Groups, Orders, and Dynamics
First Lecture

Andrés Navas, USACH

YGGT meeting
HAIFA, February 2013

References

- *Probabilistic and dynamical aspects of (left) orderable groups*. A. Navas; see http://imerl.fing.edu.uy/coloquio2/materiales/Curso_Navas.pdf

- *Orderable Groups*. R. Botto-Mura, A. Rhemtulla.

- *Right-ordered Groups*. V. Kopytov, V. Medvedev.

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$$f^n = id \implies 0 = f^n - 1 = (f-1)(f^{n-1} + f^{n-2} + \cdots + 1)$$

The Unique Product Property (UPP)

For all finite subsets A, B, there is an element

$$f \in AB := \{gh : g \in A, h \in B\}$$

that can be written uniquely as $f = gh$, with $g \in A$ and $h \in B$.

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No possible cancellation of the term gh above !

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- No torsion: Γ_P is a crystallographic group...
- The UPP fails for $A = B$ being equal to

$$\{(ba)^2, (ab)^2, a^2b, aba^{-1}, b, ab^{-1}a, b^{-1}, aba, ab^{-2}, b^2a^{-1}, a(ba)^2, bab, a, a^{-1}\}$$

Computations in Γ_P

Consider triplets $(u\,v\,w)$, where each u,v,w is either an integer or of the form $\hat{m}$.

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$$(002), (00\underline{2}), (\hat{2}1\hat{1}), (\hat{2}\underline{1}\underline{\hat{1}}), (\hat{0}1\hat{1}), (\hat{0}1\underline{\hat{1}}), (\hat{0}\underline{1}\hat{1}),$$
$$(\hat{0}\underline{1}\underline{\hat{1}}), (1\hat{2}\hat{0}), (\underline{1}\hat{2}\hat{0}), (1\hat{0}\underline{\hat{2}}), (\underline{1}\hat{0}\hat{2}), (1\hat{0}\hat{0}), (\underline{1}\hat{0}\hat{0})$$

(part of) the multiplication table

	(002)	$(00\underline{2})$	$(\hat{2}1\hat{1})$	$(\hat{2}\underline{1}\underline{\hat{1}})$	$(\hat{0}1\hat{1})$	$(\hat{0}1\underline{\hat{1}})$	$(\hat{0}\underline{1}\hat{1})$	$(\hat{0}\underline{1}\underline{\hat{1}})$	$(1\hat{2}\hat{0})$	$(\underline{1}\hat{2}\hat{0})$	$(1\hat{0}\underline{\hat{2}})$	$(\underline{1}\hat{0}\hat{2})$	$(1\hat{0}\hat{0})$	$(\underline{1}\hat{0}\hat{0})$
(002)	(004)	(000)	$(\hat{2}1\hat{3})$	$(\hat{2}\underline{1}\hat{1})$	$(\hat{0}1\hat{3})$	$(\hat{0}1\hat{1})$	$(\hat{0}\underline{1}\hat{3})$	$(\hat{0}\underline{1}\hat{1})$	$(1\hat{2}\hat{2})$	$(\underline{1}\hat{2}\hat{2})$	$(1\hat{0}\hat{0})$	$(\underline{1}\hat{0}\hat{4})$	$(1\hat{0}\hat{2})$	$(\underline{1}\hat{0}\hat{2})$
$(00\underline{2})$	(000)	$(00\underline{4})$	$(\hat{2}1\underline{\hat{1}})$	$(\hat{2}\underline{1}\underline{\hat{3}})$	$(\hat{0}1\underline{\hat{1}})$	$(\hat{0}1\underline{\hat{3}})$	$(\hat{0}\underline{1}\underline{\hat{1}})$	$(\hat{0}\underline{1}\underline{\hat{3}})$	$(1\hat{2}\underline{\hat{2}})$	$(\underline{1}\hat{2}\underline{\hat{2}})$	$(1\hat{0}\underline{\hat{4}})$	$(\underline{1}\hat{0}\hat{0})$	$(1\hat{0}\underline{\hat{2}})$	$(\underline{1}\hat{0}\underline{\hat{2}})$
$(\hat{2}1\hat{1})$	$(\hat{2}1\underline{\hat{1}})$	$(\hat{2}1\hat{3})$	(020)	(002)	(220)	(222)	(200)	(202)	$(\hat{1}\hat{3}1)$	$(\hat{3}\hat{3}1)$	$(\hat{1}\hat{1}3)$	$(\hat{3}\hat{1}\underline{1})$	$(\hat{1}\hat{1}1)$	$(\hat{3}\hat{1}1)$
$(\hat{2}\underline{1}\underline{\hat{1}})$	$(\hat{2}\underline{1}\underline{\hat{3}})$	$(\hat{2}\underline{1}\hat{1})$	$(00\underline{2})$	$(0\underline{2}0)$	$(20\underline{2})$	(200)	$(2\underline{2}\underline{2})$	$(2\underline{2}0)$	$(\hat{1}\hat{1}\underline{1})$	$(\hat{3}\hat{1}\underline{1})$	$(\hat{1}\underline{\hat{1}}1)$	$(\hat{3}\underline{\hat{1}}\underline{3})$	$(\hat{1}\underline{\hat{1}}\underline{1})$	$(\hat{3}\underline{\hat{1}}\underline{1})$
$(\hat{0}1\hat{1})$	$(\hat{0}1\underline{\hat{1}})$	$(\hat{0}1\hat{3})$	$(\underline{2}20)$	$(\underline{2}02)$	(020)	(022)	(000)	(002)	$(\underline{\hat{1}}\hat{3}1)$	$(\hat{1}\hat{3}1)$	$(\underline{\hat{1}}\hat{1}3)$	$(\hat{1}\hat{1}\underline{1})$	$(\underline{\hat{1}}\hat{1}1)$	$(\hat{1}\hat{1}1)$
$(\hat{0}1\underline{\hat{1}})$	$(\hat{0}1\underline{\hat{3}})$	$(\hat{0}1\hat{1})$	$(\underline{2}2\underline{2})$	$(\underline{2}00)$	$(02\underline{2})$	(020)	$(00\underline{2})$	(000)	$(\underline{\hat{1}}\hat{3}\underline{1})$	$(\hat{1}\hat{3}\underline{1})$	$(\underline{\hat{1}}\hat{1}1)$	$(\hat{1}\hat{1}\underline{3})$	$(\underline{\hat{1}}\hat{1}\underline{1})$	$(\hat{1}\hat{1}\underline{1})$
$(\hat{0}\underline{1}\hat{1})$	$(\hat{0}\underline{1}\underline{\hat{1}})$	$(\hat{0}\underline{1}\hat{3})$	$(\underline{2}00)$	$(\underline{2}\underline{2}2)$	(000)	(002)	$(0\underline{2}0)$	$(0\underline{2}2)$	$(\underline{\hat{1}}\hat{1}1)$	$(\hat{1}\hat{1}1)$	$(\underline{\hat{1}}\underline{\hat{1}}3)$	$(\hat{1}\underline{\hat{1}}\underline{1})$	$(\underline{\hat{1}}\underline{\hat{1}}1)$	$(\hat{1}\underline{\hat{1}}1)$
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$(1\hat{2}\hat{0})$	$(1\hat{2}\underline{\hat{2}})$	$(1\hat{2}\hat{2})$	$(\hat{3}\hat{1}\underline{1})$	$(\hat{3}\hat{3}1)$	$(\hat{1}\hat{1}\underline{1})$	$(\hat{1}\hat{1}1)$	$(\hat{1}\hat{3}\underline{1})$	$(\hat{1}\hat{3}1)$	(200)	(000)	(222)	$(02\underline{2})$	(220)	(020)
$(\underline{1}\hat{2}\hat{0})$	$(\underline{1}\hat{2}\underline{\hat{2}})$	$(\underline{1}\hat{2}\hat{2})$	$(\hat{1}\hat{1}\underline{1})$	$(\hat{1}\hat{3}1)$	$(\underline{\hat{1}}\hat{1}\underline{1})$	$(\underline{\hat{1}}\hat{1}1)$	$(\underline{\hat{1}}\hat{3}\underline{1})$	$(\underline{\hat{1}}\hat{3}1)$	(000)	$(\underline{2}00)$	(022)	$(\underline{2}2\underline{2})$	(020)	$(\underline{2}20)$
$(1\hat{0}\underline{\hat{2}})$	$(1\hat{0}\underline{\hat{4}})$	$(1\hat{0}\hat{0})$	$(\hat{3}\underline{\hat{1}}\underline{3})$	$(\hat{3}\hat{1}\underline{1})$	$(\hat{1}\underline{\hat{1}}\underline{3})$	$(\hat{1}\underline{\hat{1}}\underline{1})$	$(\hat{1}\hat{1}\underline{3})$	$(\hat{1}\hat{1}\underline{1})$	$(2\underline{2}\underline{2})$	$(0\underline{2}\underline{2})$	(200)	$(00\underline{4})$	$(20\underline{2})$	$(00\underline{2})$
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$(\underline{1}\hat{0}\hat{0})$	$(\underline{1}\hat{0}\underline{\hat{2}})$	$(\underline{1}\hat{0}\hat{2})$	$(\hat{1}\underline{\hat{1}}\underline{1})$	$(\hat{1}\hat{1}1)$	$(\underline{\hat{1}}\underline{\hat{1}}\underline{1})$	$(\underline{\hat{1}}\underline{\hat{1}}1)$	$(\underline{\hat{1}}\hat{1}\underline{1})$	$(\underline{\hat{1}}\hat{1}1)$	$(0\underline{2}0)$	$(\underline{2}\underline{2}0)$	(002)	$(\underline{2}0\underline{2})$	(000)	$(\underline{2}00)$

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$(1\hat{2}\hat{0})$	$(1\hat{2}\underline{\hat{2}})$	$(1\hat{2}\hat{2})$	$(\hat{3}\hat{1}\underline{1})$	$(\hat{3}\hat{3}1)$	$(\hat{1}\hat{1}\underline{1})$	$(\hat{1}\hat{1}1)$	$(\hat{1}\hat{3}\underline{1})$	$(\hat{1}\hat{3}1)$	(200)	(000)	(222)	$(02\underline{2})$	(220)	(020)
$(\underline{1}\hat{2}\hat{0})$	$(\underline{1}\hat{2}\underline{\hat{2}})$	$(\underline{1}\hat{2}\hat{2})$	$(\hat{1}\hat{1}\underline{1})$	$(\hat{1}\hat{3}1)$	$(\underline{\hat{1}}\hat{1}\underline{1})$	$(\underline{\hat{1}}\hat{1}1)$	$(\underline{\hat{1}}\hat{3}\underline{1})$	$(\underline{\hat{1}}\hat{3}1)$	(000)	$(\underline{2}00)$	(022)	$(\underline{2}2\underline{2})$	(020)	$(\underline{2}20)$
$(1\hat{0}\underline{\hat{2}})$	$(1\hat{0}\underline{\hat{4}})$	$(1\hat{0}\hat{0})$	$(\hat{3}\underline{\hat{1}}\underline{3})$	$(\hat{3}\hat{1}\underline{1})$	$(\hat{1}\underline{\hat{1}}\underline{3})$	$(\hat{1}\underline{\hat{1}}\underline{1})$	$(\hat{1}\hat{1}\underline{3})$	$(\hat{1}\hat{1}\underline{1})$	$(2\underline{2}\underline{2})$	$(0\underline{2}\underline{2})$	(200)	$(00\underline{4})$	$(20\underline{2})$	$(00\underline{2})$
$(\underline{1}\hat{0}\hat{2})$	$(\underline{1}\hat{0}\hat{0})$	$(\underline{1}\hat{0}\hat{4})$	$(\hat{1}\underline{\hat{1}}1)$	$(\hat{1}\hat{1}3)$	$(\underline{\hat{1}}\underline{\hat{1}}1)$	**$(\underline{\hat{1}}\underline{\hat{1}}3)$**	$(\underline{\hat{1}}\hat{1}1)$	$(\underline{\hat{1}}\hat{1}3)$	$(0\underline{2}2)$	$(\underline{2}\underline{2}2)$	(004)	$(\underline{2}00)$	(002)	$(\underline{2}02)$
$(1\hat{0}\hat{0})$	$(1\hat{0}\underline{\hat{2}})$	$(1\hat{0}\hat{2})$	$(\hat{3}\underline{\hat{1}}\underline{1})$	$(\hat{3}\hat{1}1)$	$(\hat{1}\underline{\hat{1}}\underline{1})$	$(\hat{1}\underline{\hat{1}}1)$	$(\hat{1}\hat{1}\underline{1})$	$(\hat{1}\hat{1}1)$	$(2\underline{2}0)$	$(0\underline{2}0)$	(202)	$(00\underline{2})$	(200)	(000)
$(\underline{1}\hat{0}\hat{0})$	$(\underline{1}\hat{0}\underline{\hat{2}})$	$(\underline{1}\hat{0}\hat{2})$	$(\hat{1}\underline{\hat{1}}\underline{1})$	$(\hat{1}\hat{1}1)$	$(\underline{\hat{1}}\underline{\hat{1}}\underline{1})$	$(\underline{\hat{1}}\underline{\hat{1}}1)$	$(\underline{\hat{1}}\hat{1}\underline{1})$	$(\underline{\hat{1}}\hat{1}1)$	$(0\underline{2}0)$	$(\underline{2}\underline{2}0)$	(002)	$(\underline{2}0\underline{2})$	(000)	$(\underline{2}00)$

(part of) the multiplication table

	(002)	$(00\underline{2})$	$(\hat{2}1\hat{1})$	$(\hat{2}\underline{1}\underline{\hat{1}})$	$(\hat{0}1\hat{1})$	$(\hat{0}1\underline{\hat{1}})$	$(\hat{0}\underline{1}\hat{1})$	$(\hat{0}\underline{1}\underline{\hat{1}})$	$(1\hat{2}\hat{0})$	$(\underline{1}\hat{2}\hat{0})$	$(1\hat{0}\underline{\hat{2}})$	$(\underline{1}\hat{0}\hat{2})$	$(1\hat{0}\hat{0})$	$(\underline{1}\hat{0}\hat{0})$
(002)	(004)	(000)	$(\hat{2}1\hat{3})$	$(\hat{2}\underline{1}\hat{1})$	$(\hat{0}1\hat{3})$	$(\hat{0}1\hat{1})$	$(\hat{0}\underline{1}\hat{3})$	$(\hat{0}\underline{1}\hat{1})$	$(1\hat{2}\hat{2})$	$(\underline{1}\hat{2}\hat{2})$	$(1\hat{0}\hat{0})$	$(\underline{1}\hat{0}\hat{4})$	$(1\hat{0}\hat{2})$	$(\underline{1}\hat{0}\hat{2})$
$(00\underline{2})$	(000)	$(00\underline{4})$	$(\hat{2}1\underline{\hat{1}})$	$(\hat{2}\underline{1}\underline{\hat{3}})$	$(\hat{0}1\underline{\hat{1}})$	$(\hat{0}1\underline{\hat{3}})$	$(\hat{0}\underline{1}\underline{\hat{1}})$	$(\hat{0}\underline{1}\underline{\hat{3}})$	$(1\hat{2}\underline{\hat{2}})$	$(\underline{1}\hat{2}\underline{\hat{2}})$	$(1\hat{0}\underline{\hat{4}})$	$(\underline{1}\hat{0}\hat{0})$	$(1\hat{0}\underline{\hat{2}})$	$(\underline{1}\hat{0}\underline{\hat{2}})$
$(\hat{2}1\hat{1})$	$(\hat{2}1\underline{\hat{1}})$	$(\hat{2}1\hat{3})$	(020)	(002)	(220)	(222)	(200)	(202)	$(\hat{1}\hat{3}1)$	$(\hat{3}\hat{3}1)$	$(\hat{1}\hat{1}3)$	$(\hat{3}\hat{1}\underline{1})$	$(\hat{1}\hat{1}1)$	$(\hat{3}\hat{1}1)$
$(\hat{2}\underline{1}\underline{\hat{1}})$	$(\hat{2}\underline{1}\underline{\hat{3}})$	$(\hat{2}\underline{1}\hat{1})$	$(00\underline{2})$	$(0\underline{2}0)$	$(20\underline{2})$	(200)	$(2\underline{2}\underline{2})$	$(2\underline{2}0)$	$(\hat{1}\hat{1}\underline{1})$	$(\hat{3}\hat{1}\underline{1})$	$(\hat{1}\underline{\hat{1}}1)$	$(\hat{3}\underline{\hat{1}}\underline{3})$	$(\hat{1}\underline{\hat{1}}\underline{1})$	$(\hat{3}\underline{\hat{1}}\underline{1})$
$(\hat{0}1\hat{1})$	$(\hat{0}1\underline{\hat{1}})$	$(\hat{0}1\hat{3})$	$(\underline{2}20)$	$(\underline{2}02)$	(020)	(022)	(000)	(002)	$(\underline{\hat{1}}\hat{3}1)$	$(\hat{1}\hat{3}1)$	$(\underline{\hat{1}}\hat{1}3)$	$(\hat{1}\hat{1}\underline{1})$	$(\underline{\hat{1}}\hat{1}1)$	$(\hat{1}\hat{1}1)$
$(\hat{0}1\underline{\hat{1}})$	$(\hat{0}1\underline{\hat{3}})$	$(\hat{0}1\hat{1})$	$(\underline{2}2\underline{2})$	$(\underline{2}00)$	$(02\underline{2})$	(020)	$(00\underline{2})$	(000)	$(\underline{\hat{1}}\hat{3}\underline{1})$	$(\hat{1}\hat{3}\underline{1})$	$(\underline{\hat{1}}\hat{1}1)$	$(\hat{1}\hat{1}\underline{3})$	$(\underline{\hat{1}}\hat{1}\underline{1})$	$(\hat{1}\hat{1}\underline{1})$
$(\hat{0}\underline{1}\hat{1})$	$(\hat{0}\underline{1}\underline{\hat{1}})$	$(\hat{0}\underline{1}\hat{3})$	$(\underline{2}00)$	$(\underline{2}\underline{2}2)$	(000)	(002)	$(0\underline{2}0)$	$(0\underline{2}2)$	$(\underline{\hat{1}}\hat{1}1)$	$(\hat{1}\hat{1}1)$	**$(\underline{\hat{1}}\underline{\hat{1}}3)$**	$(\hat{1}\underline{\hat{1}}\underline{1})$	$(\underline{\hat{1}}\underline{\hat{1}}1)$	$(\hat{1}\underline{\hat{1}}1)$
$(\hat{0}\underline{1}\underline{\hat{1}})$	$(\hat{0}\underline{1}\underline{\hat{3}})$	$(\hat{0}\underline{1}\hat{1})$	$(\underline{2}0\underline{2})$	$(\underline{2}\underline{2}0)$	$(00\underline{2})$	(000)	$(0\underline{2}\underline{2})$	$(0\underline{2}0)$	$(\underline{\hat{1}}\hat{1}\underline{1})$	$(\hat{1}\hat{1}\underline{1})$	$(\underline{\hat{1}}\underline{\hat{1}}1)$	$(\hat{1}\underline{\hat{1}}\underline{3})$	$(\underline{\hat{1}}\underline{\hat{1}}\underline{1})$	$(\hat{1}\underline{\hat{1}}\underline{1})$
$(1\hat{2}\hat{0})$	$(1\hat{2}\underline{\hat{2}})$	$(1\hat{2}\hat{2})$	$(\hat{3}\hat{1}\underline{1})$	$(\hat{3}\hat{3}1)$	$(\hat{1}\hat{1}\underline{1})$	$(\hat{1}\hat{1}1)$	$(\hat{1}\hat{3}\underline{1})$	$(\hat{1}\hat{3}1)$	(200)	(000)	(222)	$(02\underline{2})$	(220)	(020)
$(\underline{1}\hat{2}\hat{0})$	$(\underline{1}\hat{2}\underline{\hat{2}})$	$(\underline{1}\hat{2}\hat{2})$	$(\hat{1}\hat{1}\underline{1})$	$(\hat{1}\hat{3}1)$	$(\underline{\hat{1}}\hat{1}\underline{1})$	$(\underline{\hat{1}}\hat{1}1)$	$(\underline{\hat{1}}\hat{3}\underline{1})$	$(\underline{\hat{1}}\hat{3}1)$	(000)	$(\underline{2}00)$	(022)	$(\underline{2}2\underline{2})$	(020)	$(\underline{2}20)$
$(1\hat{0}\underline{\hat{2}})$	$(1\hat{0}\underline{\hat{4}})$	$(1\hat{0}\hat{0})$	$(\hat{3}\underline{\hat{1}}\underline{3})$	$(\hat{3}\hat{1}\underline{1})$	$(\hat{1}\underline{\hat{1}}\underline{3})$	$(\hat{1}\underline{\hat{1}}\underline{1})$	$(\hat{1}\hat{1}\underline{3})$	$(\hat{1}\hat{1}\underline{1})$	$(2\underline{2}\underline{2})$	$(0\underline{2}\underline{2})$	(200)	$(00\underline{4})$	$(20\underline{2})$	$(00\underline{2})$
$(\underline{1}\hat{0}\hat{2})$	$(\underline{1}\hat{0}\hat{0})$	$(\underline{1}\hat{0}\hat{4})$	$(\hat{1}\underline{\hat{1}}1)$	$(\hat{1}\hat{1}3)$	$(\underline{\hat{1}}\underline{\hat{1}}1)$	**$(\underline{\hat{1}}\underline{\hat{1}}3)$**	$(\underline{\hat{1}}\hat{1}1)$	$(\underline{\hat{1}}\hat{1}3)$	$(0\underline{2}2)$	$(\underline{2}\underline{2}2)$	(004)	$(\underline{2}00)$	(002)	$(\underline{2}02)$
$(1\hat{0}\hat{0})$	$(1\hat{0}\underline{\hat{2}})$	$(1\hat{0}\hat{2})$	$(\hat{3}\underline{\hat{1}}\underline{1})$	$(\hat{3}\hat{1}1)$	$(\hat{1}\underline{\hat{1}}\underline{1})$	$(\hat{1}\underline{\hat{1}}1)$	$(\hat{1}\hat{1}\underline{1})$	$(\hat{1}\hat{1}1)$	$(2\underline{2}0)$	$(0\underline{2}0)$	(202)	$(00\underline{2})$	(200)	(000)
$(\underline{1}\hat{0}\hat{0})$	$(\underline{1}\hat{0}\underline{\hat{2}})$	$(\underline{1}\hat{0}\hat{2})$	$(\hat{1}\underline{\hat{1}}\underline{1})$	$(\hat{1}\hat{1}1)$	$(\underline{\hat{1}}\underline{\hat{1}}\underline{1})$	$(\underline{\hat{1}}\underline{\hat{1}}1)$	$(\underline{\hat{1}}\hat{1}\underline{1})$	$(\underline{\hat{1}}\hat{1}1)$	$(0\underline{2}0)$	$(\underline{2}\underline{2}0)$	(002)	$(\underline{2}0\underline{2})$	(000)	$(\underline{2}00)$

(part of) the multiplication table

	(002)	$(00\underline{2})$	$(\hat{2}1\hat{1})$	$(\hat{2}\underline{1}\underline{\hat{1}})$	$(\hat{0}1\hat{1})$	$(\hat{0}1\underline{\hat{1}})$	$(\hat{0}\underline{1}\hat{1})$	$(\hat{0}\underline{1}\underline{\hat{1}})$	$(1\hat{2}\hat{0})$	$(\underline{1}\hat{2}\hat{0})$	$(1\hat{0}\underline{\hat{2}})$	$(\underline{1}\hat{0}\hat{2})$	$(1\hat{0}\hat{0})$	$(\underline{1}\hat{0}\hat{0})$
(002)	(004)	(000)	$(\hat{2}1\hat{3})$	$(\hat{2}\underline{1}\hat{1})$	$(\hat{0}1\hat{3})$	$(\hat{0}1\hat{1})$	$(\hat{0}\underline{1}\hat{3})$	$(\hat{0}\underline{1}\hat{1})$	$(1\hat{2}\hat{2})$	$(\underline{1}\hat{2}\hat{2})$	$(1\hat{0}\hat{0})$	$(\underline{1}\hat{0}\hat{4})$	$(1\hat{0}\hat{2})$	$(\underline{1}\hat{0}\hat{2})$
$(00\underline{2})$	(000)	$(00\underline{4})$	$(\hat{2}1\underline{\hat{1}})$	$(\hat{2}\underline{1}\underline{\hat{3}})$	$(\hat{0}1\underline{\hat{1}})$	$(\hat{0}1\underline{\hat{3}})$	$(\hat{0}\underline{1}\underline{\hat{1}})$	$(\hat{0}\underline{1}\underline{\hat{3}})$	$(1\hat{2}\underline{\hat{2}})$	$(\underline{1}\hat{2}\underline{\hat{2}})$	$(1\hat{0}\underline{\hat{4}})$	$(\underline{1}\hat{0}\hat{0})$	$(1\hat{0}\underline{\hat{2}})$	$(\underline{1}\hat{0}\underline{\hat{2}})$
$(\hat{2}1\hat{1})$	$(\hat{2}1\underline{\hat{1}})$	$(\hat{2}1\hat{3})$	(020)	(002)	(220)	(222)	(200)	(202)	$(\hat{1}\hat{3}1)$	$(\hat{3}\hat{3}1)$	$(\hat{1}\hat{1}3)$	$(\hat{3}\hat{1}\underline{1})$	$(\hat{1}\hat{1}1)$	$(\hat{3}\hat{1}1)$
$(\hat{2}\underline{1}\underline{\hat{1}})$	$(\hat{2}\underline{1}\underline{\hat{3}})$	$(\hat{2}\underline{1}\hat{1})$	$(00\underline{2})$	$(0\underline{2}0)$	$(20\underline{2})$	(200)	$(2\underline{2}\underline{2})$	$(2\underline{2}0)$	$(\hat{1}\hat{1}\underline{1})$	$(\hat{3}\hat{1}\underline{1})$	$(\hat{1}\underline{\hat{1}}1)$	$(\hat{3}\underline{\hat{1}}\underline{3})$	$(\hat{1}\underline{\hat{1}}\underline{1})$	$(\hat{3}\underline{\hat{1}}\underline{1})$
$(\hat{0}1\hat{1})$	$(\hat{0}1\underline{\hat{1}})$	$(\hat{0}1\hat{3})$	$(\underline{2}20)$	$(\underline{2}02)$	(020)	(022)	(000)	(002)	$(\underline{\hat{1}}\hat{3}1)$	$(\hat{1}\hat{3}1)$	$(\underline{\hat{1}}\hat{1}3)$	$(\hat{1}\hat{1}\underline{1})$	$(\underline{\hat{1}}\hat{1}1)$	$(\hat{1}\hat{1}1)$
$(\hat{0}1\underline{\hat{1}})$	$(\hat{0}1\underline{\hat{3}})$	$(\hat{0}1\hat{1})$	$(\underline{2}2\underline{2})$	$(\underline{2}00)$	$(02\underline{2})$	**(020)**	$(00\underline{2})$	(000)	$(\underline{\hat{1}}\hat{3}\underline{1})$	$(\hat{1}\hat{3}\underline{1})$	$(\underline{\hat{1}}\hat{1}1)$	$(\hat{1}\hat{1}\underline{3})$	$(\underline{\hat{1}}\hat{1}\underline{1})$	$(\hat{1}\hat{1}\underline{1})$
$(\hat{0}\underline{1}\hat{1})$	$(\hat{0}\underline{1}\underline{\hat{1}})$	$(\hat{0}\underline{1}\hat{3})$	$(\underline{2}00)$	$(\underline{2}\underline{2}2)$	(000)	(002)	$(0\underline{2}0)$	$(0\underline{2}2)$	$(\underline{\hat{1}}\hat{1}1)$	$(\hat{1}\hat{1}1)$	$(\underline{\hat{1}}\underline{\hat{1}}3)$	$(\hat{1}\underline{\hat{1}}\underline{1})$	$(\underline{\hat{1}}\underline{\hat{1}}1)$	$(\hat{1}\underline{\hat{1}}1)$
$(\hat{0}\underline{1}\underline{\hat{1}})$	$(\hat{0}\underline{1}\underline{\hat{3}})$	$(\hat{0}\underline{1}\hat{1})$	$(\underline{2}0\underline{2})$	$(\underline{2}\underline{2}0)$	$(00\underline{2})$	(000)	$(0\underline{2}\underline{2})$	$(0\underline{2}0)$	$(\underline{\hat{1}}\hat{1}\underline{1})$	$(\hat{1}\hat{1}\underline{1})$	$(\underline{\hat{1}}\underline{\hat{1}}1)$	$(\hat{1}\underline{\hat{1}}\underline{3})$	$(\underline{\hat{1}}\underline{\hat{1}}1)$	$(\hat{1}\underline{\hat{1}}\underline{1})$
$(1\hat{2}\hat{0})$	$(1\hat{2}\underline{\hat{2}})$	$(1\hat{2}\hat{2})$	$(\hat{3}\hat{1}\underline{1})$	$(\hat{3}\hat{3}1)$	$(\hat{1}\hat{1}\underline{1})$	$(\hat{1}\hat{1}1)$	$(\hat{1}\hat{3}\underline{1})$	$(\hat{1}\hat{3}1)$	(200)	(000)	(222)	$(02\underline{2})$	(220)	(020)
$(\underline{1}\hat{2}\hat{0})$	$(\underline{1}\hat{2}\underline{\hat{2}})$	$(\underline{1}\hat{2}\hat{2})$	$(\hat{1}\hat{1}\underline{1})$	$(\hat{1}\hat{3}1)$	$(\underline{\hat{1}}\hat{1}\underline{1})$	$(\underline{\hat{1}}\hat{1}1)$	$(\underline{\hat{1}}\hat{3}\underline{1})$	$(\underline{\hat{1}}\hat{3}1)$	(000)	$(\underline{2}00)$	(022)	$(\underline{2}2\underline{2})$	(020)	$(\underline{2}20)$
$(1\hat{0}\underline{\hat{2}})$	$(1\hat{0}\underline{\hat{4}})$	$(1\hat{0}\hat{0})$	$(\hat{3}\underline{\hat{1}}\underline{3})$	$(\hat{3}\hat{1}\underline{1})$	$(\hat{1}\underline{\hat{1}}\underline{3})$	$(\hat{1}\underline{\hat{1}}\underline{1})$	$(\hat{1}\hat{1}\underline{3})$	$(\hat{1}\hat{1}\underline{1})$	$(2\underline{2}\underline{2})$	$(0\underline{2}\underline{2})$	(200)	$(00\underline{4})$	$(20\underline{2})$	$(00\underline{2})$
$(\underline{1}\hat{0}\hat{2})$	$(\underline{1}\hat{0}\hat{0})$	$(\underline{1}\hat{0}\hat{4})$	$(\hat{1}\underline{\hat{1}}1)$	$(\hat{1}\hat{1}3)$	$(\underline{\hat{1}}\underline{\hat{1}}1)$	$(\underline{\hat{1}}\underline{\hat{1}}3)$	$(\underline{\hat{1}}\hat{1}1)$	$(\underline{\hat{1}}\hat{1}3)$	$(0\underline{2}2)$	$(\underline{2}\underline{2}2)$	(004)	$(\underline{2}00)$	(002)	$(\underline{2}02)$
$(1\hat{0}\hat{0})$	$(1\hat{0}\underline{\hat{2}})$	$(1\hat{0}\hat{2})$	$(\hat{3}\underline{\hat{1}}\underline{1})$	$(\hat{3}\hat{1}1)$	$(\hat{1}\underline{\hat{1}}\underline{1})$	$(\hat{1}\underline{\hat{1}}1)$	$(\hat{1}\hat{1}\underline{1})$	$(\hat{1}\hat{1}1)$	$(2\underline{2}0)$	$(0\underline{2}0)$	(202)	$(00\underline{2})$	(200)	(000)
$(\underline{1}\hat{0}\hat{0})$	$(\underline{1}\hat{0}\underline{\hat{2}})$	$(\underline{1}\hat{0}\hat{2})$	$(\hat{1}\underline{\hat{1}}\underline{1})$	$(\hat{1}\hat{1}1)$	$(\underline{\hat{1}}\underline{\hat{1}}\underline{1})$	$(\underline{\hat{1}}\underline{\hat{1}}1)$	$(\underline{\hat{1}}\hat{1}\underline{1})$	$(\underline{\hat{1}}\hat{1}1)$	$(0\underline{2}0)$	$(\underline{2}\underline{2}0)$	(002)	$(\underline{2}0\underline{2})$	(000)	$(\underline{2}00)$

(part of) the multiplication table

	(002)	$(00\underline{2})$	$(\hat{2}1\hat{1})$	$(\hat{2}\underline{1}\underline{\hat{1}})$	$(\hat{0}1\hat{1})$	$(\hat{0}1\underline{\hat{1}})$	$(\hat{0}\underline{1}\hat{1})$	$(\hat{0}\underline{1}\underline{\hat{1}})$	$(1\hat{2}\hat{0})$	$(\underline{1}\hat{2}\hat{0})$	$(1\hat{0}\underline{\hat{2}})$	$(\underline{1}\hat{0}\hat{2})$	$(1\hat{0}\hat{0})$	$(\underline{1}\hat{0}\hat{0})$
(002)	(004)	(000)	$(\hat{2}1\hat{3})$	$(\hat{2}\underline{1}\underline{\hat{1}})$	$(\hat{0}1\hat{3})$	$(\hat{0}1\hat{1})$	$(\hat{0}\underline{1}\hat{3})$	$(\hat{0}\underline{1}\hat{1})$	$(1\hat{2}\hat{2})$	$(\underline{1}\hat{2}\hat{2})$	$(1\hat{0}\hat{0})$	$(\underline{1}\hat{0}\hat{4})$	$(1\hat{0}\hat{2})$	$(\underline{1}\hat{0}\hat{2})$
$(00\underline{2})$	(000)	$(00\underline{4})$	$(\hat{2}1\underline{\hat{1}})$	$(\hat{2}\underline{1}\underline{\hat{3}})$	$(\hat{0}1\underline{\hat{1}})$	$(\hat{0}1\underline{\hat{3}})$	$(\hat{0}\underline{1}\underline{\hat{1}})$	$(\hat{0}\underline{1}\underline{\hat{3}})$	$(1\hat{2}\underline{\hat{2}})$	$(\underline{1}\hat{2}\underline{\hat{2}})$	$(1\hat{0}\underline{\hat{4}})$	$(\underline{1}\hat{0}\hat{0})$	$(1\hat{0}\underline{\hat{2}})$	$(\underline{1}\hat{0}\underline{\hat{2}})$
$(\hat{2}1\hat{1})$	$(\hat{2}1\underline{\hat{1}})$	$(\hat{2}1\hat{3})$	**(020)**	(002)	(220)	(222)	(200)	(202)	$(\hat{1}\hat{3}1)$	$(\hat{3}\hat{3}1)$	$(\hat{1}\hat{1}3)$	$(\hat{3}\hat{1}\underline{1})$	$(\hat{1}\hat{1}1)$	$(\hat{3}\hat{1}1)$
$(\hat{2}\underline{1}\underline{\hat{1}})$	$(\hat{2}\underline{1}\underline{\hat{3}})$	$(\hat{2}\underline{1}\hat{1})$	$(00\underline{2})$	$(0\underline{2}0)$	$(20\underline{2})$	(200)	$(2\underline{2}\underline{2})$	$(2\underline{2}0)$	$(\hat{1}\hat{1}\underline{1})$	$(\hat{3}\hat{1}\underline{1})$	$(\hat{1}\underline{\hat{1}}1)$	$(\hat{3}\underline{\hat{1}}\underline{3})$	$(\hat{1}\underline{\hat{1}}\underline{1})$	$(\hat{3}\underline{\hat{1}}\underline{1})$
$(\hat{0}1\hat{1})$	$(\hat{0}1\underline{\hat{1}})$	$(\hat{0}1\hat{3})$	$(\underline{2}20)$	$(\underline{2}02)$	**(020)**	(022)	(000)	(002)	$(\underline{\hat{1}}\hat{3}1)$	$(\hat{1}\hat{3}1)$	$(\underline{\hat{1}}\hat{1}3)$	$(\hat{1}\hat{1}\underline{1})$	$(\underline{\hat{1}}\hat{1}1)$	$(\hat{1}\hat{1}1)$
$(\hat{0}1\underline{\hat{1}})$	$(\hat{0}1\underline{\hat{3}})$	$(\hat{0}1\hat{1})$	$(\underline{2}2\underline{2})$	$(\underline{2}00)$	$(02\underline{2})$	**(020)**	$(00\underline{2})$	(000)	$(\underline{\hat{1}}\hat{3}\underline{1})$	$(\hat{1}\hat{3}\underline{1})$	$(\underline{\hat{1}}\hat{1}1)$	$(\hat{1}\hat{1}\underline{3})$	$(\underline{\hat{1}}\hat{1}\underline{1})$	$(\hat{1}\hat{1}\underline{1})$
$(\hat{0}\underline{1}\hat{1})$	$(\hat{0}\underline{1}\underline{\hat{1}})$	$(\hat{0}\underline{1}\hat{3})$	$(\underline{2}00)$	$(\underline{2}\underline{2}2)$	(000)	(002)	$(0\underline{2}0)$	$(0\underline{2}2)$	$(\underline{\hat{1}}\hat{1}1)$	$(\hat{1}\hat{1}1)$	$(\underline{\hat{1}}\underline{\hat{1}}3)$	$(\hat{1}\underline{\hat{1}}\underline{1})$	$(\underline{\hat{1}}\underline{\hat{1}}1)$	$(\hat{1}\underline{\hat{1}}1)$
$(\hat{0}\underline{1}\underline{\hat{1}})$	$(\hat{0}\underline{1}\underline{\hat{3}})$	$(\hat{0}\underline{1}\hat{1})$	$(\underline{2}0\underline{2})$	$(\underline{2}\underline{2}0)$	$(00\underline{2})$	(000)	$(0\underline{2}\underline{2})$	$(0\underline{2}0)$	$(\underline{\hat{1}}\hat{1}\underline{1})$	$(\hat{1}\hat{1}\underline{1})$	$(\underline{\hat{1}}\underline{\hat{1}}1)$	$(\hat{1}\underline{\hat{1}}\underline{3})$	$(\underline{\hat{1}}\underline{\hat{1}}\underline{1})$	$(\hat{1}\underline{\hat{1}}\underline{1})$
$(1\hat{2}\hat{0})$	$(1\hat{2}\underline{\hat{2}})$	$(1\hat{2}\hat{2})$	$(\hat{3}\hat{1}\underline{1})$	$(\hat{3}\hat{3}1)$	$(\hat{1}\hat{1}\underline{1})$	$(\hat{1}\hat{1}1)$	$(\hat{1}\hat{3}\underline{1})$	$(\hat{1}\hat{3}1)$	(200)	(000)	(222)	$(02\underline{2})$	(220)	**(020)**
$(\underline{1}\hat{2}\hat{0})$	$(\underline{1}\hat{2}\underline{\hat{2}})$	$(\underline{1}\hat{2}\hat{2})$	$(\hat{1}\hat{1}\underline{1})$	$(\hat{1}\hat{3}1)$	$(\underline{\hat{1}}\hat{1}\underline{1})$	$(\underline{\hat{1}}\hat{1}1)$	$(\underline{\hat{1}}\hat{3}\underline{1})$	$(\underline{\hat{1}}\hat{3}1)$	(000)	$(\underline{2}00)$	(022)	$(\underline{2}2\underline{2})$	**(020)**	$(\underline{2}20)$
$(1\hat{0}\underline{\hat{2}})$	$(1\hat{0}\underline{\hat{4}})$	$(1\hat{0}\hat{0})$	$(\hat{3}\underline{\hat{1}}\underline{3})$	$(\hat{3}\hat{1}\underline{1})$	$(\hat{1}\underline{\hat{1}}\underline{3})$	$(\hat{1}\underline{\hat{1}}\underline{1})$	$(\hat{1}\hat{1}\underline{3})$	$(\hat{1}\hat{1}\underline{1})$	$(2\underline{2}\underline{2})$	$(0\underline{2}\underline{2})$	(200)	$(00\underline{4})$	$(20\underline{2})$	$(00\underline{2})$
$(\underline{1}\hat{0}\hat{2})$	$(\underline{1}\hat{0}\hat{0})$	$(\underline{1}\hat{0}\hat{4})$	$(\hat{1}\underline{\hat{1}}1)$	$(\hat{1}\hat{1}3)$	$(\underline{\hat{1}}\underline{\hat{1}}1)$	$(\underline{\hat{1}}\underline{\hat{1}}3)$	$(\underline{\hat{1}}\hat{1}1)$	$(\underline{\hat{1}}\hat{1}3)$	$(0\underline{2}2)$	$(\underline{2}\underline{2}2)$	(004)	$(\underline{2}00)$	(002)	$(\underline{2}02)$
$(1\hat{0}\hat{0})$	$(1\hat{0}\underline{\hat{2}})$	$(1\hat{0}\hat{2})$	$(\hat{3}\underline{\hat{1}}\underline{1})$	$(\hat{3}\hat{1}1)$	$(\hat{1}\underline{\hat{1}}\underline{1})$	$(\hat{1}\underline{\hat{1}}1)$	$(\hat{1}\hat{1}\underline{1})$	$(\hat{1}\hat{1}1)$	$(2\underline{2}0)$	$(0\underline{2}0)$	(202)	$(00\underline{2})$	(200)	(000)
$(\underline{1}\hat{0}\hat{0})$	$(\underline{1}\hat{0}\underline{\hat{2}})$	$(\underline{1}\hat{0}\hat{2})$	$(\hat{1}\underline{\hat{1}}\underline{1})$	$(\hat{1}\hat{1}1)$	$(\underline{\hat{1}}\underline{\hat{1}}\underline{1})$	$(\underline{\hat{1}}\underline{\hat{1}}1)$	$(\underline{\hat{1}}\hat{1}\underline{1})$	$(\underline{\hat{1}}\hat{1}1)$	$(0\underline{2}0)$	$(\underline{2}\underline{2}0)$	(002)	$(\underline{2}0\underline{2})$	(000)	$(\underline{2}00)$

(part of) the multiplication table

	(002)	$(00\underline{2})$	$(\hat{2}1\hat{1})$	$(\hat{2}\underline{1}\underline{\hat{1}})$	$(\hat{0}1\hat{1})$	$(\hat{0}1\underline{\hat{1}})$	$(\hat{0}\underline{1}\hat{1})$	$(\hat{0}\underline{1}\underline{\hat{1}})$	$(1\hat{2}\hat{0})$	$(\underline{1}\hat{2}\hat{0})$	$(1\hat{0}\underline{\hat{2}})$	$(\underline{1}\hat{0}\hat{2})$	$(1\hat{0}\hat{0})$	$(\underline{1}\hat{0}\hat{0})$
(002)	(004)	(000)	$(\hat{2}1\hat{3})$	$(\hat{2}\underline{1}\hat{1})$	$(\hat{0}1\hat{3})$	$(\hat{0}1\hat{1})$	$(\hat{0}\underline{1}\hat{3})$	$(\hat{0}\underline{1}\hat{1})$	$(1\hat{2}\hat{2})$	$(\underline{1}\hat{2}\hat{2})$	$(1\hat{0}\hat{0})$	$(\underline{1}\hat{0}\hat{4})$	$(1\hat{0}\hat{2})$	$(\underline{1}\hat{0}\hat{2})$
$(00\underline{2})$	(000)	$(00\underline{4})$	$(\hat{2}1\underline{\hat{1}})$	$(\hat{2}\underline{1}\underline{\hat{3}})$	$(\hat{0}1\underline{\hat{1}})$	$(\hat{0}1\underline{\hat{3}})$	$(\hat{0}\underline{1}\underline{\hat{1}})$	$(\hat{0}\underline{1}\underline{\hat{3}})$	$(1\hat{2}\underline{\hat{2}})$	$(\underline{1}\hat{2}\underline{\hat{2}})$	$(1\hat{0}\underline{\hat{4}})$	$(\underline{1}\hat{0}\hat{0})$	$(1\hat{0}\underline{\hat{2}})$	$(\underline{1}\hat{0}\underline{\hat{2}})$
$(\hat{2}1\hat{1})$	$(\hat{2}1\underline{\hat{1}})$	$(\hat{2}1\hat{3})$	**(020)**	(002)	(220)	(222)	(200)	(202)	$(\hat{1}\hat{3}1)$	$(\hat{3}\hat{3}1)$	$(\hat{1}\hat{1}3)$	$(\hat{3}\hat{1}\underline{1})$	$(\hat{1}\hat{1}1)$	$(\hat{3}\hat{1}1)$
$(\hat{2}\underline{1}\underline{\hat{1}})$	$(\hat{2}\underline{1}\underline{\hat{3}})$	$(\hat{2}\underline{1}\hat{1})$	$(00\underline{2})$	$(0\underline{2}0)$	$(20\underline{2})$	(200)	$(2\underline{2}\underline{2})$	$(2\underline{2}0)$	$(\hat{1}\hat{1}\underline{1})$	$(\hat{3}\hat{1}\underline{1})$	$(\hat{1}\underline{\hat{1}}1)$	$(\hat{3}\underline{\hat{1}}\underline{3})$	$(\hat{1}\underline{\hat{1}}\underline{1})$	$(\hat{3}\underline{\hat{1}}\underline{1})$
$(\hat{0}1\hat{1})$	$(\hat{0}1\underline{\hat{1}})$	$(\hat{0}1\hat{3})$	$(\underline{2}20)$	$(\underline{2}02)$	**(020)**	(022)	(000)	(002)	$(\underline{\hat{1}}\hat{3}1)$	$(\hat{1}\hat{3}1)$	$(\underline{\hat{1}}\hat{1}3)$	$(\hat{1}\hat{1}\underline{1})$	$(\underline{\hat{1}}\hat{1}1)$	$(\hat{1}\hat{1}1)$
$(\hat{0}1\underline{\hat{1}})$	$(\hat{0}1\underline{\hat{3}})$	$(\hat{0}1\hat{1})$	$(\underline{2}2\underline{2})$	$(\underline{2}00)$	$(02\underline{2})$	**(020)**	$(00\underline{2})$	(000)	$(\underline{\hat{1}}\hat{3}\underline{1})$	$(\hat{1}\hat{3}\underline{1})$	$(\underline{\hat{1}}\hat{1}1)$	$(\hat{1}\hat{1}\underline{3})$	$(\underline{\hat{1}}\hat{1}\underline{1})$	$(\hat{1}\hat{1}\underline{1})$
$(\hat{0}\underline{1}\hat{1})$	$(\hat{0}\underline{1}\underline{\hat{1}})$	$(\hat{0}\underline{1}\hat{3})$	$(\underline{2}00)$	$(\underline{2}\underline{2}2)$	(000)	(002)	$(0\underline{2}0)$	$(0\underline{2}2)$	$(\underline{\hat{1}}\hat{1}1)$	$(\hat{1}\hat{1}1)$	$(\underline{\hat{1}}\underline{\hat{1}}3)$	$(\hat{1}\underline{\hat{1}}\underline{1})$	$(\underline{\hat{1}}\underline{\hat{1}}1)$	$(\hat{1}\underline{\hat{1}}1)$
$(\hat{0}\underline{1}\underline{\hat{1}})$	$(\hat{0}\underline{1}\underline{\hat{3}})$	$(\hat{0}\underline{1}\hat{1})$	$(\underline{2}0\underline{2})$	$(\underline{2}\underline{2}0)$	$(00\underline{2})$	(000)	$(0\underline{2}\underline{2})$	$(0\underline{2}0)$	$(\underline{\hat{1}}\hat{1}\underline{1})$	$(\hat{1}\hat{1}\underline{1})$	$(\underline{\hat{1}}\underline{\hat{1}}1)$	$(\hat{1}\underline{\hat{1}}\underline{3})$	$(\underline{\hat{1}}\underline{\hat{1}}\underline{1})$	$(\hat{1}\underline{\hat{1}}\underline{1})$
$(1\hat{2}\hat{0})$	$(1\hat{2}\underline{\hat{2}})$	$(1\hat{2}\hat{2})$	$(\hat{3}\hat{1}\underline{1})$	$(\hat{3}\hat{3}1)$	$(\hat{1}\hat{1}\underline{1})$	$(\hat{1}\hat{1}1)$	$(\hat{1}\hat{3}\underline{1})$	$(\hat{1}\hat{3}1)$	(200)	(000)	(222)	$(02\underline{2})$	(220)	**(020)**
$(\underline{1}\hat{2}\hat{0})$	$(\underline{1}\hat{2}\underline{\hat{2}})$	$(\underline{1}\hat{2}\hat{2})$	$(\hat{1}\hat{1}\underline{1})$	$(\hat{1}\hat{3}1)$	$(\underline{\hat{1}}\hat{1}\underline{1})$	$(\underline{\hat{1}}\hat{1}1)$	$(\underline{\hat{1}}\hat{3}\underline{1})$	$(\underline{\hat{1}}\hat{3}1)$	(000)	$(\underline{2}00)$	(022)	$(\underline{2}2\underline{2})$	**(020)**	$(\underline{2}20)$
$(1\hat{0}\underline{\hat{2}})$	$(1\hat{0}\underline{\hat{4}})$	$(1\hat{0}\hat{0})$	$(\hat{3}\underline{\hat{1}}\underline{3})$	$(\hat{3}\hat{1}\underline{1})$	$(\hat{1}\underline{\hat{1}}\underline{3})$	$(\hat{1}\underline{\hat{1}}\underline{1})$	$(\hat{1}\hat{1}\underline{3})$	$(\hat{1}\hat{1}\underline{1})$	$(2\underline{2}\underline{2})$	$(0\underline{2}\underline{2})$	(200)	$(00\underline{4})$	$(20\underline{2})$	$(00\underline{2})$
$(\underline{1}\hat{0}\hat{2})$	$(\underline{1}\hat{0}\hat{0})$	$(\underline{1}\hat{0}\hat{4})$	$(\hat{1}\underline{\hat{1}}1)$	$(\hat{1}\hat{1}3)$	$(\underline{\hat{1}}\underline{\hat{1}}1)$	$(\underline{\hat{1}}\underline{\hat{1}}3)$	$(\underline{\hat{1}}\hat{1}1)$	$(\underline{\hat{1}}\hat{1}3)$	$(0\underline{2}2)$	$(\underline{2}\underline{2}2)$	(004)	$(\underline{2}00)$	(002)	$(\underline{2}02)$
$(1\hat{0}\hat{0})$	$(1\hat{0}\underline{\hat{2}})$	$(1\hat{0}\hat{2})$	$(\hat{3}\underline{\hat{1}}\underline{1})$	$(\hat{3}\hat{1}1)$	$(\hat{1}\underline{\hat{1}}\underline{1})$	$(\hat{1}\underline{\hat{1}}1)$	$(\hat{1}\hat{1}\underline{1})$	$(\hat{1}\hat{1}1)$	$(2\underline{2}0)$	$(0\underline{2}0)$	(202)	$(00\underline{2})$	(200)	(000)
$(\underline{1}\hat{0}\hat{0})$	$(\underline{1}\hat{0}\underline{\hat{2}})$	$(\underline{1}\hat{0}\hat{2})$	$(\hat{1}\underline{\hat{1}}\underline{1})$	$(\hat{1}\hat{1}1)$	$(\underline{\hat{1}}\underline{\hat{1}}\underline{1})$	$(\underline{\hat{1}}\underline{\hat{1}}1)$	$(\underline{\hat{1}}\hat{1}\underline{1})$	$(\underline{\hat{1}}\hat{1}1)$	$(0\underline{2}0)$	$(\underline{2}\underline{2}0)$	(002)	$(\underline{2}0\underline{2})$	(000)	$(\underline{2}00)$

there is no “direction” to “escape”…

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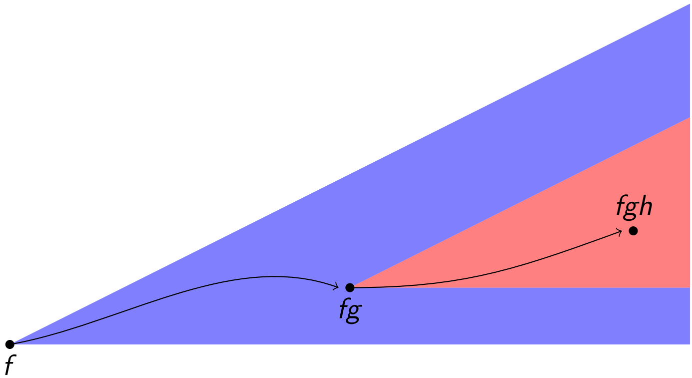
Good situation:



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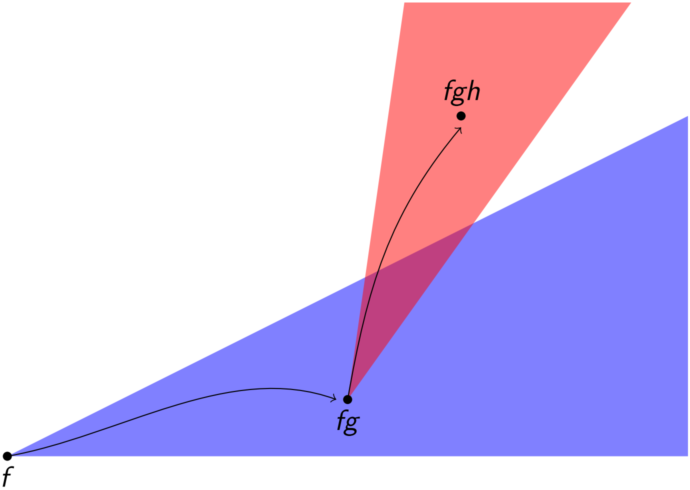

$g \in P_f$, $h \in P_{fg}$, but $gh \notin P_f$

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Question: does the converse holds ?

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Field of cones $\longrightarrow$ (local) order:

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Main property: for all $f \in \Gamma$ and $g \neq id$, either $fg \succ f$ or $fg^{-1} \succ f$

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- Many examples of left-orderable and non left-orderable groups (D.Calegari-Dunfield, Dabkowski-J.Przytycki-Togha, Boyer-Rolfsen-Wiest).
- (Agol) These groups contain a finite-index left-orderable subgroup.

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Question

Does there exist a sequence of closed hyperbolic 3-manifolds with non left-orderable π_1 and whose injectivity radii go to infinite ?

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For a large-enough k, the following elements are in Γ:

$$g_1 = \begin{pmatrix} 1 & k & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad g_2 = \begin{pmatrix} 1 & 0 & k \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad g_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & k \\ 0 & 0 & 1 \end{pmatrix},$$

$$g_4 = \begin{pmatrix} 1 & 0 & 0 \\ k & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad g_5 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ k & 0 & 1 \end{pmatrix}, \quad g_6 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & k & 1 \end{pmatrix}.$$

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$$a = g_{i-1}^{\pm 1}, b = g_{i+1}^{\pm 1}, c = g_i^{\pm k} \text{ or } a = g_{i+1}^{\pm 1}, b = g_{i-1}^{\pm 1}, c = g_i^{\pm k},$$

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Remark

Morris-Witte proves non left-orderability for many other higher-rank lattices (see also Lifschitz-Witte), but the question is still open in general.