphism $\phi : F_2 \to L$.

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- Fix a lexicographic type order relation on L which is bi-invariant under multiplication by elements in L (notice that this order will be not invariant under multiplication by elements in $\mathbb{A}$). Then induce a lexicographically type bi-invariant order.

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– Given an ordering $\preceq$ on a <u>countable</u> group Γ, let $p \colon \Gamma \to \mathbb{R}$ be an order-preserving map (with $p(id) = 0$). Define an action of Γ on $p(\Gamma)$ by letting $g(p(h)) = p(gh)$. This action may be extended continuously to the whole line... ("dynamical realization").

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Remark. These constructions are not in correspondence (*e.g.* there are actions that are not dynamical realizations). This may be used to create many new orders on a given group !

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- The space of orders of a free product is a Cantor set (Rivas).

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– Do this carefully in order to ensure that the centers of these windows lie in the same orbit (so that in particular there is no global fixed point for $\langle f, g \rangle$).

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– Since the conjugacy action corresponds to moving the reference point, under this action one can detect all orders $\preceq_k$ over large balls as (restrictions of) conjugates of $\preceq$.