Sharp regularity for certain nilpotent group actions on the interval

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Abstract

According to the classical Plante-Thurston Theorem, all nilpotent groups of C^2-diffeomorphisms of the closed interval are Abelian. Using techniques coming from the works of Denjoy and Pixton, Farb and Franks constructed a faithful action by C^1-diffeomorphisms of $[0,1]$ for every finitely-generated, torsion-free, non-Abelian nilpotent group. In this work, we give a version of this construction that is sharp in what concerns the Hölder regularity of the derivatives. Half of the proof relies on results on random paths on Heisenberg-like groups that are interesting by themselves.

Contents

1 Introduction

Much work has been done on centralizers of C^2-diffeomorphisms of the interval [3, 9, 16, 17]. This theory has been extensively used for studying the algebraic constraints of finitely-generated subgroups of $\mathrm{Diff}_+^2([0,1])$. For example, using the famous Kopell lemma [9], Plante and Thurston showed that nilpotent groups of C^2-diffeomorphisms of $[0,1[$ (resp. $]0,1[$) are Abelian (resp. metabelian); see [14].

As is well known, most of the rigidity properties are lost when we consider centralizers of C^1-diffeomorphisms. In relation to Plante-Thurston's theorem, this fact is corroborated by the work of Farb and Franks. In [4], they construct an embedding ϕ_{FF} of N_d into $\mathrm{Diff}_+^1([0,1])$, where N_d denotes the (nilpotent) group of $(d+1) \times (d+1)$ lower-triangular matrices whose entries are integers which equal 1 on the diagonal (see §2.1 for the details). Since every finitely-generated, torsion-free, nilpotent group embeds into N_d for some $d \geq 1$ (see [15]), one concludes that all these groups can be realized as groups of C^1-diffeomorphisms of the (closed) interval (compare [7]).

Major progress has been recently made in the understanding of the loss of rigidity for centralizers in intermediate differentiability classes, that is, between C^1 and C^2 (see [2, 8, 10]). Recall that, for $0 < \alpha < 1$, a diffeomorphism f is said to be of class $C^{1+\alpha}$ if its derivative is α-Hölder continuous. In other words, there exists a constant M such that for all x, y,

$$|f'(x) - f'(y)| \leq M|x - y|^{\alpha}. \tag{1}$$

We denote the group of $C^{1+\alpha}$-diffeomorphisms of $[0,1]$ by $\mathrm{Diff}_+^{1+\alpha}([0,1])$. In the first part of this work we show the following result. (Notice that for $d = 2$, the theorem below still holds and follows from Plante-Thurston's theorem.)

Theorem A. *If $d \geq 3$ and $\alpha > \frac{2}{d(d-1)}$, then the action ϕ_{FF} is not topologically conjugated to an action by $C^{1+\alpha}$-diffeomorphisms of $[0,1]$.*

This theorem should be considered as a partial complement to [10, Theorem B] which establishes that, for all $0 < \alpha < 1$, every subgroup Γ of $\mathrm{Diff}_+^{1+\alpha}([0,1])$ without free subsemigroups is virtually nilpotent. (Although the last result still holds for the open interval $]0,1[$, Theorem A above fails to be true in this context, but it extends –with the very same proof– to the case of the half-closed interval). For the proof of our theorem, the main technical achievement consists in controlling the distortion of suitable compositions of elements in any regularity larger than the critical one. To do this, we develop a nontrivial modification of the probabilistic techniques of [2, 8]. Recall that [2, Theorem B] deals with Abelian group actions that are dynamically very similar to ϕ_{FF}, and a direct application of it shows that ϕ_{FF} is not conjugated to an action by $C^{1+\alpha}$-diffeomorphisms of $[0,1[$ for any $\alpha > \frac{1}{d-1}$. The fact that our critical regularity here is actually smaller relies on that compared to the Abelian actions of [2], the action ϕ_{FF} has a more complicated combinatorial dynamics in that the growth of certain orbits is polynomial with degree precisely equal to $\frac{d(d-1)}{2}$. We should point out that similar combinatorial dynamics appear for the actions of the natural quotients of the Grigorchuk-Machi's group [5] for which the method of this article should also provide the best possible regularity (compare [10, Theorem A]). Moreover, it is worth mentioning that the very same arguments show that Theorem A above still applies to topological *semiconjugacies.*

Although not directly related, all the results described above should be compared to (and have potential relations with) Borichev's extension [1] to intermediate regularity of Polterovich-Sodin's theorem [13] concerning distortion of interval diffeomorphisms.

The second part of this work is devoted to a converse of Theorem A. The next theorem improves the main result of [4].

Theorem B. *For each $d \geq 2$ and $\alpha < \frac{2}{d(d-1)}$, the action ϕ_{FF} is topologically conjugated to an action by $C^{1+\alpha}$-diffeomorphisms of $[0,1]$.*

The proof of this theorem is based on classical constructions of Denjoy and Pixton (a clever exposition of these techniques appears in [18]; see also [11]). Nevertheless, putting these methods in practice in the present case is far from being straightforward. The computations are quite involved, and in this part of the work some of them are just sketched.

As in [2, 8], here we were unable to settle the $C^{1+\frac{2}{d(d-1)}}$ case, though we conjecture that the rigidity (*i.e.* Theorem A) still holds for this critical regularity.

Theorems A and B suggest that, attached to each finitely-generated, torsion-free nilpotent group Γ, there should be a positive exponent $\alpha(\Gamma) \leq 1$ that is critical for embedding Γ into $\mathrm{Diff}_+^{1+\alpha}([0,1])$. However, it is still unclear to us what should be the value of $\alpha(\Gamma)$. Indeed, Theorem B only deals with very particular actions, and many nilpotent groups admit actions that are fairly different from these. In order to corroborate this point, in the last part of this work we improve another construction of [4], thus proving the next

Theorem C. *For every $\alpha < 1$ and each $d \geq 1$, the group $\mathrm{Diff}_+^{1+\alpha}([0,1])$ contains a metabelian subgroup of nilpotence degree d.*

Acknowledgments. We are indebted to Yves de Cornulier and Romain Tessera for a crucial remark concerning the growth of certain orbits of the action of N_d, to Bertrand Deroin and Victor Kleptsyn for many valuable remarks to the second part of this article, and to Takashi Tsuboi for a clever hint for the constructions therein. The second-named author was funded by the Fondecyt Iniciación Project 11121316, while third-named author was partially funded by the Fondecyt Project 1100536. All authors were also funded by the Center of Dynamical Systems and Related Fields DySyRF (Anillo Project 1103, CONICYT).

2 Non-existence of smoothing for $\alpha > \frac{2}{d(d-1)}$

2.1 A reminder on Farb-Franks' action ϕ_{FF}

We deal with the group N_d of $(d+1)\times(d+1)$ lower-triangular matrices with integer entries, all of which are equal to 1 on the diagonal. Notice that N_2 corresponds to the Heisenberg group. In general, N_d is a nilpotent group of nilpotence degree d. A nice system of generators of N_d is $\{f_{2,1},\ldots,f_{d+1,d}\}$, where $f_{i,j}$ is the elementary matrix whose unique nonzero entry outside the diagonal is the (i,j)-entry (with $i>j$).

The group N_d acts linearly on $\mathbb{Z}^{d+1}$ with the affine hyperplane $1\times\mathbb{Z}^d$ remaining invariant. The thus-induced action on $\mathbb{Z}^d$ allows producing an action on the interval as follows. Let $\{I_{i_1,\ldots,i_d}\colon (i_1,\ldots,i_d)\in\mathbb{Z}^d\}$ be a family of intervals such that the sum $\sum_{i_1,\ldots,i_d}|I_{i_1,\ldots,i_d}|$ is finite, say equal to 1 after normalization. We join these intervals lexicographically on the closed interval $[0,1]$, and we identify $f_{j+1,j}$ to a certain homeomorphism sending each interval $I=I_{i_1,\ldots,i_d}$ into the interval J given by:

- $J := I_{i_1+1,i_2,\ldots,i_{d-1},i_d}$, for $j=1$,
- $J := I_{i_1,\ldots,i_{j-1},i_j+i_{j-1},i_{j+1},\ldots,i_d}$, for $2\le j\le d$.

It is not hard to perform this procedure in a equivariant way (for instance, using piecewise-affine maps), thus preserving the group structure and hence obtaining an embedding of N_d into $\mathrm{Homeo}_+([0,1])$. (Much harder is to obtain an embedding into the group of diffeomorphisms.) For this action, an interval of the form $I_{i_1,\ldots,i_d}$ is sent by $f\in N_d$ into $I_{j_1,\ldots,j_d}$, where $f\big((1,i_1,\ldots,i_d)^T\big)=(1,j_1,\ldots,j_d)^T$. Notice that up to topological conjugacy, all the actions obtained by this procedure are equivalent. This includes Farb-Franks' action ϕ_{FF}, which is obtained via this method for a well-chosen family of diffeomorphisms between the intervals of type I,J above so that the resulting $f_{i,j}$'s are C^1-diffeomorphisms.

2.2 From control of distortion to the proof of Theorem A

Let us begin by stating a general principle from [2] in the form of the following

Proposition 2.1. *Let $f_1,\ldots,f_k$ be C^1-diffeomorphisms of the interval $[0,1]$ that commute with a C^1-diffeomorphism g. Assume that g fixes a subinterval I of $[0,1]$ and its restriction to I is nontrivial. Assume moreover that for a certain $0<\alpha<1$ and a sequence of indexes $i_j\in\{1,\ldots,k\}$, the sum*

$$L_\alpha := \sum_{j\geq 0}\big|f_{i_j}\cdots f_{i_1}(I)\big|^\alpha \tag{2}$$

is finite. Then $f_1,\ldots,f_k$ cannot be all of class $C^{1+\alpha}$.

Proof. Let $x_0\in I$ be such that $g(x_0)\neq x_0$. Denote by $[a,b]$ the shortest interval containing x_0 that is fixed by g. For each $j\geq 1, n\geq 1$ and $z\in[a,b]$, the equality $g^n=(f_{i_j}\cdots f_{i_1})^{-1}\circ g^n\circ(f_{i_j}\cdots f_{i_1})$ yields

$$\log Dg^n(z)=\log D(f_{i_j}\cdots f_{i_1})(z)+\log Dg^n(f_{i_j}\cdots f_{i_1}(z))-\log D(f_{i_j}\cdots f_{i_1})(g^n(z)).$$

Fix a constant M such that (1) holds for all $f\in\{f_1,\ldots,f_k\}$ and all x,y in $[0,1]$. Letting $z_n:=g^n(z)$ and noticing that z_n belongs to $[a,b]\subset I$ for all $n\geq 1$, we obtain

$$\begin{aligned}
|\log Dg^n(z)| &\leq |\log Dg^n(f_{i_j}\cdots f_{i_1}(z))| + \sum_{m=1}^{j}\big|\log Df_{i_m}(f_{i_{m-1}}\cdots f_{i_1}(z))-\log Df_{i_m}(f_{i_{m-1}}\cdots f_{i_1}(z_n))\big|\\
&\leq |\log Dg^n(f_{i_j}\cdots f_{i_1}(z))| + \sum_{m=1}^{j} M\big|f_{i_{m-1}}\cdots f_{i_1}(z)-f_{i_{m-1}}\cdots f_{i_1}(z_n)\big|^\alpha\\
&\leq |\log Dg^n(f_{i_j}\cdots f_{i_1}(z))| + M\sum_{m=1}^{j}|f_{i_{m-1}}\cdots f_{i_1}(I)|^\alpha\\
&\leq |\log Dg^n(f_{i_j}\cdots f_{i_1}(z))| + ML_\alpha.
\end{aligned}$$

The length of the intervals $f_{i_j}\cdots f_{i_1}(I)$ must necessarily converge to zero as j goes to infinite. Moreover, since g^n fixes I and commutes with $f_1,\ldots,f_k$, on each of these intervals there must be a point at which its derivative equals 1. By the continuity of Dg^n, we conclude that the value of $Dg^n(f_{i_j}\cdots f_{i_1}(z))$ converges to 1 as j goes to infinite. Hence we obtain $Dg^n(z)\leq e^{ML_\alpha}$ for all $n\geq 1$ and all $z\in[a,b]$, which certainly contradicts the fact that the restriction of g to $[a,b]$ is nontrivial. □

Let us come back to the action $\phi_{_{FF}}$. Notice that the group N_{d-1} can be naturally viewed as the subgroup of N_d formed by the elements whose last row coincide with that of the identity. We will denote by N_{d-1}^* the copy of N_{d-1} inside N_d.

Notice that the element $g := f_{d+1,1}\in N_d$ is centralized by N_{d-1}^*. Under the action $\phi_{_{FF}}$, this element fixes the interval

$$I^* := \bigcup_{j\in\mathbb{Z}} I_{0,\ldots,0,j}. \tag{3}$$

Moreover, this interval is sent into a disjoint one by any nontrivial element of N_{d-1}^*. We are hence in a situation close to that of the preceding proposition. Thus, we need to ensure the existence of a systems of generators for N_{d-1}^* and a sequence of compositions for which the associated sum (2) is finite provided that $\alpha > \frac{2}{d(d-1)}$. To do this, we will use the system of generators $\{f_{2,1},f_{3,1},\ldots,f_{d,1}\}\cup\{f_{2,1},f_{3,2},\ldots,f_{d,d-1}\}$.

It is worth mentioning that this is an analogous problem to that of the $\mathbb{Z}^d$-actions on the interval considered in [2, Théorème B]. However, the $\mathbb{Z}^d$-case is easier in that the generators of the dynamics commute, hence the orbit graph of the associated interval I^* has a simpler structure. Indeed, the space of infinite paths of this graph can be endowed of a natural probability measure such that for appropriately large values of α (namely, for $\alpha > 1/d$), almost every path has a finite L_α-series. In order to establish this, besides the restriction on the exponent α, the main property of the underlying process is that the arrival probabilities up to time k are equidistributed along the sphere of radius k (centered at the origin) for every $k\geq 1$. Although in [2] this is modeled via a Polya urn like model that charges only the positive powers of the generators, an alternative model sharing this property that charges both positive and negative powers of the generators is the Markov process depicted in Figure 1 below for the case $d=2$ (the reader will easily check the equidistribution property along spheres as well as the general rule for the transition probabilities; the generalization for higher values of d is not very hard).

Remark 2.2. It seems to be an interesting and nontrivial problem to determine general conditions for an infinite graph ensuring the existence of a Markov process satisfying the equidistribution property above.

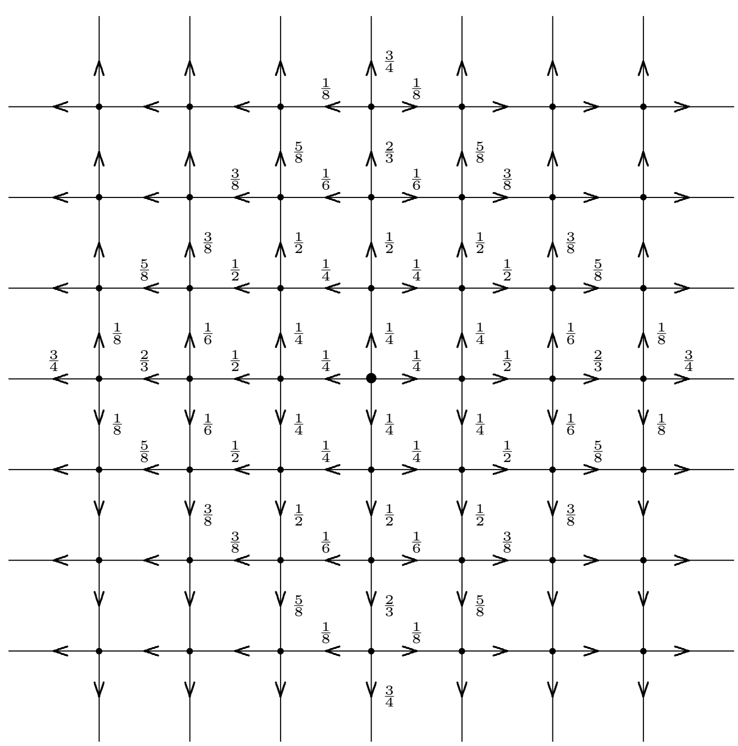

Figure 1

Let us now consider the orbit of the interval I^* defined by (3) for the action of N^*_{d-1}. For simplicity, let us first deal with the case $d = 3$. With respect to the generators $f_{2,1}, f_{3,1}, f_{3,2}$ of N^*_2, the orbit graph is depicted in Figure 2 below. Here, $f_{2,1}$ corresponds to the generator whose action on the the graph is moving to the right, whereas the action of both $f_{3,1}$ and $f_{3,2}$ consists in moving up, the former by one unit and the latter with an amplitude that depends on the position. (Notice that the directions of the arrows mean that we are only considering positive powers of the generators.)

Now, the difficulty comes from that, as the reader may easily check, it is impossible to put probability distributions on this graph yielding the equidistribution property along the spheres centered at the origin. (This is already impossible for the sphere of radius 4.) To overcome this problem, we will use the counting argument of (the first part of) [8], which actually corresponds to a deterministic counterpart of the random walk argument above. Indeed, this argument is more robust in that it does not need any equidistribution property, though it requires a certain extra argument to obtain our desired infinite path as a concatenation of finite paths that behave nicely for certain finite processes.

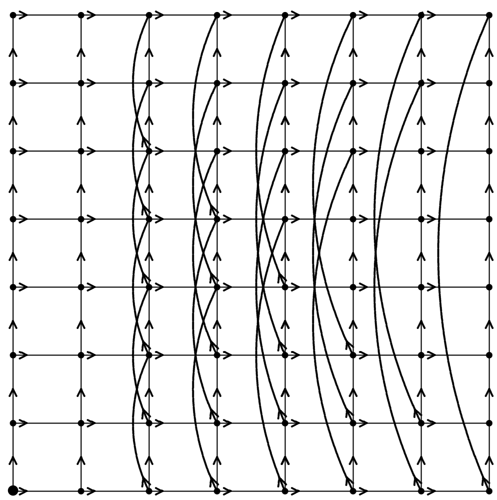

Figure 2

To close this section, let us finally explain why the exponent $\frac{2}{d(d-1)}$ is critical for the action ϕ_{FF}. For simplicity, let us first consider the case $d = 3$. Looking at the graph of Figure 2 above, one easily computes the growth of the balls. This appears to be cubic, in the sense that the number of points at distance $\leq n$ from the origin is $\frac{n^3+11n+6}{6} \sim n^3$. These points correspond to intervals in the orbit of I^* obtained up to $\leq n$ compositions of the generators. Since these intervals are disjoint, the length of a typical one should be of order $\sim 1/n^3$. Hence, along a generic sequence of compositions, the value of the corresponding sum L_α should be of order

$$\sum_{n \geq 1} \left(\frac{1}{n^3} \right)^\alpha,$$

which is finite for $\alpha > \frac{1}{3} = \frac{2}{3(3-1)}$, as expected.

For the case of a general $d \geq 3$, it is very tempting trying to argue as before for any α larger than the inverse of the degree of growth of the graph of the orbit of I. Now, recall that according to the Bass-Guivarch formula (see [6, Appendix]), the growth of N^*_{d-1} is polynomial of degree $\sum_{i=1}^{d-1} i(d-i)$. Moreover, the stabilizer of I under the action of N^*_{d-1} is the subgroup of N^*_{d-1} made of the matrices whose first column is $(1, 0, 0, \ldots, 0)^T$. This subgroup naturally identifies with N_{d-2}, whose growth is polynomial of degree $\sum_{i=1}^{d-2} i(d-i-1)$. The difference of these degrees equals

$$\sum_{i=1}^{d-1} i(d-i) - \sum_{i=1}^{d-1} i(d-i-1) = \sum_{i=1}^{d-1} i = \frac{d(d-1)}{2}. \tag{4}$$

Since the graph of the orbit of I identifies with the space of cosets N_{d-1}/N_{d-2}, one should expect that its growth is polynomial of degree given by (4), and this is actually the case.

2.3 Proof of Theorem A: the case $d = 3$

The proof of Theorem A is somewhat technical and requires hard notation. This is the reason why we have chosen to first give the proof for the case $d = 3$, where most of the ideas become more transparent and an important technical problem is overcomed by a trick consisting in the introduction of a small parameter $\varepsilon > 0$. For the general case, we use a slightly modified construction keeping essentially the same arguments. We begin with a lemma in the spirit of [8, Lemma 2.2].

Lemma 2.3. *Let $n \geq 1$ be an integer and let C_1, C_2, ε be positive constants. Let P be a set of $\leq C_1 n^{3+\varepsilon}$ pairs of non-negative integers (i,j) associated to which there is a number $\ell_{i,j} > 0$ such that $\sum_{(i,j)\in P} \ell_{i,j} \leq 1$. Suppose that P partitioned into $n' \geq n^2/C_2$ (resp. $n' \geq n^{2+\varepsilon}/C_2$) disjoint subsets $P_1, \ldots, P_{n'}$. Then, given $A > 1$ and $1 > \alpha > 0$, the proportion of indexes $m \in \{1, \ldots, n'\}$ for which*

$$\sum_{(i,j)\in P_m} \ell_{i,j}^{\alpha} \leq \frac{A C_1^{1-\alpha} C_2}{n^{3\alpha - 1 - \varepsilon(1-\alpha)}} \qquad \Big(\text{resp.} \sum_{(i,j)\in P_m} \ell_{i,j}^{\alpha} \leq \frac{A C_1^{1-\alpha} C_2}{n^{3\alpha-1+\varepsilon\alpha}}\Big)$$

is at least $1 - 1/A$.

Proof. Since $\sum_{(i,j)\in P} \ell_{i,j} \leq 1$ and P consists of at most $C_1 n^{3+\varepsilon}$ pairs, a direct application of Hölder's inequality yields

$$\sum_{(i,j)\in P} \ell_{i,j}^{\alpha} \leq (C_1 n^{3+\varepsilon})^{1-\alpha}.$$

Hence,

$$\frac{1}{n'} \sum_{m=1}^{n'} \sum_{(i,j)\in P_m} \ell_{i,j}^{\alpha} \leq \frac{C_1^{1-\alpha} n^{(3+\varepsilon)(1-\alpha)}}{n'},$$

and the latter expression is less than or equal to $C_1^{1-\alpha} C_2 n^{1-3\alpha+\varepsilon(1-\alpha)}$ (resp. $C_1^{1-\alpha} C_2 n^{1-3\alpha-\varepsilon\alpha}$). The lemma then follows as a direct application of Chebyshev's inequality. $\square$

Let us now come back to the graph associated to the action ϕ_{FF} depicted in Figure 2, and let us set $\ell_{i,j} := |f_{2,1}^i f_{3,1}^j (I^*)|$. Fix positive constants α, ε such that

$$\alpha > \frac{1}{3} = \frac{2}{(3-1)(3-2)}, \qquad \varepsilon < \max\left\{\frac{3\alpha - 1}{1-\alpha}, 1\right\}. \tag{5}$$

For any real numbers $M \leq N$, we let $[[M, N]] := [M, N] \cap \mathbb{Z}$. Given an integer $n \geq 2$, we consider the set $P(n) := [[n, 8n-1]] \times [[0, n^{2+\varepsilon}]]$. This set $P(n)$ consists of $7n([n^{2+\varepsilon}] + 1) \leq 10\, n^{3+\varepsilon}$ points (with $[\cdot]$ standing for the integer part), and is partitioned into the $n' = [n^{2+\varepsilon}] + 1 \geq n^{2+\varepsilon}$ disjoint sets (horizontal paths) $P(n,1), P(n,2), \ldots, P(n,n')$ given by

$$P(n,m) := \big\{(n,m), (n+1,m), \ldots, (8n-1,m)\big\}.$$

By the preceding lemma, for each $0 < A_n < 1$, the proportion of indexes $m \in \{1, \ldots, n'\}$ for which

$$\sum_{i=n}^{8n-1} \ell_{i,m}^{\alpha} = \sum_{(i,j)\in P(n,m)} \ell_{i,j}^{\alpha} \leq \frac{A_n 10^{1-\alpha}}{n^{3\alpha-1+\varepsilon\alpha}} \tag{6}$$

is at least $1 - 1/A_n$. Notice that each path $P(n,m)$ comes from the action of the generator $f_{2,1}$.

Similarly, for each integer $n \geq 2$, let us consider the set $Q(n) := [[n, 2n-1]] \times [[0, n^{2+\varepsilon}]]$ consisting of $n([n^{2+\varepsilon}] + 1) \leq 2n^{3+\varepsilon}$ points. Although in general there is no partition of $Q(n)$ into paths induced by the action of $f_{3,1}, f_{3,2}$ all of them having the same number of points, a partition that almost satisfies this property (and that will be sufficient for our purposes) can be defined as follows. For each $n \leq m \leq 2n-1$ we divide the set $\{(m,0), (m,1), \ldots\}$ into n paths via the following rules:

– For each $0 \leq j \leq n-2$, there is a path starting at (m,j) jumping upwards of m units;

– The path starting at $(m, n-1)$ makes $m - n$ jumps upwards of 1 unit and then makes a jump of m units;

– The picture repeats "periodically", so that each infinite path is made of $n-1$ consecutive jumps of m units followed by $m-n$ jumps of 1 unit.

Figure 3 illustrates the case where $n=3$ and $m=5$ though the resulting paths are disposed horizontally instead of vertically by obvious reasons. Although one may give precise formulas for the points in each of these paths, this is not completely necessary. The main property that we will retain is the obvious fact that the number of points of each of them inside any rectangle $[[n,2n-1]] \times [[0,K-1]]$ lies between $\frac{K}{n}-2n$ and $\frac{K}{n}+2n$. (An alternative construction leading to a much better -logarithmic- control of the deviation will be given in §2.4.) In particular, we have an induced partition of $Q(n)$ into $n''=n^2$ paths $Q(n,1), Q(n,2), \ldots, Q(n,n'')$ for which the preceding lemma yields that for each $A_n > 0$, the proportion of indexes $m \in \{1,\ldots,n''\}$ satisfying

$$\sum_{(i,j)\in Q(n,m)} \ell_{i,j}^{\alpha} \leq \frac{A_n 2^{1-\alpha}}{n^{3\alpha-1-\varepsilon(1-\alpha)}} \tag{7}$$

is at least $1-1/A_n$. Notice again that each of these paths comes from the action of the generators $f_{3,1}$ and $f_{3,2}$ according to the amplitude of the jump.

Figure 3

We will apply the preceding construction for each integer $n=n_k := 4^k$, where $k \geq 1$. The choice of the constants A_{n_k} is as follows. First, we let r_k (resp. s_k) be the minimum (resp. maximum) number of points of a path of the form $Q(n_k,m)$ inside $Q(n_k)$. Similarly, we let r'_k (resp. s'_k) be the minimum (resp. maximum) number of points in a path of the form $Q(n_k,m)$ inside $P(n_{k-1}) \cap Q(n_k)$. Finally, we let

$$B := \prod_{k\geq 2} \frac{s_k}{r_k}\frac{s'_k}{r'_k}. \tag{8}$$

Notice that the value of B is finite. Indeed, by the discussion above, we have

$$4^{k(1+\varepsilon)} - 2\cdot 4^k = n_k^{1+\varepsilon} - 2n_k \leq r_k \leq s_k \leq n_k^{1+\varepsilon} + 2n_k = 4^{k(1+\varepsilon)} + 2\cdot 4^k$$

and

$$4^{k+k\varepsilon-1} - 2\cdot 4^{k+1} = \frac{n_k^{2+\varepsilon}}{n_{k+1}} - 2n_{k+1} \leq r'_k \leq s'_k \leq \frac{n_k^{2+\varepsilon}}{n_{k+1}} + 2n_{k+1} = 4^{k+k\varepsilon-1} + 2\cdot 4^{k+1},$$

which easily yield the convergence of the infinite product in the definition of B. We will also use the constant

$$C := 4\sum_{k\geq 1} \frac{1}{2^{k(3\alpha-1-\varepsilon(1-\alpha))}}. \tag{9}$$

Notice again that since (5) implies that $3\alpha-1-\varepsilon(1-\alpha) > 0$, we have $C < \infty$.

We now fix $A_{n_1} \geq 2^{2+k(3\alpha-1-\varepsilon(1-\alpha))} BC$ such that (6) holds for $n=n_1$ and *every* m in the corresponding range. Finally, for $k \geq 2$, we set

$$A_{n_k} := B\,C\,2^{k(3\alpha-1-\varepsilon(1-\alpha))}.$$

We next state a key lemma whose proof is postponed in order to proceed immediately to the proof of Theorem A in the case $d=3$.

Lemma 2.4. *There are two infinite sequences of paths $P(n_k,m'_k)$ and $Q(n_k,m''_k)$ such that* (6) *(resp.* (7)*) holds for $n=n_k$ and $m=m'_k$ (resp. $m=m''_k$) and such that $P(n_k,m'_k)$ intersects both $Q(n_k,m''_k)$ and $Q(n_{k+1},m''_{k+1})$ for all $k\geq 1$.*

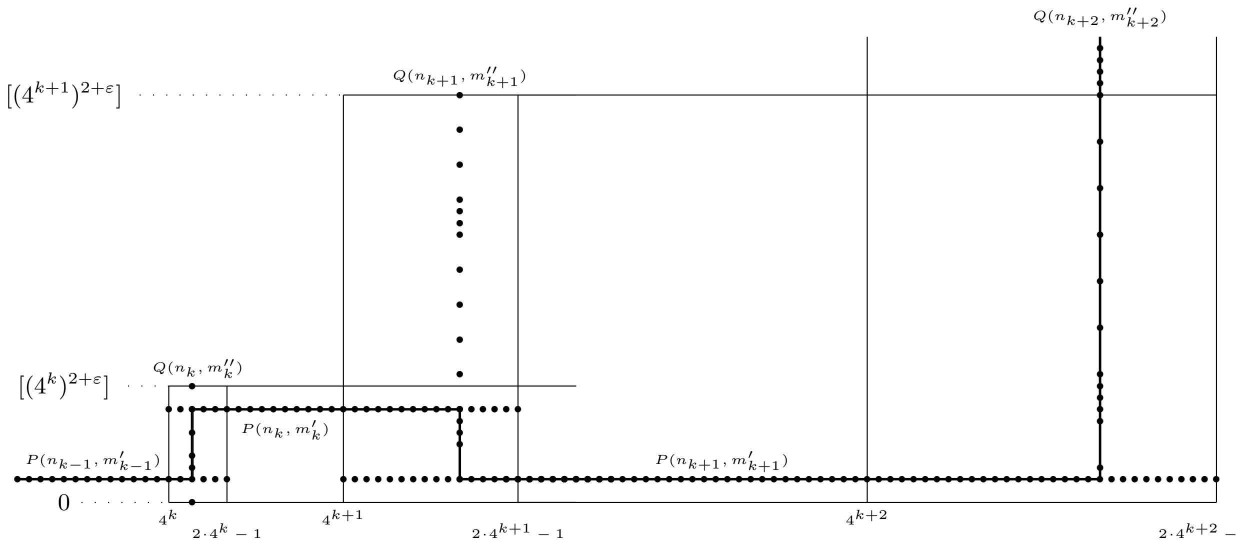

Figure 4

Assuming this lemma, the proof of Theorem A in the case $d = 3$ is at hand. Indeed, the concatenation of the sequence of finite paths provided by the lemma naturally yields an infinite path without loops which is in correspondence with a sequence of compositions by $f_{2,1}, f_{3,1}, f_{3,1}^{-1}, f_{3,2}, f_{3,2}^{-1}$ (see Figure 4). By construction, for this sequence of iterations, the value of the corresponding L_α-sum (2) for the interval I^{**} corresponding to the initial point of $Q(n_1, m'_1)$ is less than or equal to

$$10^{1-\alpha} \sum_{k \geq 1} \frac{A_{n_k}}{n_k^{3\alpha-1+\varepsilon\alpha}} + 2^{1-\alpha} \sum_{k \geq 1} \frac{A_{n_k}}{n_k^{3\alpha-1-\varepsilon(1-\alpha)}} \quad \leq \quad \frac{20 A_{n_1}}{4^{3\alpha-1-\varepsilon(1-\alpha)}} + \sum_{k \geq 2} \frac{40BC}{2^{k(3\alpha-1-\varepsilon(1-\alpha))}}$$
$$\leq \quad 80 A_{n_1} 4^{\varepsilon(1-\alpha)} + 40BC^2.$$

This interval I^{**} is in the orbit of I^*, from which it can be reached in no more than $(2 \cdot 4^1 - 1) + 4 = 11$ iterations of the generator $f_{2,1}$. By concatenating this finite path to the previous one, we obtain an infinite path associated to which the L_α-sum corresponding to I^* is finite, which allows to conclude the proof by the arguments developed in §2.2.

All that remains for completing the proof of Theorem A in the case $d = 3$ is the

Proof of Lemma 2.4. The argument is similar to that of [8, Lemma 2.3], but it needs a slight modification. Namely, for each $k \geq 1$, we let D'_k be the density of indexes $m' \in \{1, \ldots, [n_k^{2+\varepsilon}] + 1\}$ such that $P(n_k, m')$ is "reached" by a sequence of paths $Q(n_1, m''_1), P(n_1, m'_1), \ldots, Q(n_k, m''_k)$ satisfying:

– $P(n_i, m'_i)$ intersects both $Q(n_i, m''_i)$ and $Q(n_{i+1}, m''_{i+1})$ for all $1 \leq i \leq k-1$, whereas $P(n_k, m')$ intersects $Q(n_k, m''_k)$;

– Inequality (6) (resp. (7)) holds for $(n, m) = (n_i, m'_i)$ whenever $1 \leq i \leq k-1$ as well as for $(n, m) = (n_k, m')$ (resp. for $(n, m) = (n_i, m''_i)$ whenever $1 \leq i \leq k$).

Similarly, we denote by D''_k the density of indexes $m'' \in \{1, \ldots, n_k^2\}$ such that $Q(n_k, m'')$ is reached by a sequence of paths $Q(n_1, m''_1), P(n_1, m'_1), \ldots, P(n_{k-1}, m'_{k-1})$ satisfying:

– $P(n_i, m'_i)$ intersects both $Q(n_i, m''_i)$ and $Q(n_{i+1}, m''_{i+1})$ for all $1 \leq i \leq k-1$;

– Inequality (6) (resp. (7)) holds for $(n, m) = (n_i, m'_i)$ (resp. for $(n, m) = (n_i, m''_i)$) whenever $1 \leq i \leq k-1$ as well as for $(n, m) = (n_k, m'')$.

We claim that the following relations hold:

$$1 - D'_k \leq (1 - D''_k) \frac{s_k}{r_k} + \frac{1}{A_{n_k}}, \qquad 1 - D''_{k+1} \leq (1 - D'_k) \frac{s'_{k+1}}{r'_{k+1}} + \frac{1}{A_{n_{k+1}}}. \tag{10}$$

Assuming this for a while, we obtain for each $k \geq 1$,

$$1 - D'_k \leq (1 - D'_{k-1}) \frac{s_k}{r_k} \frac{s'_k}{r'_k} + \frac{1}{A_{n_k}} \frac{s_k}{r_k} + \frac{1}{A_{n_k}}.$$

Using induction, this easily yields

$$1 - D'_k \leq (1 - D'_1) \prod_{i=2}^{k} \frac{s_i}{r_i} \frac{s'_i}{r'_i} + 2 \sum_{i=2}^{k} \frac{1}{A_{n_i}} \prod_{j=2}^{i} \frac{s_j}{r_j}.$$

From the definition $n_i := 4^i$ and that of the constant B in (8), one concludes that for each $k \geq 1$,

$$1 - D'_k \leq (1 - D'_1) B + 2B \sum_{i=1}^{k} \frac{1}{A_{n_i}}.$$

Now, the choice of A_{n_1} was made so that $D'_1 = 1$, hence

$$1 - D'_k \leq 2B \sum_{i \geq 1} \frac{1}{A_{n_i}} \leq \frac{1}{2}.$$

Thus, $D'_k \geq 1/2$ holds for all $k \geq 1$, which provides finite paths satisfying the desired properties of length as large as we want. The infinite path claimed to exist is obtained easily from this by means of a Cantor diagonal type argument.

Finally, it remains to show (10). The proof follows the same principle of that of [8, Lemma 2.3] but requires a little adjustment. First, we denote by $\hat{D}''_k$ the density of points in $Q(n_k)$ that are "well-attainable" in the sense that they belong to the last of a sequence of consecutively intersecting paths $Q(n_1, m''_1), P(n_1, m'_1), \ldots, P(n_{k-1}, m'_{k-1}), Q(n_k, m''_k)$ for which inequalities of type (6) or (7) hold according to the case. We have

$$(1 - D'_k) \leq (1 - \hat{D}''_k) + \frac{1}{A_{n_k}}. \tag{11}$$

Indeed, the term $1/A_{n_k}$ corresponds to the density of horizontal paths in $P(n_k)$ that are "bad by themselves" in the sense that the corresponding type (6) inequality does not hold for them. The term $(1-\hat{D}''_k)$ corresponds to the density of paths in $P(n_k)$ that may be good by themselves but intersect $Q(n_k)$ at a set formed only by non-well-attainable points. (Notice that we are using the fact that all horizontal paths in $P(n_k)$ have the same number of points in $Q(n_k)$.) The left-side inequality in (10) then follows as a combination of (11) and the inequality

$$1 - \hat{D}''_k \leq (1 - D''_k) \frac{s_k}{r_k},$$

where the correction factor comes from the fact that although the number of points in each path of the form $Q(n_k, m)$ is not constant, it varies between r_k and s_k.

Similarly, in the right-side inequality, the term $1/A_{n_{k+1}}$ corresponds to the density of bad-by-themselves paths of the form $Q(n_{k+1}, m)$ in $Q(n_{k+1})$. The term $(1 - D'_k)$ corresponds to the "accumulated density of bad paths" up to $P(n_k)$, and equals the density of "non-well-attainable" points in $P(n_k) \cap Q(n_{k+1})$. Finally, the correction factor comes from the fact that the number of points in $P(n_k) \cap Q(n_{k+1})$ contained in each path of the form $Q(n_{k+1}, m)$ lies between r'_{k+1} and s'_{k+1}.

2.4 Proof of Theorem A: the general case

To deal with the general case we will follow a similar strategy, though most of the computations become more involved. We will now consider paths inside parallelepipeds of dimension $d-1$ having sides of length of (relative) order $k, k^2, \ldots, k^{d-1}$. This will make naturally appear the exponent $\frac{d(d-1)}{2}$ in relation to the total number of points in the parallelepiped. The most relevant difficulty will be related to the decomposition into paths. Indeed, the construction of the preceding section illustrated by Figure 3 is no longer satisfactory, and we will need to carry out a nontrivial modification of it. Since this is of independent interest and has potential applications in other contexts, the discussion of the new construction will be the subject of §2.5. Here we content ourselves in stating what we need for our purposes, which is summarized in the next

Lemma 2.5. *Let $M > N$ be positive integer numbers, with N of the form $1+2^k$. There exists a decomposition of $\mathbb{N}_0 := \{0, 1, \ldots\}$ into N subsets (paths) satisfying:*

(i) *The distance (jump) between two consecutive points of each path is either M or 1;*

(ii) *For all $0 \leq K_1 < K_2$, the maximal number of points of a path contained in $[[K_1, K_2]]$ differs from the minimal one by at most $4 + 2\frac{M-1}{N-1} + 4\log_2(N-1)$.*

We now proceed to the proof of Theorem A. Recall that the graph of the N^*_{d-1}-orbit of the interval I^* defined by (3) has $\mathbb{Z}^{d-1}$ as its set of vertices. We will hence inductively define parallelepipeds $Q(n) \subset \mathbb{Z}^{d-1}$. We start with $Q(0) := [[1, 1+4^{d+1}]]^{d-1}$. Assuming that $Q(n) := [[x_{1,n}, y_{1,n}]] \times \cdots \times [[x_{d-1,n}, y_{d-1,n}]]$ has been already defined, we let $i(n) \in \{1, \ldots, d-1\}$ be the residue class (mod. $d-1$) of n, and we set $Q(n+1) := \cdots \times [[1+4^{i(n)}(x_{i(n),n}-1), y_{i(n),n}]] \times [[x_{i(n)+1,n}, 1+4^{i(n)+1}(y_{i(n)+1,n}-1)]] \times \cdots$, where the dots mean that the corresponding factors remain untouched. (See Figure 5 for an illustration of the case $d=4$, with $n \equiv 1$ (mod. 3).)

Notice that $x_{i,n}, y_{i,n}$ are of the form $1+2^k$ for all i, n. Although one may give precise formulas for $x_{i,n}, y_{i,n}$, we will only need to record the (easy to check) fact that for some universal constants C_1, C_2, C_3, C_4, we have the estimates

$$C_1 4^{\frac{in}{d-1}} \leq y_{i,n} - x_{i,n} \leq C_2 4^{\frac{in}{d-1}} \tag{12}$$

and

$$C_3 4^{\frac{in}{d-1}} \leq x_{i,n} \leq C_4 4^{\frac{in}{d-1}}. \tag{13}$$

In particular, the number of points in $Q(n)$ is

$$|Q(n)| = \prod_{j=1}^{d-1} (y_{j,n} - x_{j,n}) \leq \prod_{j=1}^{d-1} C_2 4^{\frac{jn}{d-1}} = C_2^{d-1} 4^{\frac{n}{d-1}\sum_{j=1}^{d-1} j} = C_2^{d-1} 4^{\frac{nd}{2}}. \tag{14}$$

Each $Q(n)$ is decomposed into paths pointing in the $i(n)^{th}$-direction as follows. If $i(n)=1$, then we decompose $Q(n)$ into "horizontal" paths of jump 1 at each step, so that the number of paths is

$$\prod_{j\neq 1} (y_{j,n} - x_{j,n}) \geq \prod_{j\neq 1} C_1 4^{\frac{jn}{d-1}} = C_1^{d-2} 4^{\frac{n}{d-1}\sum_{j\neq 1} j} = C_1^{d-2} 4^{n\left[\frac{d}{2} - \frac{1}{d-1}\right]}.$$

If $i(n) \neq 1$, then for each fixed coordinates $z_j \in [[x_{j,n}, y_{j,n}]]$, with $j \neq i(n)$, we identify

$$\{z_1\} \times \cdots \times \{z_{i(n)-1}\} \times [[x_{i(n),n}, y_{i(n),n}]] \times \{z_{i(n)+1}\} \times \cdots \times \{z_{d-1}\} \;\sim\; [[x_{i(n),n}, y_{i(n),n}]] \subset \mathbb{N}$$

and we decompose this set into $N := x_{i(n)-1,n}$ paths making jumps (in the $i(n)^{th}$-direction) of either 1 or $M := z_{i(n)-1,n}$ steps following the strategy of Lemma 2.5. The corresponding number of paths now equals

$$x_{i(n)-1,n} \prod_{j\neq i(n)} (y_{j,n} - x_{j,n}) \geq C_3 4^{\frac{(i(n)-1)n}{d-1}} \prod_{j\neq i(n)} C_1 4^{\frac{jn}{d-1}} = C_3 4^{\frac{(i(n)-1)n}{d-1}} C_1^{d-2} 4^{\frac{n}{d-1}\sum_{j\neq i(n)} j} = C_3 C_1^{d-2} 4^{n\left[\frac{d}{2} - \frac{1}{d-1}\right]}.$$

In either case, we denote by $Q(n,1), \ldots, Q(n, m_n)$ these paths, and we let $C_5 := \min\{C_3 C_1^{d-2}, C_1^{d-2}\}$, so that $m_n \geq C_5 4^{n\left[\frac{d}{2} - \frac{n}{d-1}\right]}$. What is important in the construction above is that each of these paths has a concrete dynamical meaning for the action of $N^*_{d-1} \subset N_d$. Namely, if $i(n)=1$, they are induced by the action of the generator $f_{2,1}$, whereas for $i(n) \neq 1$, they are induced by the action of $f_{i(n),1}$ and $f_{i(n)-1,i(n)}$, where the first generator appears for 1-step jumps and the second one for jumps of amplitude $z_{i(n)-1,n}$.

Associated to each point $(i_1, \ldots, i_{d-1}) \in \mathbb{Z}^{d-1}$ there is a positive number $\ell_{i_1,\ldots,i_{d-1}}$, namely the length of the interval

$$I^*_{i_1,\ldots,i_{d-1}} := \bigcup_{j\in\mathbb{Z}} I_{i_1,\ldots,i_{d-1},j}.$$

Notice that the total sum of the $\ell_{i_1,\ldots,i_{d-1}}$'s equals 1. Moreover, all the intervals $I^*_{i_1,\ldots,i_{d-1}}$ are in the N^*_{d-1}-orbit of $I^* = I^*_{0,\ldots,0}$; see (3). Hence, as in the case $d=3$, what we need to do is to ensure the existence of an infinite sequence of intersecting paths in $Q(1), Q(2), \ldots$ along which the total L_α-sum is finite provided that $\alpha > \frac{2}{d(d-1)}$. To do this, we start with the next

Lemma 2.6. *Given $0 < \alpha < 1$, there exists a constant $C_6 > 0$ such that for all $A > 0$ and all $n \geq 1$, the subset of indexes $m \in \{1, \ldots, m_n\}$ satisfying*

$$\sum_{(i_1,\ldots,i_{d-1})\in Q(n,m)} \ell^\alpha_{i_1,\ldots,i_{d-1}} \leq \frac{A\, C_6}{4^{n\left[\frac{d\alpha}{2} - \frac{1}{d-1}\right]}} \tag{15}$$

has density at least $1 - 1/A$.

Proof. As in the case $d = 3$, by Hölder's inequality we have

$$\sum_{(i_1,\ldots,i_{d-1})\in Q(n)} \ell^{\alpha}_{i_1,\ldots,i_{d-1}} \leq |Q(n)|^{1-\alpha} \leq C_2^{(d-1)(1-\alpha)} 4^{\frac{nd(1-\alpha)}{2}}.$$

Hence,

$$\frac{1}{m_n}\sum_{m=1}^{m_n} \sum_{(i_1,\ldots,i_{d-1})\in Q(n,m)} \ell^{\alpha}_{i_1,\ldots,i_{d-1}} \leq \frac{C_2^{(d-1)(1-\alpha)} 4^{\frac{nd(1-\alpha)}{2}}}{C_5 4^{\frac{nd}{2}-\frac{n}{d-1}}} = \frac{C_2^{(d-1)(1-\alpha)}}{C_5 4^{\frac{nd\alpha}{2}-\frac{n}{d-1}}},$$

and the claim of the lemma follows from Chebyshev's inequality for $C_6 := C_2^{(d-1)(1-\alpha)}/C_5$. □

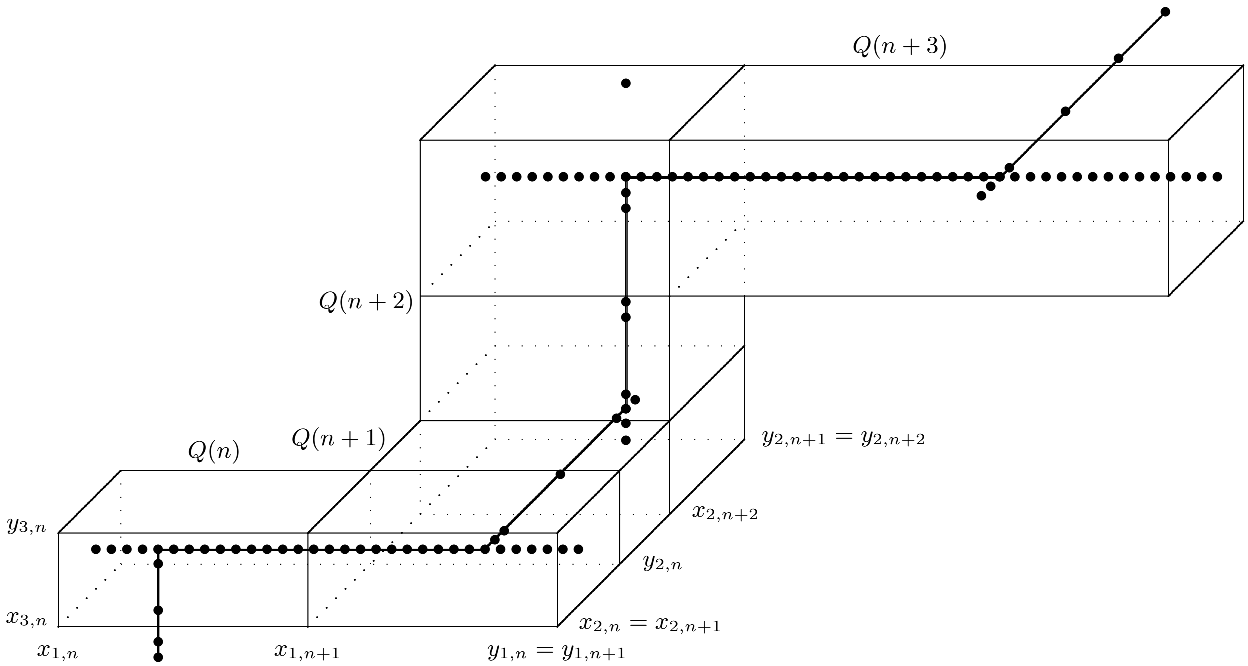

Figure 5

From now on, we fix $\alpha > \frac{d(d-1)}{2}$. We start by letting r_n (resp. s_n) be the minimum (resp. maximum) of points in a path of the form $Q(n,m)$ inside $Q(n) \cap Q(n+1)$. Similarly, we denote by r'_n (resp. s'_n) the minimum (resp. maximum) number of points of a path of the form $Q(n+1,m)$ inside $Q(n) \cap Q(n+1)$. Then we let

$$B := \prod_{n\geq 1} \frac{s_n}{r_n}\frac{s'_n}{r'_n}.$$

We claim that the value of B is finite. Indeed, we have $r_n = s_n$ whenever $i(n) = 1$, whereas $s'_n = r'_n$ whenever $i(n) = d-1$. For the other values of $i := i(n)$, the condition (ii) in Lemma 2.5 together with the inequalities $2\frac{y_{i-1,n}-1}{x_{i-1,n}-1} \leq 4^{d+2}$ and $2\frac{y_{i,n+1}-1}{x_{i,n+1}-1} \leq 4^{d+2}$ yield the estimates

$$\frac{y_{i,n+1} - x_{i,n+1}}{x_{i-1,n}} - 4 - 4^{d+2} - 4\log_2(x_{i-1,n} - 1) \leq r_n \leq s_n \leq \frac{y_{i,n+1} - x_{i,n+1}}{x_{i-1,n}} + 4 + 4^{d+2} + 4\log_2(x_{i-1,n} - 1)$$

and

$$\frac{y_{i+1,n} - x_{i+1,n}}{x_{i,n+1}} - 4 - 4^{d+2} - 4\log_2(x_{i,n+1} - 1) \leq r'_n \leq s'_n \leq \frac{y_{i+1,n} - x_{i+1,n}}{x_{i,n+1}} + 4 + 4^{d+2} + 4\log_2(x_{i,n+1} - 1),$$

which together with (12) and (13) easily imply the finiteness of B.

We will also use the (finite) constant

$$C := 2\sum_{n\geq 1} \frac{1}{2^{n\left[\frac{d\alpha}{2}-\frac{1}{d-1}\right]}}.$$

Now we fix $A_1 \geq B\,C\,2^{\frac{d\alpha}{2}-\frac{1}{d-1}}$ such (15) holds for $n=1$ and *every* $m \in \{1,\ldots,m_1\}$ when letting $A = A_1$. Finally, for $n \geq 2$, we set

$$A_n := B\,C\,2^{n\left[\frac{d\alpha}{2}-\frac{1}{d-1}\right]}.$$

Lemma 2.7. *There exists an infinite sequence of paths of the form $Q(n,m'_n)$ in $Q(n)$ such that, for all $n \geq 1$, the path $Q(n+1,m'_{n+1})$ intersects $Q(n,m'_n)$ and (15) holds for $m = m'_n$ and $A = A_n$.*

Proof. As in the case $d=3$, for each $n \geq 1$ we let D_n be the density of indexes $m \in \{1,\ldots,m_n\}$ such that there exists a finite sequence of paths $Q(1,n'_1),\ldots,Q(n,m'_n)$ satisfying:

– For each $1 \leq k \leq n-1$, the path $Q(k+1,m'_{k+1})$ intersects $Q(k,m'_k)$;

– Estimate (15) holds for each $m = m'_k$ and $A = A_k$.

Similar arguments to those leading to (10) yield

$$(1-D_{n+1}) \leq (1-D_n)\frac{s_n}{r_n}\frac{s'_n}{r'_n} + \frac{1}{A_n}.$$

Indeed, the product $\frac{s_n s'_n}{r_n r'_n}$ acts as a correction factor for the passage from $Q(n)$ to $Q(n+1)$ taking into account that the paths of the form $Q(n,m)$ do not have the same number of points in $Q(n)\cap Q(n+1)$, and similarly for those of the form $Q(n+1,m)$. By induction, the preceding inequality yields

$$1-D_n \leq (1-D_1)\prod_{k=1}^{n-1}\frac{s_k}{r_k}\frac{s'_k}{r'_k} + \sum_{k=1}^{n-1}\frac{1}{A_k}\prod_{j=1}^{k-1}\frac{s_j}{r_j}\frac{s'_j}{r'_j} \leq (1-D_1)B + B\sum_{k\geq 1}\frac{1}{A_k}.$$

The choice of A_1 was made so that $D_1 = 1$, hence

$$1-D_n \leq B\sum_{k\geq 1}\frac{1}{A_k} \leq \frac{1}{2}.$$

As a consequence, $D_n \geq 1/2$, which implies that for each n we may obtain a finite sequence of n paths with the desired properties. The infinite sequence is obtained via a Cantor diagonal type argument. □

The proof of Theorem A is now at hand. Indeed, the concatenation of the paths provided by the preceding lemma yields an infinite sequence of points in $\mathbb{Z}^{d-1}$ along which the value of the L_α-sum is bounded from above by

$$\sum_{n\geq 1}\frac{A_n C_6}{4^{n\left[\frac{d\alpha}{2}-\frac{1}{d-1}\right]}} \leq \frac{A_1 C_6}{4^{\frac{d\alpha}{2}-\frac{1}{d-1}}} + \sum_{n\geq 2}\frac{B\,C\,C_6}{2^{n\left[\frac{d\alpha}{2}-\frac{1}{d-1}\right]}} \leq 2A_1C_6 + 2BC_6.$$

This is in correspondence to a sequence of intervals of the form $I_{i_1,\ldots,i_{d-1}}$ each of which is obtained from the preceding one by applying one of the generators in $\{f_{2,1}, f_{3,1},\ldots,f_{d,1}\} \cup \{f_{2,1}, f_{3,2}.\ldots,f_{d,d-1}\}$. Joining this infinite sequence to a finite one from the origin to a point in $Q(1,n'_1)$, we obtain an infinite sequence of intervals in the N^*_{d-1}-orbit of the interval I^* for which the L_α-sum is finite, and hence the arguments of §2.2 may be applied. This concludes the proof.

2.5 An independent combinatorial lemma

The aim of this Section is to give the proof of Lemma 2.5. We first give the details of the construction of the partition of $\mathbb{N}_0$ into N sets (paths) $P_1,\ldots.,P_N$, and latter we check the desired properties. The construction is made in two steps, the former of which applies to arbitrary values of N, whereas the latter is restricted to integers of the form $1+2^k$.

Step 1. Let $M > N$ be positive integers. Assume that we are given a partition

$$[[0,M-1]] = R_0 \bigcup R_1 \bigcup \ldots \bigcup R_{N-1}$$

into "consecutive" sets, that is, such that $1+\max R_i = \min R_{i+1}$ holds for all $0 \leq i \leq N-2$. Then this induces a partition of $\mathbb{N}_0$ as follows. Denoting $R \oplus k := \{n+k \colon n \in R\}$, we define

- $S_1 := \bigcup_{j=1}^{N-1} R_j \oplus j(M-1)$,
- $S_i := \bigcup_{j=i-1}^{N-1} R_j \oplus (j-i+1)(M-1) \bigcup \bigcup_{j=1}^{i-2} R_j \oplus (j-i+N)(M-1)$, for $2 \le i \le N$.

(Notice that, by definition, the second term in the definition of S_i above is empty for $i = 2$.) Now, what defines our partition of $\mathbb{N}_0$ is the "periodic repetition" of the sets $S_1, ..., S_N$. More precisely, we let

- $P_1 := R_0 \bigcup \bigcup_{k=0}^{\infty} S_1 \oplus kN(M-1)$,
- $P_i := \bigcup_{k=0}^{\infty} S_i \oplus kN(M-1)$, for $2 \le i \le N$.

To have a clearer view of this construction, the reader may easily check that for the particular choice $R_0 := \{0\}$, $R_1 := \{1\}, \ldots, R_{N-2} := \{N-2\}$ and $R_{N-1} := \{N-1, N, N+1, \ldots, M-1\}$, it yields to the paths constructed in §2.3 (see again Figure 3 for an illustration).

It is sometimes better to think on our paths as concatenations of "patches". In this view, for $2 \le i \le N$, the sequence representing S_i is $R_{i-1}R_i \ldots R_{N-1}R_1R_2 \ldots R_{i-2}$, which in notation modulo $N-1$ may be rewritten as $R_{i-1}R_i \ldots R_{i+N-2}$. This means that S_i is made of a copy of R_{i-1} followed by a copy of R_i translated by $M-1$ units, a copy of R_{i+1} translated by another $M-1$ units, and so on. Similarly, our paths P_i may be seen as infinite sequences of patches. Thinking on each S_i as a patch as well, for $2 \le i \le N$, the path P_i is represented by $S_iS_iS_i \ldots$. The sequence representing P_1 corresponds to $R_0S_1S_1S_1 \ldots$.

Step 2. Assuming that N has the form $1 + 2^k$, we will associate to it a particular choice of sets $R_1, ..., R_N$. Let $p \ge 1$ and $q \ge 0$ be the integers such that

$$M - 1 = (N-1)p + q, \quad \text{with } q < N-1.$$

Let us consider the binary expansion of q:

$$q = 2^{r_1} + \ldots + 2^{r_l}, \quad \text{with } r_1 > .... > r_l \ge 0.$$

(Notice that since $q < N - 1 = 2^k$, we have $k > r_1$.) Now, for $1 \le i \le N-1$, define s_i as being the largest integer s such that 2^{k-r_s} divides i whenever there is such an index s, and as being equal to zero otherwise. We claim that the following relation holds:

$$s_1 + s_2 + \ldots + s_{N-1} = q. \tag{16}$$

Indeed, by definition, s_i equals $s > 0$ if and only if i is a multiple of 2^{k-r_s} but not a multiple of $2^{k-r_{s+1}}$. Now, in $\{1, 2, \ldots, N-1\}$, there are exactly 2^{r_s} multiples of 2^{k-r_s}, namely the products of 2^{k-r_s} with the integers in $\{1, 2, 3, \ldots, 2^{r_s}\}$. Hence, the left-side expression in (16) equals

$$\sum_{s=1}^{l} s \left|\{i \colon s_i = s\}\right| = \sum_{s=1}^{l-1} s\big(2^{r_s} - 2^{r_{s+1}}\big) \, + \, l2^{r_l} \, = \sum_{s=1}^{l} 2^{r_s} \, = \, q. \tag{17}$$

Finally, let us inductively define:

- $R_0 := \{0\}$,
- $R_i := \{1 + \max R_{i-1}, ..., p + s_i + \max R_{i-1}\}$, where $1 \le i \le N-1$.

Notice that for $1 \le i \le N-1$, the number of points of R_i equals

$$p + s_i \le p + l \le p + k = p + \log_2(N-1). \tag{18}$$

Using (16), we conclude that the number of points contained in the union of the R_i's equals

$$1 + p(N-1) + s_1 + \ldots + s_{N-1} \, = \, 1 + p(N-1) + q \, = \, M.$$

Thus, the R_i's yield a partition of $[[0, M-1]]$ into consecutive sets. We claim that the corresponding partition of $\mathbb{N}_0$ into the paths $P_1, \ldots, P_N$ produced as in Step 1 satisfies the desired properties.

Step 3. We first notice that in order to prove property (ii) of Lemma 2.5, we may restrict ourselves to intervals of the form $[[0,K]]$ instead of general intervals $[[K_1,K_2]]$ provided we obtain the better bound $2+\frac{M-1}{N-1}+2\log_2(N)$ for the maximal difference of points in $[[0,K]]$ among our N paths. This is what we now proceed to do.

Let a,b be non-negative integers such that

$$K = aN(M-1)+b, \quad \text{with } b < N(M-1),$$

Let us first consider a path P_i such that $2 \le i \le N$. In terms of patch sequences, and using notation modulo $N-1$, the intersection of P_i with $[[0,K]]$ has the form

$$S_i \ldots S_i R_{i-1} R_i \ldots R_{i-1+t} T, \quad \text{with } t \le N-1.$$

Here, the patch T is a starting part of the patch R_{i+t}. Moreover, the patch S_i appears precisely a times.

By construction, the number of points in the set represented above is a times the number of points in S_i plus the sum of the number of points in $R_{i-1}\ldots R_{i-1+t}$ plus the number of points in T. The former equals $a(M-1)$, hence it is independent of $i \in \{2,\ldots,N\}$, whereas the latter is smaller than or equal to $p+s_{i+t} \le p+\log_2(N-1)$; see (18). As a consequence, the difference with respect to the number of points in $[[0,K]] \cap P_j$ (with $2 \le j \le N$) is at most $p+\log_2(N-1)$ plus the difference between the number of points in $R_{i-1}\ldots R_{i-1+t}$ and $R_{j-1}\ldots R_{j-1+t}$. Since $p \le 1+\frac{M-1}{N-1}$, our task reduces to show that the last difference is at most $\log_2(N-1)$.

Now, the number of points in the first (resp. second) sequence above equals

$$(p+s_{i-1})+(p+s_i)+\ldots+(p+s_{i-1+t}) \;=\; tp+s_{i-1}+\ldots+s_{i-1+t}$$

$$\big(\text{resp. } (p+s_{j-1})+(p+s_j)+\ldots+(p+s_{j-1+t}) \;=\; tp+s_{j-1}+\ldots+s_{j-1+t}\big).$$

Define $\rho_{s,i}$ (resp. $\rho_{s,j}$) as being the number of indexes in $\{i-1,\ldots,i-1+t\}$ (resp. $\{j-1,\ldots,j-1+t\}$) that are multiples of 2^{k-r_s}. A similar argument to that leading to (17) yields

$$s_{i-1}+\ldots+s_{i-1+t} = \rho_{1,i}+\rho_{2,i}+\ldots+\rho_{l,i} \quad \big(\text{resp. } s_{j-1}+\ldots+s_{j-1+t} = \rho_{1,j}+\rho_{2,j}+\ldots+\rho_{l,j}\big).$$

Since

$$\frac{t}{2^{k-r_s}} \le \rho_{s,i} \le 1+\frac{t}{2^{k-r_s}} \quad \Big(\text{resp. } \frac{t}{2^{k-r_s}} \le \rho_{s,j} \le 1+\frac{t}{2^{k-r_s}}\Big),$$

the value of $|\rho_{s,i}-\rho_{s,j}|$ equals either zero or 1. We thus conclude that

$$\big|s_{i-1}+\ldots+s_{i-1+t}-s_{j-1}-\ldots-s_{j-1+t}\big| \le |\rho_{1,i}-\rho_{1,j}|+\ldots+|\rho_{l,i}-\rho_{l,j}| \;\le\; l \;\le\; k \;=\; \log_2(N-1),$$

as we wanted to show.

Actually, so far we have obtained the upper bound $1+\frac{M-1}{N-1}+2\log_2(N-1)$ for the difference between the number of points in $P_i \cap [[0,K]]$ and $P_j \cap [[0,K]]$. The extra 1 which lacks appears when making comparisons with the path P_1, taking into account that P_1 starts with $R_0=\{0\}$. The proof of this follows the same ideas above. We leave the details to the reader.

3 Construction of smoothings for $\alpha < \frac{2}{d(d-1)}$

3.1 A reminder on Denjoy-Pixton actions

For the constructions leading to the proofs of Theorems B and C, we will use Pixton's technique [12]. The main technical tool will be the following lemma from [18].

Lemma 3.1. *For a certain universal constant M there exists a family of diffeomorphisms $\varphi_{I',I}^{J',J}: I \to J$ where I,I',J,J' are non-degenerate intervals and I' (resp. J') is contiguous by the left to I (resp. J), satisfying $\varphi_{J',J}^{K',K} \circ \varphi_{I',I}^{J',J} = \varphi_{I',I}^{K',K}$ and*

$$\frac{\big|\log\big(D\varphi_{I',I}^{J',J}(u)\big)-\log\big(D\varphi_{I',I}^{J',J}(v)\big)\big|}{|u-v|} \le \frac{M}{|I|}\left|\frac{|I||J'|}{|J||I'|}-1\right|$$

for all u,v in I provided that $\max\{|I|,|I'|,|J|,|J'|\} \le 2\min\{|I|,|I'|,|J|,|J'|\}$.

The proof of this lemma proceeds as follows. Following [18], let $\xi(x)(\frac{\partial}{\partial x})$ be a C^∞ vector field on $[0,1]$ such that $\xi(x) = x$ near 0, and $\xi(x) = 0$ on $[1/2, 1]$. Moreover, assume that for all x,

$$|D\xi(x)| \leq 1.$$

Let $\psi_t(x)$ be the solution of the differential equation

$$\frac{d\psi_t}{dt}(x) = \xi(\psi_t(x)), \quad \psi_0(x) = x.$$

Let us consider the diffeomorphism $x \mapsto b\,\psi_t(x/a)$ sending the interval $[0,a]$ onto the interval $[0,b]$. For any real numbers a', a, b', b such that $a' < 0 < a$ and $b' < 0 < b$, let $\phi^{b',b}_{a',a}$ be the diffeomorphism from $[0,a]$ onto $[0,b]$ defined by

$$\phi^{b',b}_{a',a}(x) = b\,\psi_{\log(b'a/a'b)}(x/a).$$

Its is easy to check that for all positive a, b, c and all negative a', b', c', one has

$$\phi^{c',c}_{b',b} \circ \phi^{b',b}_{a',a} = \phi^{c',c}_{a',a}$$

Moreover, as is shown in [18],

$$\log D\phi^{b',b}_{a',a}(x) = \log\frac{b}{a} + \log D\psi_{\log(b'a/a'b)}\left(\frac{x}{a}\right), \tag{19}$$

$$\left|\log D\psi_{\log(b'a/a'b)}\right| \leq \left|\log\frac{b'a}{a'b}\right| = \left|\log\frac{b'}{a'} - \log\frac{b}{a}\right|. \tag{20}$$

Furthermore, letting $M > 0$ be a constant such that $|D^2\xi(x)| \leq M$ for all x, we have

$$\left|D\log D\phi^{b',b}_{a',a}(x)\right| \leq \frac{M}{a}\left|\frac{b'a}{a'b} - 1\right|. \tag{21}$$

Starting with the maps $\phi^{b',b}_{a',a}$, we construct the desired family $\{\varphi^{J',J}_{I',I}\}$ as follows. Letting $I = [w, w+a]$, $I' = [w+a', w]$, $J = [w', w'+b]$, and $J' = [w'+b', w']$, where $a' < 0 < a$ and $b' < 0 < b$, we let

$$\varphi^{J',J}_{I',I} = \phi^{b',b}_{a',a}(x-w) + w'.$$

3.2 Sharp embeddings of N_d

In order to prove our Theorem B, we fix once and for all an arbitrary positive number $\alpha < \frac{2}{d(d-1)}$. Our aim is to show that for a good choice of the lengths $|I_{i_1,\ldots,i_d}|$, the maps $f_j := f_{j+1,j}$, $1 \leq j \leq d-1$, defined as in §2.1 using the maps from §3.1 instead of affine maps are $C^{1+\alpha}$-diffeomorphisms of the corresponding (non necessarily normalized) interval I. From now on, we will assume that $d \geq 3$. Although the case $d = 2$ can be ruled out by a slightly modified construction, it is also covered by the (much simpler) construction leading to Theorem C. In all what follows, M will denote a universal constant whose explicit value is irrelevant for our purposes.

We begin by choosing number $p_d \in]1, 5/4]$, and for $1 \leq j \leq d-1$ we choose $p_j > 0$ so that the following properties are satisfied:

(i$_\mathrm{B}$) $p_1 > p_2 > \ldots > p_{d-1} > p_d > 1$,

(ii$_\mathrm{B}$) $\frac{1}{p_1} + \frac{1}{p_2} + \ldots + \frac{1}{p_{d-1}} + \frac{1}{p_d} < 1$,

(iii$_\mathrm{B}$) $\alpha \leq \frac{p_d}{(p_d-1)p_1}$,

(iv$_\mathrm{B}$) $\alpha \leq \frac{p_d}{p_d-1}\left(\frac{1}{p_j} - \frac{1}{p_{j-1}}\right)$ for all $1 < j < d$,

(v$_\mathrm{B}$) $\alpha \leq \frac{1}{p_d} - \frac{1}{p_{d-1}}$.

A concrete choice is $p_j := \frac{1}{j\alpha(1-1/p_d)}$. (Hence, one may take $p_d := \frac{5}{4}$ and $p_j := \frac{5}{j\alpha}$ for $1 \leq j \leq d-1$.) Indeed, the first property is easy to check. For the second one, we have

$$\sum_{j=1}^{d} \frac{1}{p_j} = \frac{1}{p_d} + \sum_{j=1}^{d-1} j\alpha \left(1 - \frac{1}{p_d}\right) = \frac{1}{p_d} + \alpha \left(1 - \frac{1}{p_d}\right) \frac{d(d-1)}{2} < \frac{1}{p_d} + \left(1 - \frac{1}{p_d}\right) = 1,$$

where the inequality comes from the hypothesis $\alpha < \frac{2}{d(d-1)}$. For the third and fourth properties, we actually have equalities with our choice. Finally, since $d \geq 3$,

$$\alpha < \frac{2}{d(d-1)} < \frac{2}{3} \leq \frac{1/p_d}{2 - 1/p_d} \leq \frac{1/p_d}{1 + (d-1)(1 - 1/p_d)}.$$

Hence,

$$\alpha \left[1 + (d-1)\left(1 - \frac{1}{p_d}\right)\right] \leq \frac{1}{p_d},$$

that is,

$$\alpha \leq \frac{1}{p_d} - \alpha(d-1)\left(1 - \frac{1}{p_d}\right) = \frac{1}{p_d} - \frac{1}{p_{d-1}},$$

which shows ($\mathrm{v_B}$).

It is worth mentioning that for $\alpha \geq \frac{2}{d(d-1)}$, the properties above are incompatible. Indeed, from ($\mathrm{iii_B}$) we get $\frac{1}{p_1} \geq \frac{\alpha(p_d-1)}{p_d}$. Using ($\mathrm{iv_B}$) inductively, we obtain $\frac{1}{p_j} \geq \frac{j\alpha(p_d-1)}{p_d}$ for $1 \leq j \leq d-1$. This yields

$$\sum_{j=1}^{d} \frac{1}{p_j} \geq \sum_{j=1}^{d-1} \frac{j\alpha(p_d-1)}{p_d} + \frac{1}{p_d} = \frac{\alpha(p_d-1)\, d\,(d-1)}{2p_d} + \frac{1}{p_d}.$$

If $\alpha \geq \frac{2}{d(d-1)}$, the right-side expression is greater than or equal to 1, contrary to ($\mathrm{ii_B}$).

Now fixing any choice of the p_j's as above, we let

$$|I_{i_1,\ldots,i_d}| := \frac{1}{|i_1|^{p_1} + \ldots + |i_d|^{p_d} + 1}.$$

According to [8, §3], property ($\mathrm{ii_B}$) implies that the sum of the lengths $|I_{i_1,\ldots,i_d}|$ is finite. We next proceed to show that the induced maps f_j are $C^{1+\alpha}$-diffeomorphisms of the corresponding interval I.

3.3 The map f_1 is a $C^{1+\alpha}$-diffeomorphim

I. First we consider x, y in the same interval $I_{i_1,\ldots,i_d}$. We have

$$\frac{|\log Df_1(x) - \log Df_1(y)|}{|x-y|} \leq \frac{M}{|I_{i_1,\ldots,i_d}|} \left| \frac{|I_{i_1,\ldots,i_d}|}{|I_{i_1+1,\ldots,i_d}|} \frac{|I_{i_1+1,\ldots,i_d-1}|}{|I_{i_1,\ldots,i_d-1}|} - 1 \right|.$$

Hence,

$$\frac{|\log Df_1(x) - \log Df_1(y)|}{|x-y|^\alpha} \leq M \left| \frac{|I_{i_1,\ldots,i_d}|}{|I_{i_1+1,\ldots,i_d}|} \frac{|I_{i_1+1,\ldots,i_d-1}|}{|I_{i_1,\ldots,i_d-1}|} - 1 \right| |I_{i_1,\ldots,i_d}|^{-\alpha}.$$

The right-side expression is bounded from above by

$$M \left| \frac{(|i_1|^{p_1} + \ldots + |i_d - 1|^{p_d} + 1)(|i_1+1|^{p_1} + \ldots + |i_d|^{p_d} + 1)}{(|i_1+1|^{p_1} + \ldots + |i_d - 1|^{p_d} + 1)(|i_1|^{p_1} + \ldots + |i_d|^{p_d} + 1)} - 1 \right| (|i_1|^{p_1} + \ldots + |i_d|^{p_d} + 1)^\alpha,$$

which equals

$$M \left| \frac{(|i_d|^{p_d} - |i_d-1|^{p_d})(|i_1|^{p_1} - |i_1+1|^{p_1})}{(|i_1+1|^{p_1} + |i_2|^{p_2} + \ldots + |i_d-1|^{p_d} + 1)(|i_1|^{p_1} + |i_2|^{p_2} + \ldots + |i_d|^{p_d} + 1)} \right| (|i_1|^{p_1} + \ldots + |i_d|^{p_d} + 1)^\alpha.$$

By the Mean Value Theorem, this expression is bounded from above by

$$M\frac{(|i_d|+1)^{p_d-1}(|i_1|+1)^{p_1-1}}{(|i_1|^{p_1}+|i_2|^{p_2}+\ldots+|i_d|^{p_d}+1)^{2-\alpha}}. \tag{22}$$

In the case $|i_1|^{p_1}\leq|i_d|^{p_d}$, this is bounded by

$$M\frac{(|i_d|+1)^{p_d-1}(|i_d|^{\frac{p_d}{p_1}}+1)^{p_1-1}}{(|i_d|^{p_d}+1)^{2-\alpha}}.$$

This expression is uniformly bounded when $p_d-1+\frac{p_d}{p_1}(p_1-1)\leq p_d(2-\alpha)$, that is, when $\alpha\leq\frac{1}{p_1}+\frac{1}{p_d}$, which is ensured by the condition ($\mathrm{v_B}$). In the case $|i_d|^{p_d}\leq|i_1|^{p_1}$, we have the upper bound

$$M\frac{(|i_1|^{\frac{p_1}{p_d}}+1)^{p_d-1}(|i_1|+1)^{p_1-1}}{(|i_1|^{p_1}+1)^{2-\alpha}}.$$

This expression is uniformly bounded when $p_1-1+\frac{p_1}{p_d}(p_d-1)\leq p_1(2-\alpha)$, that is, when $\alpha\leq\frac{1}{p_1}+\frac{1}{p_d}$, which –as we have already seen– is ensured by the condition ($\mathrm{v_B}$).

II. Now we consider x,y so that $x\in I_{i_1,\ldots,i_{d-1},i_d}$ and $y\in I_{i_1,\ldots,i_{d-1},i'_d}$ for some $i_d<i'_d$. To simplify, we will just deal with positive i_d,i'_d, the other cases being analogous.

If $i'_d=i_d+1$, then letting z be the right endpoint of the interval $I_{i_1,\ldots,i_{d-1},i_d}$, we have

$$\frac{|\log Df_1(x)-\log Df_1(y)|}{|x-y|^\alpha}\leq\frac{|\log Df_1(x)-\log Df_1(z)|}{|x-z|^\alpha}+\frac{|\log Df_1(z)-\log Df_1(y)|}{|z-y|^\alpha},$$

and both terms of the sum above are uniformly bounded by the previous case.

Assume henceforth that $i'_d-i_d\geq 2$. By property (20), the value of $|\log Df_1(x)-\log Df_1(y)|$ is bounded from above by

$$\left|\log\frac{|I_{i_1+1,\ldots,i_d}|}{|I_{i_1,\ldots,i_d}|}-\log\frac{|I_{i_1+1,\ldots,i'_d}|}{|I_{i_1,\ldots,i'_d}|}\right|+\left|\log\frac{|I_{i_1+1,\ldots,i_d}|}{|I_{i_1,\ldots,i_d}|}-\log\frac{|I_{i_1+1,\ldots,i_d-1}|}{|I_{i_1,\ldots,i_d-1}|}\right|+\left|\log\frac{|I_{i_1+1,\ldots,i'_d}|}{|I_{i_1,\ldots,i'_d}|}-\log\frac{|I_{i_1+1,\ldots,i'_d-1}|}{|I_{i_1,\ldots,i'_d-1}|}\right|.$$

Since $i\mapsto\frac{|I_{i_1+1,\ldots,i_{d-1},i}|}{|I_{i_1,\ldots,i_{d-1},i}|}$ is a monotonous function, this expression is smaller than or equal to

$$3\left|\log\left(\frac{|I_{i_1+1,\ldots,i'_d}|}{|I_{i_1,\ldots,i'_d}|}\right)-\log\left(\frac{|I_{i_1+1,\ldots,i_d-1}|}{|I_{i_1,\ldots,i_d-1}|}\right)\right|=$$

$$=\left|\log\frac{(|i_1|^{p_1}+|i_2|^{p_2}+\ldots+|i_d-1|^{p_d}+1)(|i_1+1|^{p_1}+|i_2|^{p_2}+\ldots+|i'_d|^{p_d}+1)}{(|i_1+1|^{p_1}+|i_2|^{p_2}+\ldots+|i_d-1|^{p_d}+1)(|i_1|^{p_1}+|i_2|^{p_2}+\ldots+|i'_d|^{p_d}+1)}\right|=$$

$$=\left|\log\left(1+\frac{(i'^{p_d}_d-(i_d-1)^{p_d})(|i_1|^{p_1}-|i_1+1|^{p_1})}{(|i_1+1|^{p_1}+|i_2|^{p_2}+\ldots+|i_d-1|^{p_d}+1)(|i_1|^{p_1}+|i_2|^{p_2}+\ldots+|i'_d|^{p_d}+1)}\right)\right|.$$

Since the expression in brackets in the right-side term equals

$$\frac{(|i_1|^{p_1}+|i_2|^{p_2}+\ldots+|i_d-1|^{p_d}+1)(|i_1+1|^{p_1}+|i_2|^{p_2}+\ldots+|i'_d|^{p_d}+1)}{(|i_1+1|^{p_1}+|i_2|^{p_2}+\ldots+|i_d-1|^{p_d}+1)(|i_1|^{p_1}+|i_2|^{p_2}+\ldots+|i'_d|^{p_d}+1)},$$

it is bounded from below by a positive number. Therefore,

$$\big|\log Df_1(x)-\log Df_1(y)\big|\leq M\left|\frac{(i'^{p_d}_d-i^{p_d}_d)(|i_1|^{p_1}-|i_1+1|^{p_1})}{(|i_1+1|^{p_1}+|i_2|^{p_2}+\ldots+|i_d|^{p_d}+1)(|i_1|^{p_1}+|i_2|^{p_2}+\ldots+|i'_d|^{p_d}+1)}\right|.$$

By The Mean Value Theorem, the last expression is bounded from above by

$$M\frac{i'^{p_d-1}_d(i'_d-i_d)(|i_1|+1)^{p_1-1}}{(|i_1|^{p_1}+\ldots+i^{p_d}_d+1)(|i_1|^{p_1}+\ldots+i'^{p_d}_d+1)}.$$

Thus, in order to get an upper bound for $\frac{|\log f_1'(x) - \log f_1'(y)|}{|x-y|^\alpha}$, we need to estimate the expression

$$\frac{i_d'^{p_d-1}(i_d' - i_d)(|i_1|+1)^{p_1-1}}{(|i_1|^{p_1} + \ldots + i_d^{p_d} + 1)(|i_1|^{p_1} + \ldots + i_d'^{p_d} + 1)|x-y|^\alpha}. \tag{23}$$

We will split the general case into four ones:

(a) $i_d' \leq 2i_d + 1$,

(b) $i_d'^{p_d} \leq |i_1|^{p_1} + \ldots + |i_{d-1}|^{p_{d-1}}$,

(c) $i_d' \geq 2i_d + 2$ and $i_d^{p_d} \geq |i_1|^{p_1} + \ldots + |i_{d-1}|^{p_{d-1}}$,

(d) $i_d' \geq 2i_d + 2$ and $i_d^{p_d} \leq |i_1|^{p_1} + \ldots + |i_{d-1}|^{p_{d-1}} \leq i_d'^{p_d}$.

In case (a), the estimate $|x-y| \geq (i_d' - i_d - 1)|I_{i_1, i_2, \ldots, i_d'}|$ shows that the expression (23) is bounded from above by

$$M \frac{i_d'^{p_d-1}(i_d' - i_d)^{1-\alpha}(|i_1|+1)^{p_1-1}}{(|i_1|^{p_1} + \ldots + i_d^{p_d} + 1)(|i_1|^{p_1} + \ldots + i_d'^{p_d} + 1)^{1-\alpha}}. \tag{24}$$

By the condition $i_d' \leq 2i_d + 1$, the latter expression is smaller than or equal to

$$M \frac{i_d^{p_d-\alpha}(|i_1|+1)^{p_1-1}}{(|i_1|^{p_1} + i_d^{p_d} + 1)^{2-\alpha}}.$$

If $|i_1|^{p_1} \leq i_d^{p_d}$, then $\frac{i_d^{p_d-\alpha}(|i_1|+1)^{p_1-1}}{(|i_1|^{p_1}+i_d^{p_d}+1)^{2-\alpha}} \leq \frac{i_d^{p_d-\alpha}(i_d^{\frac{p_d}{p_1}}+1)^{p_1-1}}{(i_d^{p_d}+1)^{2-\alpha}}$, and the last expression is uniformly bounded by condition (iii$_{\mathrm{B}}$). If $i_d^{p_d} \leq |i_1|^{p_1}$, then $\frac{i_d^{p_d-\alpha}(|i_1|+1)^{p_1-1}}{(|i_1|^{p_1}+i_d^{p_d}+1)^{2-\alpha}} \leq \frac{|i_1|^{\frac{p_1}{p_d}(p_d-\alpha)}(|i_1|+1)^{p_1-1}}{(|i_1|^{p_1}+1)^{2-\alpha}}$, and this is uniformly bounded again by condition (iii$_{\mathrm{B}}$).

In case (b), the expression (23) is still bounded from above by (24), which in its turn is smaller than or equal to

$$M \frac{i_d'^{p_d-\alpha}(|i_1|+1)^{p_1-1}}{(|i_1|^{p_1} + \ldots + |i_{d-1}|^{p_{d-1}}+1)^{2-\alpha}}.$$

Now using the condition $i_d'^{p_d} \leq |i_1|^{p_1} + \ldots + |i_{d-1}|^{p_{d-1}}$, we see that this last expression is bounded from above by

$$M \frac{(|i_1|+1)^{p_1-1}}{(|i_1|^{p_1} + \ldots + |i_{d-1}|^{p_{d-1}}+1)^{1-\alpha+\frac{\alpha}{p_d}}} \leq \frac{(|i_1|+1)^{p_1-1}}{(|i_1|^{p_1}+1)^{1-\alpha+\frac{\alpha}{p_d}}}.$$

Finally, the right-side expression is uniformly bounded by condition (iii$_{\mathrm{B}}$).

In case (c), we first need to estimate the value of $|x-y|$:

$$|x-y| \geq \sum_{i_d<j<i_d'} |I_{i_1,\ldots,i_{d-1},j}| = \sum_{i_d<j<i_d'} \frac{1}{|i_1|^{p_1} + \ldots + |i_{d-1}|^{p_{d-1}} + j^{p_d} + 1} \geq$$

$$\geq \sum_{i_d<j<i_d'} \frac{1}{i_d^{p_d} + j^{p_d} + 1} \geq \sum_{i_d<j<i_d'} \frac{1}{3j^{p_d}} \geq \int_{i_d+1}^{i_d'} \frac{1}{3x^{p_d}} dx \geq$$

$$\geq \frac{M}{(i_d+1)^{p_d-1}} \left(1 - \left(\frac{i_d+1}{i_d'}\right)^{p_d-1}\right) \geq$$

$$\geq \frac{M}{(i_d+1)^{p_d-1}} \left(1 - \left(\frac{1}{2}\right)^{p_d-1}\right) \geq \frac{M}{(i_d+1)^{p_d-1}},$$

where in the second inequality we used the hypothesis $i_d^{p_d} \geq |i_1|^{p_1} + \ldots + |i_{d-1}|^{p_{d-1}}$. Using this, the value of (23) is easily seen to be smaller than or equal to

$$M \frac{i_d'^{p_d-1}(i_d' - i_d)(|i_1|+1)^{p_1-1}(i_d+1)^{(p_d-1)\alpha}}{(|i_1|^{p_1} + \ldots + i_d^{p_d} + 1)(|i_1|^{p_1} + \ldots + i_d'^{p_d} + 1)} \leq M \frac{(|i_1|+1)^{p_1-1}(i_d+1)^{(p_d-1)\alpha}}{|i_1|^{p_1} + \ldots + i_d^{p_d} + 1}.$$

Since by hypothesis we have $i_d^{p_d} \geq |i_1|^{p_1}$, the right-side expression above is bounded from above by

$$M \frac{(i_d^{\frac{p_d}{p_1}} + 1)^{p_1 - 1} (i_d + 1)^{(p_d - 1)\alpha}}{i_d^{p_d} + 1},$$

which is uniformly bounded by the condition (iii$_\mathrm{B}$).

Let us finally consider the case (d). Letting

$$S := 1 + |i_1|^{p_1} + |i_2|^{p_2} + \ldots + |i_{d-1}|^{p_{d-1}},$$

we first observe that

$$|x - y| \geq \sum_{i_d < j < i'_d} |I_{i_1,\ldots,i_{d-1},j}| = \sum_{i_d < j < i'_d} \frac{1}{S + j^{p_d}} \geq \int_{i_d+1}^{i'_d} \frac{dx}{x^{p_d} + S} \geq \int_{i_d+1}^{i'_d} \frac{dx}{(x + S^{1/p_d})^{p_d}}.$$

The last integral equals

$$\frac{1}{(p_d - 1)} \left[\frac{1}{(i_d + 1 + S^{1/p_d})^{p_d - 1}} - \frac{1}{(i'_d + S^{1/p_d})^{p_d - 1}} \right] = \frac{1}{(p_d - 1)} \left[\frac{(i'_d + S^{1/p_d})^{p_d - 1} - (i_d + 1 + S^{1/p_d})^{p_d - 1}}{(i_d + 1 + S^{1/p_d})^{p_d - 1} (i'_d + S^{1/p_d})^{p_d - 1}} \right].$$

Using the Mean Value Theorem, we conclude that

$$|x - y| \geq \frac{i'_d - i_d - 1}{(i_d + 1 + S^{1/p_d})^{p_d - 1} \, (i'_d + S^{1/p_d})}.$$

Using this, we conclude that (23) is smaller than or equal to

$$\frac{i_d'^{p_d - 1} (i'_d - i_d) (|i_1| + 1)^{p_1 - 1} (i_d + 1 + S^{1/p_d})^{\alpha(p_d - 1)} (i'_d + S^{1/p_d})^{\alpha}}{(|i_1|^{p_1} + \ldots + i_d^{p_d} + 1)(|i_1|^{p_1} + \ldots + i_d'^{p_d} + 1)(i'_d - i_d - 1)^{\alpha}} \leq$$

$$\leq \frac{(|i_1| + 1)^{p_1 - 1} (i_d + 1 + S^{1/p_d})^{\alpha(p_d - 1)} (i'_d + S^{1/p_d})^{\alpha}}{(|i_1|^{p_1} + \ldots + i_d^{p_d} + 1)(i'_d - i_d - 1)^{\alpha}}. \quad (25)$$

By hypothesis, $1 + i_d^{p_d} \leq S$, thus $i_d \leq S^{1/p_d}$. Since (by definition) $|i_1|^{p_1} \leq S$, this yields

$$\frac{(|i_1| + 1)^{p_1 - 1} (i_d + 1 + S^{1/p_d})^{\alpha(p_d - 1)}}{(|i_1|^{p_1} + \ldots + i_d^{p_d} + 1)} \leq M S^{\frac{p_1 - 1}{p_1} + \frac{\alpha(p_d - 1)}{p_d} - 1}. \quad (26)$$

By hypothesis, we also have $S \leq 1 + i_d'^{p_d}$ and $i'_d \geq 2 i_d + 2$, which gives

$$\frac{(i'_d + S^{1/p_d})^{\alpha}}{(i'_d - i_d - 1)^{\alpha}} \leq M. \quad (27)$$

Putting together (26) and (27), and using again that $S \leq 1 + i_d'^{p_d}$, we conclude that the expression in (25) is bounded from above by

$$M S^{\frac{p_1 - 1}{p_1} + \frac{\alpha(p_d - 1)}{p_d} - 1},$$

which is uniformly bounded by the condition (iii$_\mathrm{B}$).

III. Finally, we consider x, y so that $x \in I_{i_1,\ldots,i_{d-1},i_d}$ and $y \in I_{i'_1,\ldots,i'_{d-1},i'_d}$ for $(i_1, \ldots, i_{d-1}) \prec (i'_1, \ldots, i'_{d-1})$, where $\prec$ stands for the lexicographic ordering. Letting z (resp. z') be the right endpoint (resp. left endpoint) of $\bigcup_{j \in \mathbb{Z}} I_{i_1,\ldots,i_{d-1},j}$ (resp. $\bigcup_{j \in \mathbb{Z}} I_{i'_1,\ldots,i'_{d-1},j}$), by construction we have $Df_1(z) = Df_1(z') = 1$. Hence

$$\frac{|\log Df_1(x) - \log Df_1(y)|}{|x - y|^{\alpha}} \leq \frac{|\log Df_1(x) - \log Df_1(z)|}{|x - z|^{\alpha}} + \frac{|\log Df_1(z') - \log Df_1(y)|}{|z' - y|^{\alpha}},$$

and both terms of the sum above are uniformly bounded by the previous case.

To conclude the proof of the regularity of f_1, notice that slightly modified arguments apply to the inverse f_1^{-1}, thus showing that f_1 is a $C^{1+\alpha}$-diffeomorphism.

3.4 For $2 \leq j \leq d-1$, the map f_j is a $C^{1+\alpha}$-diffeomorphism

I. We first consider x, y in the same interval $I_{i_1,\ldots,i_d}$. We have

$$\frac{|\log Df_j(x) - \log Df_j(y)|}{|x-y|} \leq \frac{M}{|I_{i_1,\ldots,i_d}|} \left| \frac{|I_{i_1,\ldots,i_d}|}{|I_{i_1,\ldots,i_j+i_{j-1},\ldots,i_d}|} \frac{|I_{i_1,\ldots,i_j+i_{j-1},\ldots,i_d-1}|}{|I_{i_1,\ldots,i_d-1}|} - 1 \right|.$$

Hence,

$$\frac{|\log Df_j(x) - \log Df_j(y)|}{|x-y|^\alpha} \leq M \left| \frac{|I_{i_1,\ldots,i_d}|}{|I_{i_1,\ldots,i_j+i_{j-1},\ldots,i_d}|} \frac{|I_{i_1,\ldots,i_j+i_{j-1},\ldots,i_d-1}|}{|I_{i_1,\ldots,i_d-1}|} - 1 \right| |I_{i_1,\ldots,i_d}|^{-\alpha}.$$

One readily checks that the right-side expression equals

$$M \left| \frac{(|i_d|^{p_d} - |i_d-1|^{p_d})(|i_j|^{p_j} - |i_j+i_{j-1}|^{p_j})}{(|i_1|^{p_1} + \ldots + |i_j+i_{j-1}|^{p_j} + \ldots + |i_d-1|^{p_d} + 1)(|i_1|^{p_1} + \ldots + |i_j|^{p_j} + \ldots + |i_d|^{p_d} + 1)^{1-\alpha}} \right|.$$

By The Mean Value Theorem, and since $p_{j-1} > p_j$, the last expression is bounded from above by

$$M \frac{(|i_d|+1)^{p_d-1}(|i_j|+|i_{j-1}|)^{p_j-1}|i_{j-1}|}{(|i_1|^{p_1} + \ldots + |i_j|^{p_j} + \ldots + |i_d|^{p_d} + 1)^{2-\alpha}}. \tag{28}$$

To estimate this expression, let us first assume that $|i_j|^{p_j} \leq |i_{j-1}|^{p_{j-1}}$. In this case, (28) is bounded from above by

$$M \frac{(|i_d|+1)^{p_d-1}(|i_{j-1}|^{\frac{p_{j-1}}{p_j}} + |i_{j-1}|)^{p_j-1}|i_{j-1}|}{(|i_{j-1}|^{p_{j-1}} + |i_d|^{p_d} + 1)^{2-\alpha}}. \tag{29}$$

If $|i_d|^{p_d} \leq |i_{j-1}|^{p_{j-1}}$, then this expression is smaller than or equal to

$$M \frac{(|i_{j-1}|^{\frac{p_{j-1}}{p_d}} + 1)^{p_d-1}(|i_{j-1}|^{\frac{p_{j-1}}{p_j}} + |i_{j-1}|)^{p_j-1}|i_{j-1}|}{(|i_{j-1}|^{p_{j-1}} + 1)^{2-\alpha}}.$$

Since $p_{j-1}/p_j \geq 1$, this is uniformly bounded if

$$\frac{p_{j-1}}{p_d}(p_d - 1) + \frac{p_{j-1}}{p_j}(p_j - 1) + 1 - p_{j-1}(2-\alpha) \ \leq \ 0,$$

that is, $\alpha \leq \frac{1}{p_d} + \frac{1}{p_j} - \frac{1}{p_{j-1}}$, and this is ensured by conditions ($\mathrm{i_B}$) and ($\mathrm{v_B}$). If $|i_{j-1}|^{p_{j-1}} \leq |i_d|^{p_d}$, then the expression (29) is smaller than or equal to

$$M \frac{(|i_d|+1)^{p_d-1}(|i_d|^{\frac{p_d}{p_j}} + |i_d|^{\frac{p_d}{p_{j-1}}})^{p_j-1}|i_d|^{\frac{p_d}{p_{j-1}}}}{(|i_d|^{p_d} + 1)^{2-\alpha}}.$$

Since $p_d/p_{j-1} \leq p_d/p_j$, this is uniformly bounded if

$$p_d - 1 + \frac{p_d}{p_j}(p_j - 1) + \frac{p_d}{p_{j-1}} - p_d(2-\alpha) \ \leq \ 0,$$

which is again ensured by conditions ($\mathrm{i_B}$) and ($\mathrm{v_B}$).

Assume now that $|i_{j-1}|^{p_{j-1}} \leq |i_j|^{p_j}$. In this case, (28) is bounded from above by

$$M \frac{(|i_d|+1)^{p_d-1}(|i_j| + |i_j|^{\frac{p_j}{p_{j-1}}})^{p_j-1}|i_j|^{\frac{p_j}{p_{j-1}}}}{(|i_j|^{p_j} + |i_d|^{p_d} + 1)^{2-\alpha}}.$$

Proceeding as in the previous case, one readily checks that this expression is uniformly bounded when $\alpha \leq \frac{1}{p_d} + \frac{1}{p_j} - \frac{1}{p_{j-1}}$, which is ensured by conditions ($\mathrm{i_B}$) and ($\mathrm{v_B}$).

II. Now we consider the case where $x \in I_{i_1,i_2,\ldots,i_d}$ and $y \in I_{i_1,i_2,\ldots,i'_d}$ for different i_d, i'_d. As in the case of f_1, we may restrict ourselves to the case where $i'_d - i_d \geq 2$, with i_d, i'_d both positive.

Once again, property (20) implies that $|\log Df_j(x) - \log Df_j(y)|$ is smaller than or equal to the sum

$$\left|\log \frac{|I_{i_1,\ldots,i_j+i_{j-1},\ldots,i_d}|}{|I_{i_1,\ldots,i_j,\ldots,i_d}|} - \log \frac{|I_{i_1,\ldots,i_j+i_{j-1},\ldots,i'_d}|}{|I_{i_1,\ldots,i_j,\ldots,i'_d}|}\right| + \\ + \left|\log \frac{|I_{i_1,\ldots,i_j+i_{j-1},\ldots,i_d}|}{|I_{i_1,\ldots,i_j,\ldots,i_d}|} - \log \frac{|I_{i_1,\ldots,i_j+i_{j-1},\ldots,i_d-1}|}{|I_{i_1,\ldots,i_j,\ldots,i_d-1}|}\right| + \\ + \left|\log \frac{|I_{i_1,\ldots,i_j+i_{j-1},\ldots,i'_d-1}|}{|I_{i_1,\ldots,i_j,\ldots,i'_d-1}|} - \log \frac{|I_{i_1,\ldots,i_j+i_{j-1},\ldots,i'_d}|}{|I_{i_1,\ldots,i_j,\ldots,i'_d}|}\right|.$$

As in previous estimates of similar expressions, we have

$$|\log Df_j(x) - \log Df_j(y)| \leq 3 \left|\log\left(\frac{|I_{i_1,\ldots,i_j+i_{j-1},\ldots,i_d-1}|}{|I_{i_1,\ldots,i_j,\ldots,i_d-1}|}\right) - \log\left(\frac{|I_{i_1,\ldots,i_j+i_{j-1},\ldots,i'_d}|}{|I_{i_1,\ldots,i_j,\ldots,i'_d}|}\right)\right|.$$

The last expression equals

$$3\left|\log \frac{(|i_1|^{p_1}+\ldots+|i_j|^{p_j}+\ldots+|i_d-1|^{p_d}+1)(|i_1|^{p_1}+\ldots+|i_j+i_{j-1}|^{p_j}+\ldots+|i'_d|^{p_d}+1)}{(|i_1|^{p_1}+\ldots+|i_j+i_{j-1}|^{p_j}+\ldots+|i_d-1|^{p_d}+1)(|i_1|^{p_1}+\ldots+|i_j|^{p_j}+\ldots+|i'_d|^{p_d}+1)}\right|,$$

that is,

$$3\left|\log\left(1+\frac{(|i_d-1|^{p_d}-i'^{p_d}_d)(|i_j+i_{j-1}|^{p_j}-|i_j|^{p_j})}{(|i_1|^{p_1}+\ldots+|i_j+i_{j-1}|^{p_j}+\ldots+|i_d-1|^{p_d}+1)(|i_1|^{p_1}+\ldots+|i_j|^{p_j}+\ldots+|i'_d|^{p_d}+1)}\right)\right|. \tag{30}$$

The expression into brackets in the right-side term equals

$$\frac{(|i_1|^{p_1}+\ldots+|i_j|^{p_j}+\ldots+|i_d-1|^{p_d}+1)(|i_1|^{p_1}+\ldots+|i_j+i_{j-1}|^{p_j}+\ldots+|i'_d|^{p_d}+1)}{(|i_1|^{p_1}+\ldots+|i_j+i_{j-1}|^{p_j}+\ldots+|i_d-1|^{p_d}+1)(|i_1|^{p_1}+\ldots+|i_j|^{p_j}+\ldots+|i'_d|^{p_d}+1)},$$

hence it is uniformly bounded from below by a positive number. Therefore, the value of (30) is smaller than or equal to

$$M\left|\frac{(i^{p_d}_d-i'^{p_d}_d)(|i_j+i_{j-1}|^{p_j}-|i_j|^{p_j})}{(|i_1|^{p_1}+\ldots+|i_j+i_{j-1}|^{p_j}+\ldots+|i_d|^{p_d}+1)(|i_1|^{p_1}+\ldots+|i_j|^{p_j}+\ldots+|i'_d|^{p_d}+1)}\right|.$$

Using the Mean Value Theorem and the condition $p_{j-1} > p_j$, this last expression is easily seen to be bounded from above by

$$M\frac{i'^{p_d-1}_d(i'_d-i_d)(|i_j|+|i_{j-1}|)^{p_j-1}|i_{j-1}|}{(|i_1|^{p_1}+\ldots+|i_j|^{p_j}+\ldots+i^{p_d}_d+1)(|i_1|^{p_1}+\ldots+|i_j|^{p_j}+\ldots+i'^{p_d}_d+1)}.$$

Therefore, $\frac{|\log Df_j(x)-\log Df_j(y)|}{|x-y|^\alpha}$ is smaller than or equal to

$$M\frac{i'^{p_d-1}_d(i'_d-i_d)(|i_j|+|i_{j-1}|)^{p_j-1}|i_{j-1}|}{(|i_1|^{p_1}+\ldots+|i_j|^{p_j}+\ldots+i^{p_d}_d+1)(|i_1|^{p_1}+\ldots+|i_j|^{p_j}+\ldots+i'^{p_d}_d+1)|x-y|^\alpha}. \tag{31}$$

In order to estimate this expression, we will again consider separately the cases (a), (b), (c) and (d) of the previous section.

The case (a) is $i'_d \leq 2i_d+1$. Here the estimate $|x-y| \geq (i'_d-i_d-1)|I_{i_1,i_2,\ldots,i'_d}|$ shows that (31) is bounded from above by

$$M\frac{i'^{p_d-1}_d(i'_d-i_d)^{1-\alpha}(|i_j|+|i_{j-1}|)^{p_j-1}|i_{j-1}|}{(|i_1|^{p_1}+\ldots+|i_j|^{p_j}+\ldots+i^{p_d}_d+1)(|i_1|^{p_1}+\ldots+|i_j|^{p_j}+\ldots+i'^{p_d}_d+1)^{1-\alpha}}, \tag{32}$$

which is smaller than or equal to

$$M\frac{i^{p_d-\alpha}_d(|i_j|+|i_{j-1}|)^{p_j-1}|i_{j-1}|}{(|i_1|^{p_1}+\ldots+|i_j|^{p_j}+\ldots+i^{p_d}_d+1)^{2-\alpha}}. \tag{33}$$

There are three subcases:
– If $|i_j|^{p_j} \le |i_{j-1}|^{p_{j-1}}$ and $i_d^{p_d} \le |i_{j-1}|^{p_{j-1}}$, then

$$M\frac{i_d^{p_d-\alpha}(|i_j|+|i_{j-1}|)^{p_j-1}|i_{j-1}|}{(|i_1|^{p_1}+\ldots+|i_j|^{p_j}+\ldots+i_d^{p_d}+1)^{2-\alpha}} \le M\frac{|i_{j-1}|^{\frac{p_{j-1}}{p_d}(p_d-\alpha)}(|i_{j-1}|^{\frac{p_{j-1}}{p_j}}+|i_{j-1}|)^{p_j-1}|i_{j-1}|}{(|i_{j-1}|^{p_{j-1}}+1)^{2-\alpha}}.$$

The last expression is easily seen to be uniformly bounded by condition ($\mathrm{iv_B}$).
– If $|i_j|^{p_j} \le i_d^{p_d}$ and $|i_{j-1}|^{p_{j-1}} \le i_d^{p_d}$, then

$$M\frac{i_d^{p_d-\alpha}(|i_j|+|i_{j-1}|)^{p_j-1}|i_{j-1}|}{(|i_1|^{p_1}+\ldots+|i_j|^{p_j}+\ldots+i_d^{p_d}+1)^{2-\alpha}} \le M\frac{i_d^{p_d-\alpha}(i_d^{\frac{p_d}{p_j}}+i_d^{\frac{p_d}{p_{j-1}}})^{p_j-1}i_d^{\frac{p_d}{p_{j-1}}}}{(i_d^{p_d}+1)^{2-\alpha}},$$

and the last expression is uniformly bounded by condition ($\mathrm{iv_B}$).
– If $|i_{j-1}|^{p_{j-1}} \le |i_j|^{p_j}$ and $i_d^{p_d} \le |i_j|^{p_j}$, then one has

$$M\frac{i_d^{p_d-\alpha}(|i_j|+|i_{j-1}|)^{p_j-1}|i_{j-1}|}{(|i_1|^{p_1}+\ldots+|i_j|^{p_j}+\ldots+i_d^{p_d}+1)^{2-\alpha}} \le M\frac{|i_j|^{\frac{p_j}{p_d}(p_d-\alpha)}(|i_j|+|i_j|^{\frac{p_j}{p_{j-1}}})^{p_j-1}|i_j|^{\frac{p_j}{p_{j-1}}}}{(|i_j|^{p_j}+1)^{2-\alpha}},$$

and the last expression is uniformly bounded by condition ($\mathrm{iv_B}$).

In case (b), we still have the upper bound (32) for (31). Now, using the condition

$$i_d'^{p_d} \le |i_1|^{p_1}+\ldots+|i_{d-1}|^{p_{d-1}},$$

the value of (32) is easily seen to be bounded from above by

$$M\frac{i_d'^{p_d-\alpha}(|i_j|+|i_{j-1}|)^{p_j-1}|i_{j-1}|}{(|i_1|^{p_1}+\ldots+|i_{d-1}|^{p_{d-1}}+1)^{2-\alpha}} \;\le\; M\frac{(|i_j|+|i_{j-1}|)^{p_j-1}|i_{j-1}|}{(|i_1|^{p_1}+\ldots+|i_{d-1}|^{p_{d-1}}+1)^{1-\alpha+\frac{\alpha}{p_d}}}.$$

To estimate the right-side expression of this inequality, we consider two subcases:
– If $|i_{j-1}|^{p_{j-1}} \le |i_j|^{p_j}$, then

$$M\frac{(|i_j|+|i_{j-1}|)^{p_j-1}|i_{j-1}|}{(|i_1|^{p_1}+\ldots+|i_{d-1}|^{p_{d-1}}+1)^{1-\alpha+\frac{\alpha}{p_d}}} \le M\frac{(|i_j|+|i_j|^{\frac{p_j}{p_{j-1}}})^{p_j-1}|i_j|^{\frac{p_j}{p_{j-1}}}}{(|i_j|^{p_j}+1)^{1-\alpha+\frac{\alpha}{p_d}}},$$

and the last expression is easily seen to be uniformly bounded by condition ($\mathrm{iv_B}$).
– If $|i_j|^{p_j} \le |i_{j-1}|^{p_{j-1}}$, then

$$M\frac{(|i_j|+|i_{j-1}|)^{p_j-1}|i_{j-1}|}{(|i_1|^{p_1}+\ldots+|i_{d-1}|^{p_{d-1}}+1)^{1-\alpha+\frac{\alpha}{p_d}}} \le M\frac{(|i_{j-1}|^{\frac{p_{j-1}}{p_j}}+|i_{j-1}|)^{p_j-1}|i_{j-1}|}{(|i_{j-1}|^{p_{j-1}}+1)^{1-\alpha+\frac{\alpha}{p_d}}},$$

and the last expression is easily seen to be uniformly bounded by condition ($\mathrm{iv_B}$).

In case (c), we had the estimate

$$|x-y| \;\ge\; \frac{M}{(i_d+1)^{p_d-1}}, \tag{34}$$

which shows that (31) is bounded from above by

$$M\frac{i_d'^{p_d-1}(i_d'-i_d)(|i_j|+|i_{j-1}|)^{p_j-1}|i_{j-1}|(i_d+1)^{(p_d-1)\alpha}}{(|i_1|^{p_1}+\ldots+|i_j|^{p_j}+\ldots+i_d^{p_d}+1)(|i_1|^{p_1}+\ldots+|i_j|^{p_j}+\ldots+i_d'^{p_d}+1)}.$$

This is smaller than or equal to

$$M\frac{(|i_j|+|i_{j-1}|)^{p_j-1}|i_{j-1}|(i_d+1)^{(p_d-1)\alpha}}{|i_1|^{p_1}+\ldots+|i_j|^{p_j}+\ldots+i_d^{p_d}+1}. \tag{35}$$

Now from the condition $i_d^{p_d} \geq |i_1|^{p_1} + \ldots + |i_{d-1}|^{p_{d-1}}$ it follows that $|i_j|^{p_j} \leq i_d^{p_d}$ and $|i_{j-1}|^{p_{j-1}} \leq i_d^{p_d}$. Therefore, (35) is bounded from above by

$$M \frac{(i_d^{p_d/p_j} + i_d^{p_d/p_{j-1}})^{p_j-1} i_d^{p_d/p_{j-1}} (i_d+1)^{(p_d-1)\alpha}}{i_d^{p_d}+1},$$

and this expression is easily seen to be uniformly bounded by condition ($\mathrm{iv_B}$).

In case (d), we had the estimate

$$|x-y| \geq \frac{i'_d - i_d - 1}{(i_d + 1 + S^{1/p_d})^{p_d-1}\,(i'_d + S^{1/p_d})}, \tag{36}$$

where $S := 1 + |i_1|^{p_1} + |i_2|^{p_2} \ldots |i_{d-1}|^{p_{d-1}}$. Thus, (31) is bounded from above by

$$M \frac{i_d'^{p_d-1}(i'_d - i_d)(|i_j|+|i_{j-1}|)^{p_j-1}|i_{j-1}|(i_d+1+S^{1/p_d})^{\alpha(p_d-1)}(i'_d+S^{1/p_d})^{\alpha}}{(|i_1|^{p_1}+\ldots+|i_j|^{p_j}+\ldots+i_d^{p_d}+1)(|i_1|^{p_1}+\ldots+|i_j|^{p_j}+\ldots+i_d'^{p_d}+1)(i'_d-i_d-1)^{\alpha}},$$

hence by

$$M \frac{(|i_j|+|i_{j-1}|)^{p_j-1}|i_{j-1}|(i_d+1+S^{1/p_d})^{\alpha(p_d-1)}(i'_d+S^{1/p_d})^{\alpha}}{(|i_1|^{p_1}+\ldots+|i_j|^{p_j}+\ldots+i_d^{p_d}+1)(i'_d-i_d-1)^{\alpha}}.$$

Since the condition $1 + i_d^{p_d} \leq S$ yields $i_d \leq S^{1/p_d}$, this expression is smaller than or equal to

$$M \; \frac{(|i_j|+|i_{j-1}|)^{p_j-1}|i_{j-1}|(i'_d+S^{1/p_d})^{\alpha}}{(i'_d-i_d-1)^{\alpha}} \; S^{\frac{\alpha(p_d-1)}{p_d}-1}.$$

The conditions $1 + i_d^{p_d} \leq S$ and $p_{j-1} \geq p_j$ also yield $|i_j| \leq S^{1/p_j}$ and $|i_{j-1}| \leq S^{1/p_{j-1}} \leq S^{1/p_j}$, thus showing that the last expression is smaller than or equal to

$$M \; \frac{(i'_d+S^{1/p_d})^{\alpha}}{(i'_d-i_d-1)^{\alpha}} \; S^{\frac{p_j-1}{p_j}+\frac{1}{p_{j-1}}+\frac{\alpha(p_d-1)}{p_d}-1}.$$

Using the conditions $i'_d \geq 2i_d + 2$ and $S \leq 1 + i_d'^{p_d}$, this last expression is easily seen to be bounded from above by

$$M S^{\frac{p_j-1}{p_j}+\frac{1}{p_{j-1}}+\frac{\alpha(p_d-1)}{p_d}-1},$$

which is uniformly bounded by the condition ($\mathrm{iv_B}$).

III. Finally, in the case where $x \in I_{i_1,\ldots,i_d}$ and $y \in I_{i'_1,\ldots,i'_d}$ for different $(i_1,\ldots,i_{d-1})$ and $(i'_1,\ldots,i'_{d-1})$, one may apply the same argument as that of f_1.

3.5 The map f_d is a $C^{1+\alpha}$-diffeomorphism

I. First we consider x, y in the same interval $I_{i_1,\ldots,i_d}$. We have

$$\frac{|\log Df_d(x) - \log Df_d(y)|}{|x-y|} \leq \frac{M}{|I_{i_1,\ldots,i_d}|} \left| \frac{|I_{i_1,\ldots,i_d}|}{|I_{i_1,\ldots,i_d+i_{d-1}}|} \, \frac{|I_{i_1,\ldots,i_d+i_{d-1}-1}|}{|I_{i_1,\ldots,i_d-1}|} - 1 \right|,$$

hence

$$\frac{|\log Df_d(x) - \log Df_d(y)|}{|x-y|^{\alpha}} \leq \; M \left| \frac{|I_{i_1,\ldots,i_d}|}{|I_{i_1,\ldots,i_d+i_{d-1}}|} \, \frac{|I_{i_1,\ldots,i_d+i_{d-1}-1}|}{|I_{i_1,\ldots,i_d-1}|} - 1 \right| |I_{i_1,\ldots,i_d}|^{-\alpha}.$$

The right-side term above is smaller than or equal to

$$M \left| \frac{(|i_1|^{p_1}+\ldots+|i_d-1|^{p_d}+1)(|i_1|^{p_1}+\ldots+|i_d+i_{d-1}|^{p_d}+1)}{(|i_1|^{p_1}+\ldots+|i_d+i_{d-1}-1|^{p_d}+1)(|i_1|^{p_1}+\ldots+|i_d|^{p_d}+1)} - 1 \right| (|i_1|^{p_1}+\ldots+|i_d|^{p_d}+1)^{\alpha},$$

which equals

$$M \left| \frac{\sum_{k=1}^{d-1}|i_k|^{p_k}(|i_d+i_{d-1}|^{p_d}-|i_d|^{p_d}) + \sum_{k=1}^{d-1}|i_k|^{p_k}(|i_d-1|^{p_d}-|i_d+i_{d-1}-1|^{p_d}) + C}{(|i_1|^{p_1}+\ldots+|i_d+i_{d-1}-1|^{p_d}+1)(|i_1|^{p_1}+\ldots+|i_d|^{p_d}+1)} \right| \left(S+|i_d|^{p_d}\right)^{\alpha},$$

where

$$C := |i_d - 1|^{p_d} |i_d + i_{d-1}|^{p_d} - |i_d + i_{d-1} - 1|^{p_d} |i_d|^{p_d} + |i_d - 1|^{p_d} - |i_d + i_{d-1} - 1|^{p_d} + |i_d + i_{d-1}|^{p_d} - |i_d|^{p_d}$$

and, as before, $S := 1 + |i_1|^{p_1} + |i_2|^{p_2} \ldots |i_{d-1}|^{p_{d-1}}$. By the Mean Value Theorem, the last expression is bounded from above by

$$M \frac{\sum_{k=1}^{d-1} |i_k|^{p_k} (|i_d| + |i_{d-1}|)^{p_d - 1} |i_{d-1}| + \sum_{k=1}^{d-1} |i_k|^{p_k} (|i_d| + |i_{d-1}| + 1)^{p_d - 1} |i_{d-1}| + C'}{(|i_1|^{p_1} + \ldots + |i_d + i_{d-1} - 1|^{p_d} + 1)(|i_1|^{p_1} + \ldots + |i_d|^{p_d} + 1)^{1-\alpha}},$$

where

$$C' := |i_d + i_{d-1}|^{p_d} (|i_d| + 1)^{p_d - 1} + |i_d|^{p_d} (|i_d| + |i_{d-1}| + 1)^{p_d - 1} + \\ + (|i_d| + |i_{d-1}| + 1)^{p_d - 1} |i_{d-1}| + (|i_d| + |i_{d-1}|)^{p_d - 1} |i_{d-1}|.$$

To get an upper bound for this last expression, it is enough to do so for

$$\frac{|i_d + i_{d-1}|^{p_d} (|i_d| + 1)^{p_d - 1}}{(|i_1|^{p_1} + \ldots + |i_d + i_{d-1} - 1|^{p_d} + 1)(|i_1|^{p_1} + \ldots + |i_d|^{p_d} + 1)^{1-\alpha}} \tag{37}$$

and

$$\frac{|i_k|^{p_k} (|i_d| + |i_{d-1}|)^{p_d - 1} |i_{d-1}|}{(|i_1|^{p_1} + \ldots + |i_d + i_{d-1} - 1|^{p_d} + 1)(|i_1|^{p_1} + \ldots + |i_d|^{p_d} + 1)^{1-\alpha}}, \tag{38}$$

where $1 \leq k \leq n$.

Expression (37) may be written as

$$\frac{|i_d + i_{d-1}|^{p_d}}{(|i_1|^{p_1} + \ldots + |i_d + i_{d-1} - 1|^{p_d} + 1)} \; \frac{(|i_d| + 1)^{p_d - 1}}{(|i_1|^{p_1} + \ldots + |i_d|^{p_d} + 1)^{1-\alpha}}.$$

The first factor is uniformly bounded, whereas the second is smaller than or equal to

$$\frac{(|i_d| + 1)^{p_d - 1}}{(|i_d|^{p_d} + 1)^{1-\alpha}}.$$

This last expression is uniformly bounded provided that $p_d - 1 - p_d(1-\alpha) \leq 0$, which is a consequence of condition ($\mathrm{v_B}$).

Concerning expression (38), notice that since $p_{d-1} > p_d$, it is smaller than or equal to

$$\frac{|i_k|^{p_k} (|i_d| + |i_{d-1}|)^{p_d - 1} |i_{d-1}|}{(|i_1|^{p_1} + \ldots + |i_d|^{p_d} + 1)^{2-\alpha}} \leq \frac{(|i_d| + |i_{d-1}|)^{p_d - 1} |i_{d-1}|}{(|i_1|^{p_1} + \ldots + |i_d|^{p_d} + 1)^{1-\alpha}}.$$

On the one hand, if $|i_d|^{p_d} \leq |i_{d-1}|^{p_{d-1}}$, then

$$\frac{(|i_d| + |i_{d-1}|)^{p_d - 1} |i_{d-1}|}{(|i_1|^{p_1} + \ldots + |i_d|^{p_d} + 1)^{1-\alpha}} \leq \frac{(|i_{d-1}|^{\frac{p_{d-1}}{p_d}} + |i_{d-1}|)^{p_d - 1} |i_{d-1}|}{(|i_{d-1}|^{p_{d-1}} + 1)^{1-\alpha}},$$

and the last term is uniformly bounded when $\frac{p_{d-1}}{p_d}(p_d - 1) + 1 \leq p_{d-1}(1-\alpha)$, which is ensured by condition ($\mathrm{v_B}$). On the other hand, if $|i_{d-1}|^{p_{d-1}} \leq |i_d|^{p_d}$, then

$$\frac{(|i_d| + |i_{d-1}|)^{p_d - 1} |i_{d-1}|}{(|i_1|^{p_1} + \ldots + |i_d|^{p_d} + 1)^{1-\alpha}} \leq \frac{(|i_d| + |i_d|^{\frac{p_d}{p_{d-1}}})^{p_d - 1} |i_d|^{\frac{p_d}{p_{d-1}}}}{(|i_d|^{p_d} + 1)^{1-\alpha}},$$

which is uniformly bounded when $p_d - 1 + \frac{p_d}{p_{d-1}} \leq p_d(1-\alpha)$, that is, when condition ($\mathrm{v_B}$) holds.

II. Next we consider the case where $x \in I_{i_1, i_2, \ldots, i_d}$ and $y \in I_{i_1, i_2, \ldots, i'_d}$, with $i'_d - i_d \geq 2$. (For the case where $i'_d = i_d + 1$, we apply a similar argument to that of the previous maps.) Once again, we will only deal with positive i_d, i'_d. As in previous cases, $|\log Df_d(x) - \log Df_d(y)|$ is bounded from above by

$$3 \left| \log \frac{|I_{i_1, \ldots, i_d + i_{d-1} - 1}|}{|I_{i_1, \ldots, i_d - 1}|} \frac{|I_{i_1, \ldots, i'_d}|}{|I_{i_1, \ldots, i'_d + i_{d-1}}|} \right| \leq M \left| \log \frac{(|i_1|^{p_1} + \ldots + i_d^{p_d} + 1)(|i_1|^{p_1} + \ldots + |i'_d + i_{d-1}|^{p_d} + 1)}{(|i_1|^{p_1} + \ldots + |i_d + i_{d-1}|^{p_d} + 1)(|i_1|^{p_1} + \ldots + i'^{p_d}_d + 1)} \right|.$$

Note that the right-side term may be rewritten as

$$M\left|\log\left(1+\frac{\sum_{k=1}^{d-1}|i_k|^{p_k}(|i'_d+i_{d-1}|^{p_d}-i'^{p_d}_d)+\sum_{k=1}^{d-1}|i_k|^{p_k}(i^{p_d}_d-|i_d+i_{d-1}|^{p_d})+\bar{C}}{(|i_1|^{p_1}+\ldots+|i_d+i_{d-1}|^{p_d}+1)(|i_1|^{p_1}+\ldots+i'^{p_d}_d+1)}\right)\right|, \tag{39}$$

where

$$\begin{aligned}\bar{C} &:= i^{p_d}_d|i'_d+i_{d-1}|^{p_d}-|i_d+i_{d-1}|^{p_d}i'^{p_d}_d+i^{p_d}_d-i'^{p_d}_d+|i'_d+i_{d-1}|^{p_d}-|i_d+i_{d-1}|^{p_d}\\ &= i^{p_d}_d\left[|i'_d+i_{d-1}|^{p_d}-i'^{p_d}_d\right]+i'^{p_d}_d\left[i^{p_d}_d-|i_d+i_{d-1}|^{p_d}\right]+\left[i^{p_d}_d-|i_d+i_{d-1}|^{p_d}\right]+\left[|i'_d+i_{d-1}|^{p_d}-i'^{p_d}_d\right].\end{aligned}$$

Since the expression

$$\frac{(|i_1|^{p_1}+\ldots+i^{p_d}_d+1)(|i_1|^{p_1}+\ldots+|i'_d+i_{d-1}|^{p_d}+1)}{(|i_1|^{p_1}+\ldots+|i_d+i_{d-1}|^{p_d}+1)(|i_1|^{p_1}+\ldots+i'^{p_d}_d+1)}$$

is uniformly bounded from below by a positive number, (39) is bounded from above by

$$M\left|\frac{\sum_{k=1}^{d-1}|i_k|^{p_k}(|i'_d+i_{d-1}|^{p_d}-i'^{p_d}_d)+\sum_{k=1}^{d-1}|i_k|^{p_k}(i^{p_d}_d-|i_d+i_{d-1}|^{p_d})+\bar{C}}{(|i_1|^{p_1}+\ldots+|i_d+i_{d-1}|^{p_d}+1)(|i_1|^{p_1}+\ldots+i'^{p_d}_d+1)}\right|.$$

By The Mean Value Theorem, and since $p_{d-1}>p_d$, this last expression is smaller than or equal to

$$M\frac{\sum_{k=1}^{d-1}|i_k|^{p_k}(i'_d+|i_{d-1}|)^{p_d-1}|i_{d-1}|+\sum_{k=1}^{d-1}|i_k|^{p_k}(i_d+|i_{d-1}|)^{p_d-1}|i_{d-1}|+\bar{C}'}{(|i_1|^{p_1}+\ldots+i^{p_d}_d+1)(|i_1|^{p_1}+\ldots+i'^{p_d}_d+1)},$$

where $\bar{C}'$ equals

$$i^{p_d}_d(i'_d+|i_{d-1}|)^{p_d-1}|i_{d-1}|+i'^{p_d}_d(i_d+|i_{d-1}|)^{p_d-1}|i_{d-1}|+(i_d+|i_{d-1}|)^{p_d-1}|i_{d-1}|+(i'_d+|i_{d-1}|)^{p_d-1}|i_{d-1}|.$$

Therefore, in order to get an upper bound for the value of $\frac{|\log Df_d(x)-\log Df_d(y)|}{|x-y|^\alpha}$, we only need to do so with

$$\frac{|i_k|^{p_k}(i'_d+|i_{d-1}|)^{p_d-1}|i_{d-1}|}{(|i_1|^{p_1}+\ldots+i^{p_d}_d+1)(|i_1|^{p_1}+\ldots+i'^{p_d}_d+1)|x-y|^\alpha}, \quad \text{where } 1\leq k\leq n, \tag{40}$$

and

$$\frac{i'^{p_d}_d(i_d+|i_{d-1}|)^{p_d-1}|i_{d-1}|}{(|i_1|^{p_1}+\ldots+i^{p_d}_d+1)(|i_1|^{p_1}+\ldots+i'^{p_d}_d+1)|x-y|^\alpha}. \tag{41}$$

Expression (40) is easy to deal with. Indeed, since

$$|x-y|\;\geq\;(i'_d-i_d-1)|I_{i_1,i_2,\ldots,i'_d}|\;=\;\left(\frac{i'_d-i_d-1}{|i_1|^{p_1}+\ldots+i'^{p_d}_d+1}\right), \tag{42}$$

we have

$$\begin{aligned}&\frac{|i_k|^{p_k}(i'_d+|i_{d-1}|)^{p_d-1}|i_{d-1}|}{(|i_1|^{p_1}+\ldots+i^{p_d}_d+1)(|i_1|^{p_1}+\ldots+i'^{p_d}_d+1)|x-y|^\alpha}\leq\\ &\qquad\leq\frac{|i_k|^{p_k}(i'_d+|i_{d-1}|)^{p_d-1}|i_{d-1}|}{(|i_1|^{p_1}+\ldots+i^{p_d}_d+1)(|i_1|^{p_1}+\ldots+i'^{p_d}_d+1)^{1-\alpha}}\leq\\ &\qquad\qquad\leq\frac{(i'_d+|i_{d-1}|)^{p_d-1}|i_{d-1}|}{(|i_1|^{p_1}+\ldots+i'^{p_d}_d+1)^{1-\alpha}}.\end{aligned}$$

To estimate the right-side expression, we consider two cases. If, on the one hand, we have $i'^{p_d}_d\leq|i_{d-1}|^{p_{d-1}}$, then

$$\frac{(i'_d+|i_{d-1}|)^{p_d-1}|i_{d-1}|}{(|i_1|^{p_1}+\ldots+i'^{p_d}_d+1)^{1-\alpha}}\;\leq\;\frac{(|i_{d-1}|^{\frac{p_{d-1}}{p_d}}+|i_{d-1}|)^{p_d-1}|i_{d-1}|}{(|i_{d-1}|^{p_{d-1}}+1)^{1-\alpha}}.$$

This is uniformly bounded when $\frac{p_{d-1}}{p_d}(p_d-1)+1 \leq p_{d-1}(1-\alpha)$, which is equivalent to condition ($\mathrm{v_B}$). On the other hand, if $|i_{d-1}|^{p_{d-1}} \leq i_d'^{p_d}$, then

$$\frac{(i_d' + |i_{d-1}|)^{p_d-1}|i_{d-1}|}{(|i_1|^{p_1} + \ldots + i_d'^{p_d} + 1)^{1-\alpha}} \;\leq\; \frac{(i_d' + (i_d')^{\frac{p_d}{p_{d-1}}})^{p_d-1}(i_d')^{\frac{p_d}{p_{d-1}}}}{(i_d'^{p_d}+1)^{1-\alpha}},$$

and the right-side term is uniformly bounded provided that condition ($\mathrm{v_B}$) holds.

To obtain an upper bound for (41), we will consider separately the cases (a), (b), (c) and (d) of the previous two sections.

In case (a) we have $i_d \leq i_d' \leq 2i_d + 1$. Hence, the upper bound already obtained for (40) with $k=d$ is an upper bound for (41).

In case (b), we have $i_d'^{p_d} \leq |i_1|^{p_1} + \ldots + |i_{d-1}|^{p_{d-1}}$. Hence, (41) is smaller than or equal to

$$\sum_{k=1}^{d-1} \frac{|i_k|^{p_k}(i_d' + |i_{d-1}|)^{p_d-1}|i_{d-1}|}{(|i_1|^{p_1} + \ldots + i_d^{p_d} + 1)(|i_1|^{p_1} + \ldots + i_d'^{p_d} + 1)|x-y|^{\alpha}},$$

and we have already seen that each term of this sum is uniformly bounded.

In case (c), we use (34) to obtain

$$\frac{i_d'^{p_d}(i_d + |i_{d-1}|)^{p_d-1}|i_{d-1}|}{(|i_1|^{p_1} + \ldots + i_d^{p_d} + 1)(|i_1|^{p_1} + \ldots + i_d'^{p_d} + 1)|x-y|^{\alpha}} \leq M \, \frac{(i_d + |i_{d-1}|)^{p_d-1}|i_{d-1}|}{(|i_1|^{p_1} + \ldots + i_d^{p_d} + 1)^{1-\alpha}} \, \frac{(i_d+1)^{(p_d-1)\alpha}}{(|i_1|^{p_1} + \ldots + i_d^{p_d} + 1)^{\alpha}}.$$

In the right-side expression, the second factor $\frac{(i_d+1)^{(p_d-1)\alpha}}{(i_d^{p_d}+1)^{\alpha}}$ is uniformly bounded. To show that the same holds with the first factor, one may proceed as at the end of the estimates for (40) just changing i_d' by i_d.

Finally, in case (d), the estimate (36) shows that (41) is smaller than or equal to

$$\frac{(i_d + |i_{d-1}|)^{p_d-1}|i_{d-1}|(i_d + 1 + S^{1/p_d})^{\alpha(p_d-1)} \, (i_d' + S^{1/p_d})^{\alpha}}{(|i_1|^{p_1} + \ldots + i_d^{p_d} + 1)(i_d' - i_d - 1)^{\alpha}}.$$

Since the condition $1 + i_d^{p_d} \leq S$ yields $i_d \leq S^{1/p_d}$, this expression is smaller than or equal to

$$M \, \frac{(i_d + |i_{d-1}|)^{p_d-1}|i_{d-1}|(i_d' + S^{1/p_d})^{\alpha}}{(i_d' - i_d - 1)^{\alpha}} \, S^{\frac{\alpha(p_d-1)}{p_d} - 1}.$$

Moreover, by the definition of S, we have $|i_{d-1}| \leq S^{1/p_{d-1}} \leq S^{1/p_d}$, which shows that the last expression is smaller than or equal to

$$M \, \frac{(i_d' + S^{1/p_d})^{\alpha}}{(i_d' - i_d - 1)^{\alpha}} \, S^{\frac{p_d-1}{p_d} + \frac{1}{p_{d-1}} + \frac{\alpha(p_d-1)}{p_d} - 1}.$$

Because of the conditions $i_d' \geq 2i_d + 2$ and $S \leq 1 + i_d'^{p_d}$, this last expression is bounded from above by

$$MS^{\frac{p_d-1}{p_d} + \frac{1}{p_{d-1}} + \frac{\alpha(p_d-1)}{p_d} - 1},$$

which is uniformly bounded by the condition ($\mathrm{v_B}$).

III. Finally, in the case where $x \in I_{i_1,\ldots,i_d}$ and $y \in I_{i_1',\ldots,i_d'}$ for different $(i_1,\ldots,i_{d-1})$ and $(i_1',\ldots,i_{d-1}')$, one may apply the same argument of the previous maps.

4 On a family of metabelian subgroups of $\mathrm{Diff}_+^{1+\alpha}([0,1])$

For each couple of integers (i,j), let $I_{i,j}$ be an interval of length $|I_{i,j}|$ so that the sum $\sum_{i,j}|I_{i,j}|$ is finite. Joining these intervals lexicographically, we obtain a closed interval I. Following [4, §2.3], we will deal with a particular family of nilpotent groups N_d acting on I. Each N_d has nilpotence degree $d+1$ and is metabelian. Moreover, N_1 coincides with the Heisenberg group N_2.

The group N_d has a presentation

$$\big\langle f, g_0, g_1, \ldots, g_d \colon [g_i, g_j] = id,\ [f, g_0] = id,\ [f, g_i] = g_{i-1} \text{ for all } i \geq 1 \big\rangle.$$

As maps, the generators send each interval $I_{i,j}$ into a certain $I_{i',j'}$ and coincide with the diffeomorphism $\varphi_{I_{i,j}, I_{i,j-1}}^{I_{i',j'}, I_{i',j'-1}}$ therein. The map f sends $I_{i,j}$ into $I_{i+1,j}$. The maps g_0 and g_1 send $I_{i,j}$ into $I_{i,j+1}$ or $I_{i,j+i}$, respectively. To describe the dynamics of $g_2, \ldots, g_d$, for each $0 < k \leq d$ and each $i \in \mathbb{Z}$, we let

$$r_k(i) = \frac{i(i+1)(i+2)\ldots(i+k-1)}{k!},$$

and we define $r_0(i) = 1$ for all i. (Note that $|r_k(i)| \leq |i|^k$ for $k > 0$.) Then the element g_k sends the interval $I_{i,j}$ into $I_{i,j+r_k(i)}$.

Now fix a positive number $\alpha < 1$. To carry out the preceding construction so that the resulting maps are $C^{1+\alpha}$-diffeomorphisms of I, we need to make a careful choice of the lengths $|I_{i,j}|$. We let $q > 1$ be such that the following conditions are satisfied:

(i$_\mathrm{C}$) $1 < q < 2$,

(ii$_\mathrm{C}$) $\alpha < 2 - q$,

(iii$_\mathrm{C}$) $\alpha < \frac{q}{2q-1}$,

(iv$_\mathrm{C}$) $\alpha < \frac{1}{q}$.

Note that since $\alpha < 1$ and the preceding right-side expressions go to 1 or to infinity as q tends to 1 from above, we may choose q very near to 1 in such a way that these conditions are fulfilled.

Now let $p := \frac{2q-1}{q-1}$. Clearly, we may also impose the following supplementary conditions:

(v$_\mathrm{C}$) $p > dq$,

(vi$_\mathrm{C}$) $\alpha \leq \frac{1}{q} - \frac{d}{p}$,

(vii$_\mathrm{C}$) $\alpha < \frac{1}{q-1} - \frac{dq}{2q-1}$.

We then define

$$|I_{i,j}| := \frac{1}{|i|^p + |j|^q + 1}.$$

Since $1/p + 1/q < 1$, it follows from [8, §3] that the sum $\sum_{i,j} |I_{i,j}|$ is finite. We claim that the group N_d obtained by using the maps from §3.1 is formed by $C^{1+\alpha}$-diffeomorphisms of I.

4.1 The map f is a $C^{1+\alpha}$-diffeomorphism

For simplicity, we only deal with points in the intervals $I_{i,j}$ with $i \geq 0$ and $j \geq 0$ (the other cases are analogous).

First we consider x, y in the same interval $I_{i,j}$. We have

$$\frac{|\log Df(x) - \log Df(y)|}{|x-y|} \leq \frac{M}{|I_{i,j}|} \left| \frac{|I_{i,j}|}{|I_{i+1,j}|} \frac{|I_{i+1,j-1}|}{|I_{i,j-1}|} - 1 \right|.$$

Hence,

$$\frac{|\log Df(x) - \log Df(y)|}{|x-y|^\alpha} \leq \frac{M}{|I_{i,j}|} \left| \frac{|I_{i,j}|}{|I_{i+1,j}|} \frac{|I_{i+1,j-1}|}{|I_{i,j-1}|} - 1 \right| |I_{i,j}|^{1-\alpha} = M \left| \frac{|I_{i,j}|}{|I_{i+1,j}|} \frac{|I_{i+1,j-1}|}{|I_{i,j-1}|} - 1 \right| |I_{i,j}|^{-\alpha}.$$

This yields

$$\frac{|\log Df(x) - \log Df(y)|}{|x-y|^\alpha} \leq M \left| \frac{(i+1)^p + j^q + 1}{i^p + j^q + 1} \frac{i^p + (j-1)^q + 1}{(i+1)^p + (j-1)^q + 1} - 1 \right| (i^p + j^q + 1)^\alpha.$$

Therefore, the value of $\frac{|\log Df(x) - \log Df(y)|}{|x-y|^\alpha}$ is bounded from above by

$$M \left| \frac{((i+1)^p + j^q + 1)(i^p + (j-1)^q + 1) - (i^p + j^q + 1)((i+1)^p + (j-1)^q + 1)}{(i^p + j^q + 1)^{1-\alpha}((i+1)^p + (j-1)^q + 1)} \right|,$$

which equals

$$M\frac{\big((i+1)^p-i^p\big)\big(j^q-(j-1)^q\big)}{(i^p+j^q+1)^{1-\alpha}((i+1)^p+(j-1)^q+1)}.$$

By the Mean Value Theorem, this expression is bounded from above by

$$M\frac{i^{p-1}j^{q-1}}{(i^p+j^q+1)^{1-\alpha}((i+1)^p+(j-1)^q+1)}.$$

Thus

$$\frac{|\log Df(x)-\log Df(y)|}{|x-y|^\alpha}\ \le\ M\frac{i^{p-1}}{(i+1)^p}\frac{j^{q-1}}{j^{q(1-\alpha)}}.$$

Now notice that the last expression is uniformly bounded when $q-1\le q(1-\alpha)$, which is satisfied by condition ($\mathrm{iv_C}$).

Next we consider $x\in I_{i,j}$ and $y\in I_{i,j'}$, with $j<j'$. The definition of f and property (19) yield

$$\log Df(x)=\log D\phi_{b',b}^{a',a}(x-w)=\log\frac{b}{a}+\log D\psi_{\log(b'a/a'b)}\left(\frac{x-w}{a}\right),$$

where $I_{i,j}=[w,w+a]$, $I_{i,j-1}=[w+a',w]$, $I_{i+1,j}=[w',w'+b]$, and $I_{i+1,j-1}=[w'+b',w']$. Analogously,

$$\log Df(y)=\log D\phi_{d',d}^{c',c}(y-u)=\log\frac{d}{c}+\log D\psi_{\log(d'c/c'd)}\left(\frac{y-u}{c}\right),$$

where $I_{i,j'}=[u,u+c]$, $I_{i,j'-1}=[u+c',u]$, $I_{i+1,j'}=[u',u'+d]$, and $I_{i+1,j'-1}=[u'+d',u']$. By property (20),

$$\begin{aligned}\big|\log Df(x)-\log Df(y)\big| &\le \left|\log\frac{b}{a}-\log\frac{d}{c}\right|+\left|\log D\psi_{\log(b'a/a'b)}\left(\frac{x-w}{a}\right)\right|+\left|\log D\psi_{\log(d'c/c'd)}\left(\frac{y-u}{c}\right)\right|\\ &\le \left|\log\frac{b}{a}-\log\frac{d}{c}\right|+\left|\log\frac{b'}{a'}-\log\frac{b}{a}\right|+\left|\log\frac{d'}{c'}-\log\frac{d}{c}\right|.\end{aligned}$$

Note that the last expression corresponds to

$$\left|\log\left(\frac{|I_{i+1,j}|}{|I_{i,j}|}\right)-\log\left(\frac{|I_{i+1,j'}|}{|I_{i,j'}|}\right)\right|+\left|\log\left(\frac{|I_{i+1,j-1}|}{|I_{i,j-1}|}\right)-\log\left(\frac{|I_{i+1,j}|}{|I_{i,j}|}\right)\right|+\left|\log\left(\frac{|I_{i+1,j'-1}|}{|I_{i,j'-1}|}\right)-\log\left(\frac{|I_{i+1,j'}|}{|I_{i,j'}|}\right)\right|.$$

Since the function $j\mapsto\frac{|I_{i+1,j}|}{|I_{i,j}|}$ is non-decreasing, the preceding inequality yields

$$\begin{aligned}\big|\log Df(x)-\log Df(y)\big| &\le 3\left|\log\left(\frac{|I_{i+1,j'}|}{|I_{i,j'}|}\right)-\log\left(\frac{|I_{i+1,j-1}|}{|I_{i,j-1}|}\right)\right|\\ &= 3\left|\log\frac{|I_{i+1,j'}|}{|I_{i,j'}|}\frac{|I_{i,j-1}|}{|I_{i+1,j-1}|}\right|\\ &= 3\left|\log\frac{(i^p+j'^q+1)}{((i+1)^p+j'^q+1)}\frac{((i+1)^p+(j-1)^q+1)}{(i^p+(j-1)^q+1)}\right|.\end{aligned}$$

Hence, the value of $\big|\log Df(x)-\log Df(y)\big|$ is bounded from above by

$$M\left|\log\left(1+\frac{(i^p+j'^q+1)((i+1)^p+(j-1)^q+1)-((i+1)^p+j'^q+1)(i^p+(j-1)^q+1)}{((i+1)^p+j'^q+1)(i^p+(j-1)^q+1)}\right)\right|. \qquad (43)$$

Since $\frac{i^p+j'^q+1}{(i+1)^p+j'^q+1}\frac{(i+1)^p+(j-1)^q+1}{i^p+(j-1)^q+1}$ is uniformly bounded from below, namely

$$\frac{i^p+j'^q+1}{(i+1)^p+j'^q+1}\frac{(i+1)^p+(j-1)^q+1}{i^p+(j-1)^q+1}\ge\frac{i^p+j'^q+1}{2^pi^p+j'^q+1}\ge\frac{i^p+j'^q+1}{2^pi^p+2^pj'^q+2^p}=\frac{1}{2^p},$$

the expression in (43) is bounded from above by

$$M\left|\frac{(i^p+j'^q+1)((i+1)^p+(j-1)^q+1)-((i+1)^p+j'^q+1)(i^p+(j-1)^q+1)}{((i+1)^p+j'^q+1)(i^p+(j-1)^q+1)}\right|,$$

which equals

$$M\left|\frac{(j'^q-(j-1)^q)((i+1)^p-i^p)}{((i+1)^p+j'^q+1)(i^p+(j-1)^q+1)}\right|.$$

By the Mean Value Theorem, this expression is bounded from above by

$$M\frac{i^{p-1}j'^{q-1}(j'-j+1)}{((i+1)^p+j'^q+1)(i^p+(j-1)^q+1)}.$$

Therefore,

$$\big|\log Df(x)-\log Df(y)\big|\leq M\frac{i^{p-1}j'^{q-1}(j'-j)}{(i^p+j'^q)(i^p+j^q)}. \tag{44}$$

We will split the general case into the following four cases:

(a) $j'\leq 2j+1$,

(b) $j'^q\leq i^p$,

(c) $j'>2j+1\,, j'^q>i^p\,, j^q\geq i^p$,

(d) $j'>2j+1\,, j'^q>i^p\,, j^q<i^p$.

In cases (a) and (b), notice that from

$$|x-y|\,\geq\,(j'-j-1)I_{i,j'}$$

it follows that

$$|x-y|^\alpha\,\geq\,\left(\frac{j'-j-1}{i^p+j'^q+1}\right)^\alpha.$$

Hence by (44),

$$\frac{|\log Df(x)-\log Df(y)|}{|x-y|^\alpha}\,\leq\,M\frac{i^{p-1}j'^{q-1}(j'-j)(i^p+j'^q+1)^\alpha}{(i^p+j'^q)(i^p+j^q)(j'-j-1)^\alpha},$$

that is,

$$\frac{|\log Df(x)-\log Df(y)|}{|x-y|^\alpha}\,\leq\,M\frac{i^{p-1}j'^{q-1}(j'-j)^{1-\alpha}}{(i^p+j'^q)^{1-\alpha}(i^p+j^q)}. \tag{45}$$

In case (a), we have $j'\leq 2j+1$, and hence the right-side of (45) is bounded from above by

$$M\frac{i^{p-1}j^{q-1}j^{1-\alpha}}{(i^p+j^q)^{1-\alpha}(i^p+j^q)}=M\frac{i^{p-1}j^{q-\alpha}}{(i^p+j^q)^{2-\alpha}}.$$

On the one hand, if $i\leq j^{\frac{1}{p-1}}$, then this expression is bounded by $\frac{j^{q-\alpha+1}}{j^{q(2-\alpha)}}=j^{(\alpha-1)(q-1)}$. Since $\alpha<1$, the expression is uniformly bounded. On the other hand, if $j\leq i^{p-1}$, then we have the upper bound

$$\frac{i^{p-1+(p-1)(q-\alpha)}}{i^{p(2-\alpha)}}=i^{\alpha-1-p-q+pq},$$

and this expression is uniformly bounded by the condition ($\mathrm{ii_C}$).

In case (b), we have $j'^q\leq i^p$, and hence the right-side expression in (45) is bounded from above by

$$\frac{i^{p-1}i^{\frac{p}{q}(q-1)}i^{\frac{p}{q}(1-\alpha)}}{i^{p+p(1-\alpha)}}=i^{\alpha(p-\frac{p}{q})-1},$$

which is uniformly bounded by condition ($\mathrm{iii_C}$).

In case (c), we have

$$|x-y|\geq\sum_{j<n<j'}I_{i,n}=\sum_{j<n<j'}\frac{1}{i^p+n^q+1}\geq\sum_{j<n<j'}\frac{1}{j^q+n^q+1}\geq\sum_{j<n<j'}\frac{1}{3n^q}\geq\int_{j+1}^{j'}\frac{dx}{3x^q},$$

and hence

$$|x-y| \geq M\frac{1}{(j+1)^{q-1}}\Big(1-(\frac{j+1}{j'})^{q-1}\Big) \geq \frac{1}{(j+1)^{q-1}}\Big(1-(\frac{1}{2})^{q-1}\Big).$$

Therefore,

$$|x-y|^{\alpha} \geq M\frac{1}{(j+1)^{(q-1)\alpha}}, \tag{46}$$

and this yields

$$\frac{|\log Df(x)-\log Df(y)|}{|x-y|^{\alpha}} \leq M\frac{i^{p-1}j'^{q-1}(j'-j)j^{(q-1)\alpha}}{(i^p+j'^q)(i^p+j^q)} = M\Big(\frac{j'^{q-1}(j'-j)}{i^p+j'^q}\Big)\Big(\frac{i^{p-1}j^{(q-1)\alpha}}{i^p+j^q}\Big).$$

In the last expression, the first factor is uniformly bounded, while the second one is bounded by

$$M\,\frac{j^{\frac{q}{p}(p-1)}j^{(q-1)\alpha}}{j^q}.$$

This last expression is uniformly bounded when $\frac{q}{p}(p-1)+(q-1)\alpha \leq q$, which is ensured by the condition ($\mathrm{iii_C}$).

The last case (d) is

$$j' > 2j+1\,, j'^q > i^p\,, j^q < i^p.$$

For the distance between x and y we now have the estimate

$$|x-y| \geq \sum_{j<n<j'} I_{i,n} = \sum_{j<n<j'} \frac{1}{i^p+n^q+1} \geq \int_{j+1}^{j'} \frac{1}{i^p+x^q+1}dx \geq \int_{j+1}^{j'} \frac{1}{(i^{\frac{p}{q}}+x+1)^q}dx.$$

The last integral is essentially

$$M\,\frac{(i^{\frac{p}{q}}+j'+1)^{q-1}-(i^{\frac{p}{q}}+j+2)^{q-1}}{(i^{\frac{p}{q}}+j+2)^{q-1}(i^{\frac{p}{q}}+j'+1)^{q-1}},$$

and by the Mean Value Theorem, this is larger than

$$M\,\frac{j'-j-1}{(i^{\frac{p}{q}}+j'+1)^{2-q}(i^{\frac{p}{q}}+j+2)^{q-1}(i^{\frac{p}{q}}+j'+1)^{q-1}}.$$

Therefore,

$$|x-y|^{\alpha} \;\geq\; M\frac{(j'-j-1)^{\alpha}}{(i^{\frac{p}{q}}+j'+1)^{\alpha}(i^{\frac{p}{q}}+j+2)^{(q-1)\alpha}}. \tag{47}$$

This yields

$$\begin{aligned}
\frac{|\log Df(x)-\log Df(y)|}{|x-y|^{\alpha}} &\leq M\frac{i^{p-1}j'^{q-1}(j'-j)(i^{\frac{p}{q}}+j'+1)^{\alpha}(i^{\frac{p}{q}}+j+2)^{(q-1)\alpha}}{(i^p+j'^q)(i^p+j^q)(j'-j-1)^{\alpha}}\\
&\leq M\frac{i^{p-1}(i^{\frac{p}{q}}+j'+1)^{\alpha}(i^{\frac{p}{q}}+j+2)^{(q-1)\alpha}}{(i^p+j^q)(j'-j-1)^{\alpha}}\\
&\leq M\frac{i^{p-1}(2j'+1)^{\alpha}(2i^{\frac{p}{q}}+2)^{(q-1)\alpha}}{(i^p+j^q)(\frac{j'}{2})^{\alpha}}\\
&\leq M\frac{i^{p-1}(i^{\frac{p}{q}}+1)^{(q-1)\alpha}}{(i^p+j^q)},
\end{aligned}$$

and by condition ($\mathrm{iii_C}$) the last expression is uniformly bounded.

This completes the proof of the $C^{1+\alpha}$ regularity of f. Similar arguments apply to its inverse f^{-1}, thus showing that f is a $C^{1+\alpha}$-diffeomorphism of I.

4.2 Each map g_k is a $C^{1+\alpha}$-diffeomorphism

Again, we will only consider the case of positive i, j. First, we take x, y in the same interval $I_{i,j}$. We have

$$\frac{|\log Dg_k(x) - \log Dg_k(y)|}{|x-y|} \leq \frac{M}{|I_{i,j}|} \left| \frac{|I_{i,j}|}{|I_{i,j+r_k(i)}|} \frac{|I_{i,j+r_k(i)-1}|}{|I_{i,j-1}|} - 1 \right|.$$

Hence

$$\begin{aligned} \frac{|\log Dg_k(x) - \log Dg_k(y)|}{|x-y|^\alpha} &\leq \frac{M}{|I_{i,j}|} \left| \frac{|I_{i,j}|}{|I_{i,j+r_k(i)}|} \frac{|I_{i,j+r_k(i)-1}|}{|I_{i,j-1}|} - 1 \right| |I_{i,j}|^{1-\alpha} \\ &= M \left| \frac{|I_{i,j}|}{|I_{i,j+r_k(i)}|} \frac{|I_{i,j+r_k(i)-1}|}{|I_{i,j-1}|} - 1 \right| |I_{i,j}|^{-\alpha} \\ &\leq M \left| \frac{i^p + (j+r_k(i))^q + 1}{i^p + j^q + 1} \frac{i^p + (j-1)^q + 1}{i^p + (j + r_k(i) - 1)^q + 1} - 1 \right| (i^p + j^q + 1)^\alpha. \end{aligned}$$

The last expression may be rewritten as

$$M \left| \frac{(i^p + (j+r_k(i))^q + 1)(i^p + (j-1)^q + 1) - (i^p + j^q + 1)(i^p + (j + r_k(i) - 1)^q + 1)}{(i^p + j^q + 1)^{1-\alpha}(i^p + (j + r_k(i) - 1)^q + 1)} \right|.$$

By the Mean Value Theorem, the value of this expression is bounded from above by

$$M \, \frac{i^p j^{q-1} + i^p (j + r_k(i))^{q-1} + j^{q-1} + (j+r_k(i))^{q-1} + (j+r_k(i))^q j^{q-1} + (j+r_k(i))^{q-1} j^q}{(i^p + j^q)^{1-\alpha}(i^p + (j + r_k(i) - 1)^q)},$$

and hence by

$$M \, \frac{i^p (j + r_k(i))^{q-1} + (j+r_k(i))^q j^{q-1} + (j+r_k(i))^{q-1} j^q}{(i^p + j^q)^{2-\alpha}} \leq M \, \frac{i^p (j + i^k)^{q-1} + (j+i^k)^q j^{q-1} + (j+i^k)^{q-1} j^q}{(i^p + j^q)^{2-\alpha}}.$$

We claim that the preceding right-expression is uniformly bounded. Indeed, if $i^p \leq j^q$, then it is smaller than or equal to

$$M \frac{j^q (j + j^{\frac{qk}{p}})^{q-1} + (j + j^{\frac{qk}{p}})^q j^{q-1} + (j + j^{\frac{qk}{p}})^{q-1} j^q}{j^{q(2-\alpha)}},$$

which is uniformly bounded by the conditions (iv_C) and (v_C). If $j^q \leq i^p$, then it is smaller than or equal to

$$M \frac{i^p (i^{\frac{p}{q}} + i^k)^{q-1} + (i^{\frac{p}{q}} + i^k)^q i^{\frac{p}{q}(q-1)} + (i^{\frac{p}{q}} + i^k)^{q-1} i^p}{i^{p(2-\alpha)}},$$

which is again uniformly bounded by the conditions (iv_C) and (v_C).

Next we consider the case where $x \in I_{i,j}$ and $y \in I_{i,j'}$, with $j \leq j'$. In this case, $|\log\, Dg_k(x) - \log\, Dg_k(y)|$ is smaller than or equal to

$$\left| \log \frac{|I_{i,j+r_k(i)}|}{|I_{i,j}|} - \log \frac{|I_{i,j'+r_k(i)}|}{|I_{i,j'}|} \right| + \left| \log \frac{|I_{i,j+r_k(i)-1}|}{|I_{i,j-1}|} - \log \frac{|I_{i,j+r_k(i)}|}{|I_{i,j}|} \right| + \left| \log \frac{|I_{i,j'+r_k(i)-1}|}{|I_{i,j'-1}|} - \log \frac{|I_{i,j'+r_k(i)}|}{|I_{i,j'}|} \right|.$$

The estimates for the last two terms are similar to those above, and we leave the computations to the reader. The first term equals

$$\left| \log \frac{|I_{i,j}|}{|I_{i,j+r_k(i)}|} \frac{|I_{i,j'+r_k(i)}|}{|I_{i,j'}|} \right| = \left| \log \frac{i^p + j'^q + 1}{i^p + (j' + r_k(i))^q + 1} \frac{i^p + (j + r_k(i))^q + 1}{i^p + j^q + 1} \right|,$$

that is,

$$\left| \log \left(1 + \frac{(i^p + j'^q + 1)(i^p + (j + r_k(i))^q + 1) - (i^p + (j' + r_k(i))^q + 1)(i^p + j^q + 1)}{(i^p + (j' + r_k(i))^q + 1)(i^p + j^q + 1)} \right) \right|.$$

We claim that the expression $\frac{i^p + j'^q + 1}{i^p + (j' + r_k(i))^q + 1} \frac{i^p + (j + r_k(i))^q + 1}{i^p + j^q + 1}$ is bounded from below by a positive number. Indeed, the first factor is uniformly bounded because:

– if $j'^q \le i^p$, then $\frac{i^p+j'^q+1}{i^p+(j'+r_k(i))^q+1} \ge \frac{i^p+j'^q+1}{i^p+(j'+r_k(i))^q+1} \ge \frac{i^p+1}{i^p+(i^{\frac{p}{q}}+i^k)^q+1}$, and the last expression is uniformly bounded from below by a positive number;
– if $i^p \le j'^q$, then $\frac{i^p+j'^q+1}{i^p+(j'+r_k(i))^q+1} \ge \frac{i^p+j'^q+1}{i^p+(j'+i^k)^q+1} \ge \frac{j'^q+1}{j'^q+(j'+j'^{\frac{qk}{p}})^q+1}$, which is uniformly bounded from below by a positive number.

The second factor is uniformly bounded as well because:
– if $i^p \le j^q$, then $0 \le j - j^{\frac{qk}{p}} \le j + r_k(i)$, thus $\frac{i^p+(j+r_k(i))^q+1}{i^p+j^q+1} \ge \frac{i^p+(j-j^{\frac{qk}{p}})^q+1}{i^p+j^q+1} \ge \frac{(j-j^{\frac{qk}{p}})^q+1}{2j^q+1}$, which is uniformly bounded from below by a positive number;
– if $j^q \le i^p$, then $\frac{i^p+(j+r_k(i))^q+1}{i^p+j^q+1} \ge \frac{i^p+1}{2i^p+1}$, which is uniformly bounded from below by a positive number.

From the preceding, we deduce the estimate

$$\left|\log\frac{|I_{i,j+r_k(i)}|}{|I_{i,j}|} - \log\frac{|I_{i,j'+r_k(i)}|}{|I_{i,j'}|}\right| \le M\left|\frac{(i^p+j'^q+1)(i^p+(j+r_k(i))^q+1)-(i^p+(j'+r_k(i))^q+1)(i^p+j^q+1)}{(i^p+(j'+r_k(i))^q+1)(i^p+j^q+1)}\right|.$$

Using the Mean Value Theorem and the inequalities $j < j'$ and $r_k(i) \le i^k$, the right-side term above is easily seen to be smaller than or equal to

$$M\frac{i^{p+k}(j'+i^k)^{q-1}+j^q(j'+i^k)^{q-1}i^k+j'^q(j+i^k)^{q-1}i^k}{(i^p+j'^q)(i^p+j^q)}. \tag{48}$$

To get an upper bound for this expression, we separately consider the cases (a), (b), (c), and (d), from the previous section.

The first case (a) is $j' \le 2j+1$. We have $|x-y| \ge \frac{j'-j-1}{i^p+j'^q+1}$, and hence

$$\begin{aligned}\frac{|\log Dg_k(x)-\log Dg_k(y)|}{|x-y|^\alpha} &\le M\,\frac{i^{p+k}(j'+i^k)^{q-1}+j^q(j'+i^k)^{q-1}i^k+j'^q(j+i^k)^{q-1}i^k}{(i^p+j'^q)(i^p+j^q)}\,\frac{(i^p+j'^q+1)^\alpha}{(j'-j-1)^\alpha}+M\\ &\le M\,\frac{i^{p+k}(j'+i^k)^{q-1}+j^q(j'+i^k)^{q-1}i^k+j'^q(j+i^k)^{q-1}i^k}{(i^p+j'^q)^{1-\alpha}(i^p+j^q)}+M\\ &\le M\,\frac{i^{p+k}(j+i^k)^{q-1}+2j^q(j+i^k)^{q-1}i^k}{(i^p+j'^q)^{1-\alpha}(i^p+j^q)}+M.\end{aligned}$$

We will deal with the expressions $\frac{i^{p+k}(j+i^k)^{q-1}}{(i^p+j'^q)^{1-\alpha}(i^p+j^q)}$ and $\frac{j^q(j+i^k)^{q-1}i^k}{(i^p+j'^q)^{1-\alpha}(i^p+j^q)}$ separately. For the first we have

$$\frac{i^{p+k}(j+i^k)^{q-1}}{(i^p+j'^q)^{1-\alpha}(i^p+j^q)} \le \frac{i^{p+k}(j+i^k)^{q-1}}{(i^p+j^q)^{2-\alpha}}.$$

Now notice that
– if $i^p \le j^q$, then $\frac{i^{p+k}(j+i^k)^{q-1}}{(i^p+j^q)^{2-\alpha}} \le \frac{j^{\frac{q}{p}(p+k)}(j+j^{\frac{qk}{p}})^{q-1}}{j^{q(2-\alpha)}}$, and this is bounded when $\frac{q}{p}(p+k)+q-1 \le q(2-\alpha)$, that is, when $\alpha \le \frac{1}{q}-\frac{k}{p}$, which is our condition ($\mathrm{vi_C}$);
– if $j^q \le i^p$, then $\frac{i^{p+k}(j+i^k)^{q-1}}{(i^p+j^q)^{2-\alpha}} \le \frac{i^{p+k}(i^{\frac{p}{q}}+i^k)^{q-1}}{i^{p(2-\alpha)}}$, and this is bounded when $p+k+\frac{p}{q}(q-1) \le p(2-\alpha)$, that is, when $\alpha \le \frac{1}{q}-\frac{k}{p}$.
For the second expression we have

$$\frac{j^q(j+i^k)^{q-1}i^k}{(i^p+j'^q)^{1-\alpha}(i^p+j^q)} \le \frac{j^q(j+i^k)^{q-1}i^k}{(i^p+j^q)^{2-\alpha}}.$$

Again, notice that
– if $i^p \le j^q$, then $\frac{j^q(j+i^k)^{q-1}i^k}{(i^p+j^q)^{2-\alpha}} \le \frac{j^q(j+j^{\frac{qk}{p}})^{q-1}j^{\frac{qk}{p}}}{j^{q(2-\alpha)}}$, and as before, this is bounded when $\alpha \le \frac{1}{q}-\frac{k}{p}$;
– if $j^q \le i^p$, then $\frac{j^q(j+i^k)^{q-1}i^k}{(i^p+j^q)^{2-\alpha}} \le \frac{i^p(i^{\frac{p}{q}}+i^k)^{q-1}i^k}{i^{p(2-\alpha)}}$, and as before, this is bounded when $\alpha \le \frac{1}{q}-\frac{k}{p}$.

The second case (b) is $j'^q \le i^p$. The inequality $|x-y| \ge \frac{j'-j-1}{i^p+j'^q+1}$ yields

$$\begin{aligned}
\frac{|\log Dg_k(x) - \log Dg_k(y)|}{|x-y|^\alpha} &\le M\,\frac{i^{p+k}(j'+i^k)^{q-1} + j^q(j'+i^k)^{q-1}i^k + j'^q(j+i^k)^{q-1}i^k}{(i^p+j'^q)(i^p+j^q)}\,\frac{(i^p+j'^q+1)^\alpha}{(j'-j-1)^\alpha} + M \\
&\le M\,\frac{i^{p+k}(j'+i^k)^{q-1} + j^q(j'+i^k)^{q-1}i^k + j'^q(j+i^k)^{q-1}i^k}{(i^p+j^q)^{2-\alpha}} + M \\
&\le M\,\frac{i^{p+k}(i^{\frac{p}{q}}+i^k)^{q-1} + j^q(i^{\frac{p}{q}}+i^k)^{q-1}i^k + i^p(j+i^k)^{q-1}i^k}{(i^p+j^q)^{2-\alpha}} + M.
\end{aligned}$$

To estimate the last expression, we will bound the following three expressions:

$$\frac{i^{p+k}(i^{\frac{p}{q}}+i^k)^{q-1}}{(i^p+j^q)^{2-\alpha}}, \quad \frac{j^q(i^{\frac{p}{q}}+i^k)^{q-1}i^k}{(i^p+j^q)^{2-\alpha}}, \quad \text{and} \quad \frac{i^p(j+i^k)^{q-1}i^k}{(i^p+j^q)^{2-\alpha}}.$$

For the first we have

$$\frac{i^{p+k}(i^{\frac{p}{q}}+i^k)^{q-1}}{(i^p+j^q)^{2-\alpha}} \le \frac{i^{p+k}(i^{\frac{p}{q}}+i^k)^{q-1}}{i^{p(2-\alpha)}},$$

and the right-side member is bounded provided that $p+k+\frac{p}{q}(q-1) \le p(2-\alpha)$, that is, $\alpha \le \frac{1}{q} - \frac{k}{p}$, which is our condition ($\mathrm{vi_C}$). For the second expression, notice that

– if $i^p \le j^q$, then $\frac{j^q(i^{\frac{p}{q}}+i^k)^{q-1}i^k}{(i^p+j^q)^{2-\alpha}} \le \frac{j^q(j+j^{\frac{qk}{p}})^{q-1}j^{\frac{qk}{p}}}{j^{q(2-\alpha)}}$, and this is bounded when $q+\frac{qk}{p}+q-1 \le q(2-\alpha)$, that is, $\alpha \le \frac{1}{q} - \frac{k}{p}$;

– if $j^q \le i^p$, then $\frac{j^q(i^{\frac{p}{q}}+i^k)^{q-1}i^k}{(i^p+j^q)^{2-\alpha}} \le \frac{i^p(i^{\frac{p}{q}}+i^k)^{q-1}i^k}{i^{p(2-\alpha)}}$, and the last term is bounded because $\alpha \le \frac{1}{q} - \frac{k}{p}$.

For the third expression, we have that

– if $i^p \le j^q$, then $\frac{i^p(j+i^k)^{q-1}i^k}{(i^p+j^q)^{2-\alpha}} \le \frac{j^q(j+j^{\frac{qk}{p}})^{q-1}j^{\frac{qk}{p}}}{j^{q(2-\alpha)}}$, which is bounded when $\alpha \le \frac{1}{q} - \frac{k}{p}$;

– if $j^q \le i^p$, then $\frac{i^p(j+i^k)^{q-1}i^k}{(i^p+j^q)^{2-\alpha}} \le \frac{i^p(i^{\frac{p}{q}}+i^k)^{q-1}i^k}{i^{p(2-\alpha)}}$, which is also bounded when $\alpha \le \frac{1}{q} - \frac{k}{p}$.

The third case (c) is $j' > 2j+1$, $j'^q > i^p$, $j^q \ge i^p$. Using (46) we obtain

$$\frac{|\log Dg_k(x) - \log Dg_k(y)|}{|x-y|^\alpha} \le M\,\frac{i^{p+k}(j'+i^k)^{q-1} + j^q(j'+i^k)^{q-1}i^k + j'^q(j+i^k)^{q-1}i^k}{(i^p+j'^q)(i^p+j^q)}\,j^{(q-1)\alpha} + M.$$

To estimate the preceding right-side expression, we deal separately with

$$\frac{i^{p+k}(j'+i^k)^{q-1}}{(i^p+j'^q)(i^p+j^q)}j^{(q-1)\alpha}, \quad \frac{j^q(j'+i^k)^{q-1}i^k}{(i^p+j'^q)(i^p+j^q)}j^{(q-1)\alpha}, \quad \text{and} \quad \frac{j'^q(j+i^k)^{q-1}i^k}{(i^p+j'^q)(i^p+j^q)}j^{(q-1)\alpha}.$$

For the first of these expressions one has

$$\frac{i^{p+k}(j'+i^k)^{q-1}}{(i^p+j'^q)(i^p+j^q)}j^{(q-1)\alpha} \le \frac{i^k(j'+i^k)^{q-1}j'^{(q-1)\alpha}}{i^p+j'^q} \le \frac{j'^{\frac{qk}{p}}(j'+j'^{\frac{qk}{p}})^{q-1}j'^{(q-1)\alpha}}{j'^q},$$

and the right-side term is bounded by condition ($\mathrm{vii_C}$). For the second expression one has

$$\frac{j^q(j'+i^k)^{q-1}i^k}{(i^p+j'^q)(i^p+j^q)}j^{(q-1)\alpha} \le \frac{(j'+i^k)^{q-1}i^k j'^{(q-1)\alpha}}{i^p+j'^q} \le \frac{(j'+j'^{\frac{qk}{p}})^{q-1}j'^{\frac{qk}{p}}j'^{(q-1)\alpha}}{j'^q},$$

and the right-side term is uniformly bounded by the condition ($\mathrm{vii_C}$). Finally, for the third expression one has

$$\frac{j'^q(j+i^k)^{q-1}i^k}{(i^p+j'^q)(i^p+j^q)}j^{(q-1)\alpha} \le \frac{(j+i^k)^{q-1}i^k j^{(q-1)\alpha}}{i^p+j^q} \le \frac{(j+j^{\frac{qk}{p}})^{q-1}j^{\frac{qk}{p}}j^{(q-1)\alpha}}{j^q},$$

and the right-side term is also bounded by condition ($\mathrm{vii_C}$).

The last case (d) is $j' > 2j+1\,, j'^q > i^p\,, j^q < i^p$. Inequality (47) shows that $\frac{|\log\, Dg_k(x) - \log\, Dg_k(y)|}{|x-y|^\alpha}$ is smaller than or equal to

$$M \frac{i^{p+k}(j'+i^k)^{q-1} + j^q(j'+i^k)^{q-1}i^k + j'^q(j+i^k)^{q-1}i^k}{(i^p+j'^q)(i^p+j^q)} \frac{(i^{\frac{p}{q}}+j'+1)^\alpha (i^{\frac{p}{q}}+j+2)^{(q-1)\alpha}}{(j'-j-1)^\alpha} + M.$$

In this expression, the term $\frac{(i^{\frac{p}{q}}+j'+1)^\alpha}{(j'-j-1)^\alpha}$ is bounded by $\frac{(2j'+1)^\alpha}{(\frac{j'}{2})^\alpha}$, and hence it is uniformly bounded. Therefore,

$$\frac{|\log\, Dg_k(x) - \log\, Dg_k(y)|}{|x-y|^\alpha} \leq M \frac{i^{p+k}(j'+i^k)^{q-1} + j^q(j'+i^k)^{q-1}i^k + j'^q(j+i^k)^{q-1}i^k}{(i^p+j'^q)(i^p+j^q)} (i^{\frac{p}{q}}+j)^{(q-1)\alpha}.$$

To estimate the right-side expression, we will deal separately with

$$\frac{i^{p+k}(j'+i^k)^{q-1}}{(i^p+j'^q)(i^p+j^q)}(i^{\frac{p}{q}}+j)^{(q-1)\alpha}, \quad \frac{j^q(j'+i^k)^{q-1}i^k}{(i^p+j'^q)(i^p+j^q)}(i^{\frac{p}{q}}+j)^{(q-1)\alpha}, \text{ and } \frac{j'^q(j+i^k)^{q-1}i^k}{(i^p+j'^q)(i^p+j^q)}(i^{\frac{p}{q}}+j)^{(q-1)\alpha}.$$

For the first of these expressions one has

$$\frac{i^{p+k}(j'+i^k)^{q-1}}{(i^p+j'^q)(i^p+j^q)}(i^{\frac{p}{q}}+j)^{(q-1)\alpha} \leq \frac{i^k(j'+i^k)^{q-1}}{(i^p+j'^q)}(i^{\frac{p}{q}}+j)^{(q-1)\alpha} \leq \frac{j'^{\frac{qk}{p}}(j'+j'^{\frac{qk}{p}})^{q-1}}{j'^q}(j'+j')^{(q-1)\alpha},$$

and, as before, we know that the right-side term is uniformly bounded by the condition ($\mathrm{vii}_{\mathrm{C}}$). For the second expression one has

$$\frac{j^q(j'+i^k)^{q-1}i^k}{(i^p+j'^q)(i^p+j^q)}(i^{\frac{p}{q}}+j)^{(q-1)\alpha} \leq \frac{(j'+i^k)^{q-1}i^k}{(i^p+j'^q)}(i^{\frac{p}{q}}+j)^{(q-1)\alpha} \leq \frac{(j'+j'^{\frac{qk}{p}})^{q-1}j'^{\frac{qk}{p}}}{j'^q}(j'+j')^{(q-1)\alpha},$$

and the right-side term is uniformly bounded by the condition ($\mathrm{vii}_{\mathrm{C}}$). Finally, for the third expression one has

$$\frac{j'^q(j+i^k)^{q-1}i^k}{(i^p+j'^q)(i^p+j^q)}(i^{\frac{p}{q}}+j)^{(q-1)\alpha} \leq \frac{(j+i^k)^{q-1}i^k}{(i^p+j^q)}(i^{\frac{p}{q}}+j)^{(q-1)\alpha} \leq \frac{(i^{\frac{p}{q}}+i^k)^{q-1}i^k}{i^p}(i^{\frac{p}{q}}+i^{\frac{p}{q}})^{(q-1)\alpha}.$$

Here the right-side term is bounded when $\frac{p}{q}(q-1)+k+\frac{p}{q}(q-1)\alpha \leq p$, which is true by the condition ($\mathrm{vii}_{\mathrm{C}}$).

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