On the sharp regularity for arbitrary actions of nilpotent groups on the interval: the case of N_4

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Abstract

In this work, we determine the largest α for which the nilpotent group of 4-by-4 triangular matrices with integer coefficients and 1 in the diagonal embeds into the group of $C^{1+\alpha}$ diffeomorphisms of the closed interval.

Introduction

This work deals with the next general two-fold question:

Given a group G of orientation-preserving homeomorphisms of a manifold M, is it conjugate to a group of diffeomorphisms of M? If so, how smooth can this action be made?

In dimension larger than 1, the first half of the question has, in general, a negative answer, even for the action of a single homeomorphism [5]. However, in the case where M has dimension 1, this turns out to be very interesting, and the answer deeply depends on the dynamical/algebraic structure of the action/group considered. For instance, from the dynamical point of view, the classical Denjoy theorem says that a C^2 (more generally, C^{1+bv}) orientation-preserving circle diffeomorphism with irrational rotation number is necessarily conjugate to a rotation, hence minimal. On the other hand, in lower regularity, there are the so-called Denjoy counterexamples, namely, $C^{1+\alpha}$ diffeomorphisms with irrational rotation number that admit wandering intervals; besides, every circle homeomorphism is conjugate to a C^1 diffeomorphism. From the algebraic point of view, there is an important obstruction for a group G to admit a faithful action by C^1-diffeomorphisms of a 1-manifold with boundary: every finitely-generated subgroup of G must admit a nontrivial homomorphism onto $\mathbb{Z}$ (see [14]; see also [9] and [1]).

In this article, we focus on nilpotent group actions on the closed interval $[0,1]$. (Extensions of our results to the case of the circle are left to the reader.) The picture for Abelian group actions is essentially completed by the works [3, 15]. For non-Abelian nilpotent groups, an important theorem of J.Plante and W.Thurston establishes that they do not embed in the group of C^2-diffeomorphisms of $[0,1]$ (see [12]). However, according to B.Farb and J.Franks, every finitely-generated, torsion-free nilpotent group can be realized as a group of C^1 diffeomorphisms of $[0,1]$ (see also [6]). Motivated by this, we pursue the problem below, which was first addressed in [4] and stated this way in [8]. For the statement, recall that a diffeomorphism f is said to be of class $C^{1+\alpha}$ if its derivative is α-Holder continous, that is, there exists $C>0$ such that $|f'(x)-f'(y)| \leq C|x-y|^{\alpha}$ holds for all x, y.

Problem. *Given a nilpotent subgroup G of* $\mathrm{Homeo}_+([0,1])$*, find the largest α such that G embeds into the group* $\mathrm{Diff}_+^{1+\alpha}([0,1])$ *of $C^{1+\alpha}$ diffeomorphisms.*

There are two results in this direction. First, in [2] (see also [7]), the aforementioned Farb-Franks action of N_d, the nilpotent group of d-by-d lower triangular matrices with integer entries and 1 in the diagonal, is studied in detail. In particular, it is showed that this action cannot be made of class $C^{1+\alpha}$ for $\alpha \geq \frac{2}{(d-1)(d-2)}$, yet it can be made $C^{1+\alpha}$ for any smaller α. Second, a recent result of K.Parkhe [10] establishes that any action of a finitely-generated nilpotent group on $[0,1]$ is topologically conjugate to an action by $C^{1+\alpha}$-diffeomorphisms for any $\alpha < 1/\kappa$, where κ is the polynomial growth degree of the group.

For the particular case of N_4, the regularity obtained by Parkhe is hence smaller than that of the Farb-Franks action, namely, $C^{1+\alpha}$ for $\alpha < 1/3$. Somehow surprisingly, even this regularity is not sharp, as it is shown by our

Theorem A. *The group N_4 embeds into $\mathrm{Diff}_+^{1+\alpha}([0,1])$ for every $\alpha < 1/2$.*

In [2], it is also shown that for any any $d \in \mathbb{N}$, there is a nilpotent group of nilpotence degree d embedded into $\mathrm{Diff}_+^{1+\alpha}([0,1])$, for any $\alpha < 1$. (This is for instance the case of the Heissenberg group N_3.) This suggests that the optimal regularity of a nilpotent group embedding into $\mathrm{Diff}_+([0,1])$ may not depend on the degree of nilpotence. Our second result shows that, at least, this invariant is not trivial, hence it is worth pursuing its study.

Theorem B. *The group N_4 does not embeds into $\mathrm{Diff}_+^{1+\alpha}([0,1])$ for any $\alpha > 1/2$.*

We point out that the $C^{3/2}$ regularity is not covered by our results, though we strongly believe that N_4 does not admit an embedding in such regularity (compare [7]).

This article is organized as follows. In §1, the review some basic facts about the group N_4 such as normal forms; we also construct an action of N_4 on $\mathbb{Z}^3$ that preserves the lexicographic order on $\mathbb{Z}^3$. In §3, we show that for any $\alpha < 1/2$, the action of N_4 on $\mathbb{Z}^3$ can be projected into an action of N_4 on $[0,1]$ by $C^{1+\alpha}$ diffeomorphisms, which shows Theorem A. Theorem B in turn is proved in §2.

All actions considered in this work are by orientation-preserving maps.

1 The group N_4

Throughout this work, we use the following notation. Given two group elements x, y, we let $[x,y] := xyx^{-1}y^{-1}$, and $x^y := yxy^{-1}$. Recall that the derived series of a group G is defined by $G^0 := G$ and $G^{i+1} := [G^i, G^i]$. The group G is solvable of degree d if G^d is trivial but G^{d-1} is not. The central series of G is defined by $G^{(0)} := G$ and $G^{(i+1)} := [G, G^{(i)}]$. The group G is nilpotent of degree ℓ if $G^{(\ell)}$ is trivial but $G^{(\ell-1)}$ is not.

The group N_4 is by definition the group of matrices of the form

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ e & 1 & 0 & 0 \\ a & f & 1 & 0 \\ c & b & d & 1 \end{pmatrix}, \tag{1}$$

where all the entries belong to $\mathbb{Z}$. We will use the generating set S of N_4 consisting of the matrices for which all non-diagonal entries are 0 except for one which is 1. The elements of S will be denoted by e, f, d, a, b, c, where each of these elements represent the generating matrix with a 1 in the position corresponding to the letter in (1); for example,

$$e = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The reader can easily check that N_4 is isomorphic to the (inner) semidirect product $\langle f,a,b,c\rangle \rtimes \langle e,d\rangle$, where $\langle f,a,b,c\rangle \simeq \mathbb{Z}^4$ and $\langle d,e\rangle \simeq \mathbb{Z}^2$. The conjugacy action of $\mathbb{Z}^2$ on $\mathbb{Z}^4$ is given by

$$e: \ f \mapsto fa^{-1}\ ,\ a \mapsto a\ ,\ b \mapsto bc^{-1}\ ,\ c \mapsto c, \tag{2}$$

$$d: \ f \mapsto fb\ ,\ a \mapsto ac\ ,\ b \mapsto b\ ,\ c \mapsto c. \tag{3}$$

In particular, N_4 is metabelian (*i.e.* it has solvability degree 2). Further, N_4 has nilpotence degree 3: its lower central series is given by

$$N_4^{(1)} = \langle a,b,c\rangle\ ,\ N_4^{(2)} = \langle c\rangle\ ,\ N_4^{(3)} = \{id\}.$$

It follows from equations (2) and (3) that any element of N_4 can be written in a unique way as

$$f^{n_1}e^{n_2}d^{n_3}a^{n_4}b^{n_5}c^{n_6},$$

where the exponents n_i belong to $\mathbb{Z}$. This will be our preferred normal form. It allows proving the next

Lemma 1. *Let $\phi : N_4 \to G$ be a group homomorphism such that $\phi(c)$ is a nontrivial element of G with infinite order. Then ϕ is an embedding.*

Proof: We first observe that, for $(n_1,n_2) \neq (0,0)$,

$$[\phi(d^{n_1}e^{n_2}), \phi(a^{n_1}b^{-n_2}c^{n_3})] = \phi([d^{n_1}e^{n_2}, a^{n_1}b^{-n_2}c^{n_3}]) = \phi(c^{n_1^2+n_2^2}).$$

By the hypothesis, $\phi(c^{n_1^2+n_2^2}) \neq id$, which implies that the restriction of ϕ to both $\langle a,b,c\rangle$ and $\langle d,e\rangle$ is an embedding.

Further, for $(n_1,n_2) \neq (0,0)$, we have

$$\phi([d^{n_1}e^{n_2}a^{n_3}b^{n_4}c^{n_5}, a^{n_1}b^{-n_2}]) = \phi([d^{n_1}e^{n_2}, a^{n_1}b^{-n_2}]) = \phi(c^{n_1^2+n_2^2}) \neq id,$$

thus the restriction of ϕ to $\langle d,e,a,b,c\rangle$ is an embedding. Finally we have that, for $n_0 \neq 0$,

$$\phi([f^{n_0}e^{n_1}d^{n_2}a^{n_3}b^{n_4}c^{n_5}, e]) = \phi(a^{n_0}c^{n_4}) \neq id.$$

Hence, ϕ is injective. □

Remark 1. An immediate consequence of Lemma 1 is that for every faithful action of N_4 by homeomorphisms of $[0,1]$, there is a point $x_0 \in (0,1)$ such that N_4 acts faithfully on its orbit. Indeed, it suffices to consider x_0 as any point moved by c.

We next construct an action of N_4 by homeomorphisms of $[0,1]$. Our method is close to the construction of Farb and Franks, who first built an action of N_4 on $\mathbb{Z}^3$ and then project it to an action on $[0,1]$; see [4] or [2]. However, it should be emphasized that our action is different, which allows improving the regularity. We begin with

Proposition 1. *Let e', f', d' ,a', b', and c' be the maps from $\mathbb{Z}^3$ to $\mathbb{Z}^3$ defined by:*

$$\begin{aligned} e' &: (i,j,k) \mapsto (i+1,j,k),\\ d' &: (i,j,k) \mapsto (i,j+1,k),\\ f' &: (i,j,k) \mapsto (i,j,k-ij),\\ a' &: (i,j,k) \mapsto (i,j,k-j),\\ b' &: (i,j,k) \mapsto (i,j,k+i),\\ c' &: (i,j,k) \mapsto (i,j,k+1). \end{aligned} \tag{4}$$

Then the group N' generated by $\langle e',f',d',a',b',c'\rangle$ is isomorphic to N_4.

Proof: It follows from the definition that f', a', b' and c' commute, and that the subgroup of N' that they generate is normal and isomorphic to $\mathbb{Z}^4$. Further, the subgroup generated by $\{e', d'\}$ is Abelian, and its action by conjugation on $\langle f', a', b', c' \rangle$ mimics equations (2) and (3). Therefore, by Lemma 1, the application $x \mapsto x'$, with $x \in \{e, d, f, a, b, c\}$, induces an isomorphism between N_4 and N'. $\square$

We now let $(I_{i,j,k})_{(i,j,k)\in\mathbb{Z}^3}$ be a family of disjoint open intervals disposed on $[0,1]$ respecting the (direct) lexicographic order of $\mathbb{Z}^3$, that is, $I_{i,j,k}$ is to the left of $I_{i',j',k'}$ if and only if $(i,j,k) \prec (i',j',k')$, where $\preceq$ is the lexicographic order on $\mathbb{Z}^3$. Assume further that the union of this family of intervals is dense in $[0,1]$. Then, by some abuse of notation, we can define e, d, f to be the unique homeomorphism of $[0,1]$ whose restriction to each of the intervals $I_{i,j,k}$ is affine and send, respectively,

$$\begin{aligned} e &: I_{i,j,k} \mapsto I_{i+1,j,k}, \\ d &: I_{i,j,k} \mapsto I_{i,j+1,k}, \\ f &: I_{i,j,k} \mapsto I_{i,j,k+ij}. \end{aligned} \tag{5}$$

Since an affine map fixing a bounded interval must be the identity, Proposition 1 implies that the homeomorphisms e, d, f generate a subgroup of $\mathrm{Homeo}_+([0,1])$ isomorphic to N_4. In order to show Theorem A, in §3, we will use, instead of affine maps, the so-called Pixton-Tsuboi family of local diffeomorphisms [13, 15].

2 Bounding the regularity

In this section, we show that the group N_4 does not embed in $\mathrm{Diff}_+^{1+\alpha}([0,1])$ provided that $\alpha > 1/2$. We first reduce Theorem B to a combinatorial statement, namely Lemma 2 below.

2.1 The combinatorics prevents an embedding

Recall that every finitely-generated nilpotent group G of orientation-preserving homeomorphisms of $(0,1)$ preserves a nontrivial Radon measure μ on $(0,1)$; see [11] or [8]. This measure induces a group homomorphism, the so-called *translation number homomorphism* $\tau_\mu \colon G \to \mathbb{R}$, whose kernel coincides with the set of elements in G having fixed points, and such an element must fix all points in $supp(\mu)$, the support of μ. Moreover, if $\tau_\mu(G)$ has rank 2 or more, then G is semiconjugate to a group of translations isomorphic to $\tau_\mu(G)$.

Further, as any subgroup of a finitely-generated nilpotent group is also finitely generated, by looking at the action of $Ker(\tau_\mu)$ on any connected component J of $(0,1) \setminus supp(\mu)$ we obtain a $Ker(\tau_\mu)$-invariant measure on J with its corresponding translation number homomorphism. By iterating this process, we obtain a (partial) filtration of the nilpotent group G.

Now, looking for a contradiction, we suppose that N_4 faithfully acts by $C^{1+\alpha}$ diffeomorphisms of $[0,1]$, for some $\alpha > 1/2$. In this case, the key point (whose proof is postponed to §2.2.) is the next

Lemma 2. *Suppose that N_4 is faithfully acting on $[0,1]$ by $C^{1+\alpha}$-diffeomorphisms for some $\alpha > 1/2$. Then there is a sequence of open intervals $J_{n+1} \subsetneq J_n \subsetneq \ldots \subsetneq J_0$, and a filtration*

$$K_{n+1} \le K_n \le \ldots \le K_0 = N_4,$$

with the following properties:

1. *J_i is fixed by K_i, and the induced action of K_i on J_i admits no global fixed point;*
2. *$K_{i+1} = Ker(\tau_{\mu_i})$, where μ_i is a K_i-invariant Radon measure on J_i and $\tau_{\mu_i} : K_i \to \mathbb{R}$ is the associated translation-number homomorphism;*

3. *there exist g_1, g_2, g_3 in N_4 and non-negative integers $i_1 < i_2 < i_3 \leq n$, such that $\langle g_1, g_2, g_3 \rangle \simeq \mathbb{Z}^3$ and $g_j \in K_{i_j} \setminus K_{i_j+1}$.*

Lemma 2 provides us enough combinatorial information about the dynamics of N_4. In concrete terms, if we denote by $g_1,\ g_2,\ g_3$ the elements provided by the conclusion of the lemma, and we let x_0 be a point in $[0,1]$ not fixed by g_3, then the only element in the Abelian group $\langle g_1, g_2, g_3 \rangle$ fixing x_0 is the trivial one. Further, by eventually changing some of g_1, g_2, g_3 by their inverses, we can suppose that they move x_0 to the right. Hence, if we define $I_{0,0,0}$ as the interval $(x_0, g_3(x_0))$ and $I_{n_1,n_2,n_3} := g_1^{n_1}\, g_2^{n_2}\, g_3^{n_3}\, (I_{0,0,0})$, then the intervals $I_{i,j,k}$ are pairwise disjoint, they are disposed on $[0,1]$ respecting the lexicographic order of the indices, and

$$g_1(I_{i,j,k}) = I_{i+1,j,k}\ ,\ g_2(I_{i,j,k}) = I_{i,j+1,k}\ ,\ g_3(I_{i,j,k}) = I_{i,j,k+1}.$$

A contradiction is then provided by the following theorem from [7] (see also [3])

Theorem 1. *Let $k \geq 2$ be an integer, and let $f_1, \ldots, f_k$ be commuting C^1-diffeomorphisms of $[0,1]$. Suppose that there exist disjoint open intervals $I_{n_1,\ldots,n_k}$ disposed on $(0,1)$ respecting the lexicographic order and so that for all $(n_1, \ldots, n_k) \in \mathbb{Z}^k$ and all $i \in \{1, \ldots, k\}$,*

$$f_i(I_{n_1,\ldots,n_i,\ldots,n_k}) = I_{n_1,\ldots,n_i+1,\ldots,n_k}.$$

Then $f_1, \ldots, f_{k-1}$ cannot be all simultaneously of class $C^{1+1/(k-1)}$ provided that f_k is of class $C^{1+\alpha}$ for some $\alpha > 0$.

2.2 Proof of Lemma 2

As discussed in the previous section, in order to finish the proof of Theorem B, we need to prove Lemma 2. A first crucial step is given by the next result, which can be thought of as a version of Denjoy's theorem on the interval and corresponds to an extension of [3, Theorem C] for the case where the maps are not assumed to commute.

Theorem 2. *Given an integer $d \geq 2$ and $\alpha > 1/d$, suppose that G is a subgroup of $\mathrm{Diff}_+^{1+\alpha}([0,1])$ whose action is semiconjugate to a free action by translations of $\mathbb{Z}^d$. Then G is Abelian and acts minimally on $(0,1)$.*

For the proof of Theorem 2, we state a lemma that is a special case of [3, Lemma 2.2].

Lemma 3. *Let Γ be a group of $C^{1+\alpha}$-diffeomorphism of a 1-dimensional compact variety M^1. Suppose there exists a finite subset $\mathcal{G}$ of Γ, and interval I of M^1, and a constant $S < \infty$, so that the following holds: For every $n \in \mathbb{N}$, there is an element $h_n = f_{i_n} \ldots f_{i_1} \in \Gamma$ such that each f_{i_k} belongs to $\mathcal{G}$ and*

$$\sum_{k=1}^{n-1} |f_{i_k} \ldots f_{i_1}(I)|^{\alpha} \leq S.$$

Then, there is a positive constant $L = L(\alpha, S, |I|, \mathcal{G})$ such that if $h_n(I)$ does not intersect I but is contained in the L-neighborhood of I, then h_n has an hyperbolic fixed point (inside the $2L$-neighborhood of I).

Proof of Theorem 2: Looking for a contradiction, we suppose that the action of G is not minimal. We let I be a maximal open interval that is mapped into a single point by the semiconjugacy into a group of translations, and we let $f_1, \ldots, f_d \in G$ be elements whose semiconjugate images generate $\mathbb{Z}^d$.

Following [3], let us consider the Markov process on $\mathbb{N}_0^d$ with transtion probabilities

$$p\big((n_1,\ldots,n_i,\ldots,n_d)\to(n_1,\ldots,1+n_i,\ldots,n_d)\big):=\frac{1+n_i}{d+n_1+\ldots+n_d}.$$

Let us denote by Ω the space of infinite paths ω endowed with the induced probability measure $\mathbb{P}$. Let $S\colon\Omega\to\mathbb{R}$ be defined by

$$S(w)=\sum_{k\geq 0}|I_{\omega_k}|^\alpha,$$

where $w_k=(n_{1,k},\ldots,n_{d,k})$ denotes the position of w at time k, and $I_{n_1,\ldots,n_d}:=f_1^{n_1}\ldots f_d^{n_d}(I)$. Since $\alpha>1/d$, this function has a finite expectation (see [3]). Thus, its value at a generic random sequence ω is finite. Moreover, by an easy application of the Bernoulli $0-1$ law, given $L>0$ we have that, for any generic sequence ω, infinitely many intervals I_{ω_k} are contained in the L-neighborhood of I. By Lemma 3, a generic sequence ω would hence lead to infinitely many nontrivial elements having hyperbolic fixed points, which contradicts the freeness of the action. $\square$

We now proceed to the proof of Lemma 2. Suppose N_4 is acting faithfully by $C^{1+\alpha}$ diffeomorphisms of $[0,1]$ for some $\alpha>1/2$. By Remark 1, if $x_0\in[0,1]$ is a point moved by c, then N_4 acts faithfully on its orbit.

The filtration of intervals J_i and subgroups K_i of Lemma 2 are easy to define. We let $K_0:=N_4$ and define J_0 as the smallest K_0-invariant open interval containing x_0. We also let μ_0 be a K_0-invariant Radon measure on J_0, and we denote by $\tau_{\mu_0}:N_4\to\mathbb{R}$ its associated translation-number homomorphism.

In general if J_i, K_i and $\tau_{\mu_i}:K_i\to\mathbb{R}$ have been already defined and the action of K_i on J_i has no global fixed point, then we let $K_{i+1}:=Ker(\tau_{\mu_i})$. If K_{i+1} does not fix x_0, then we let J_{i+1} be the smallest K_{i+1}-invariant open interval containing x_0, we denote by μ_{i+1} a K_{i+1}-invariant Radon measure on J_{i+1} and by $\tau_{\mu_{i+1}}:K_{i+1}\to\mathbb{R}$ its associated translation-number homomorphism. If K_{i+1} fixes x_0, then we stop the filtration at the previous step.

Notice that the procedure above has to end at some moment, because G is nilpotent, hence has finite (torsion-free) rank. Therefore, in order to finish the proof of Lemma 2, it only remains to show the third point in its conclusion, namely that the number of "levels" is at least 3, and the corresponding g_i can be chosen to commute. The rest of this section is devoted to this task.

Since N_4 is non-Abelian, Theorem 2 implies that the image $\tau_{\mu_0}(N_4)\subset\mathbb{R}$ has rank 1. Thus, as N_4 is finitely generated, up to rescaling μ_0, we have

$$\tau_{\mu_0}\colon N_4\to\mathbb{Z}.$$

As $\langle a,b,c\rangle$ is the commutator subgroup $[N_4,N_4]$, we conclude that there exist h_1 and h_2 in $N_4\setminus\langle a,b,c\rangle$ such that $K_1:=Ker(\tau_{\mu_0})=\langle h_1,h_2,a,b,c\rangle$, say

$$h_1=f^{m_1}e^{m_2}d^{m_3}\ ,\ \ h_2=f^{n_1}e^{n_2}d^{n_3},\quad \text{for some } m_i,\, n_i \text{ in } \mathbb{Z}.$$

Notice that

$$[f^{m_1}e^{m_2}d^{m_3},a]=c^{m_3}\quad\text{and}\quad[f^{m_1}e^{m_2}d^{m_3},b]=c^{-m_2}. \tag{6}$$

Therefore, the conjugacy action of $\langle h_1,h_2\rangle$ on $\langle a,b,c\rangle$ is nontrivial. To finish the proof, we will separately analyze three cases. The first one will be when h_1 and h_2 commute modulo $\langle c\rangle$. The other two cases correspond to different instances in which h_1 and h_2 do not commute modulo $\langle c\rangle$. To differentiate them, we notice that

$$[f^{m_1}e^{m_2}d^{m_3},f^{n_1}e^{n_2}d^{n_3}]=(e^{m_2}d^{m_3})^{f^{m_1}}(e^{n_2-m_2}d^{n_3-m_3})^{f^{m_1+n_1}}(d^{-n_3}e^{-n_2})^{f^{n_1}},$$

and

$$[f, e^{\ell} d^{n}] = a^{\ell} b^{-n} c^{\ell n}. \tag{7}$$

Using this, one readily checks that, modulo $\langle c \rangle$, we have that

$$\begin{aligned} [h_1, h_2] = [f^{m_1} e^{m_2} d^{m_3}, f^{n_1} e^{n_2} d^{n_3}] &\equiv (a^{m_2} b^{-m_3})^{m_1} \, (a^{n_2 - m_2} b^{m_3 - n_3})^{m_1 + n_1} (a^{-n_2} b^{n_3})^{n_1} \\ &\equiv a^{m_1 n_2 - n_1 m_2} \; b^{n_1 m_3 - m_1 n_3}. \end{aligned} \tag{8}$$

Case 1: The elements h_1 and h_2 commute modulo $\langle c \rangle$.

We first claim that, in this case, $\tau_{\mu_0}(e) = 0 = \tau_{\mu_0}(d)$, and hence $\tau_{\mu_0}(f) \neq 0$. Indeed, equation (8) yields

$$m_1 n_2 = n_1 m_2 \quad \text{and} \quad n_1 m_3 = m_1 n_3.$$

On the other hand, since $K_1 / \langle a, b, c \rangle \simeq \mathbb{Z}^2$, the set $\{(m_1, m_2, m_3), (n_1, n_2, n_3)\}$ must be linearly independent over $\mathbb{Q}$. As $n_1(m_1, m_2, m_3) - m_1(n_1, n_2, n_3) = 0$, we conclude that $n_1 = m_1 = 0$. It follows that some nontrivial power of e and some nontrivial power of d have trivial image under τ_{μ_0}. Therefore, since $\mathbb{Z}$ is torsion free, we have that $\tau_{\mu_0}(e) = 0 = \tau_{\mu_0}(d)$, and the claim follows.

Notice that, conversely, if $\tau_{\mu_0}(e) = 0 = \tau_{\mu_0}(d)$, then $m_1 = n_1 = 0$, and (8) shows that h_1 and h_2 do commute modulo $\langle c \rangle$.

In particular, in Case 1, we may actually take

$$h_1 = e \quad \text{and} \quad h_2 = d,$$

which are commuting elements of N_4.

Now, since $c \in K_1 = Ker(\tau_{\mu_0})$ does not fix x_0, we have that the action of K_1 on J_1 has no global fixed point. Further, (6) implies that K_1 is non-Abelian, so Theorem 2 implies that $\tau_{\mu_1}(K_1)$ has a cyclic image. Notice that, again by (6), the element c belongs to $Ker(\tau_{\mu_1})$.

Subcase 1.1: $\tau_{\mu_1}(\langle a, b \rangle) \neq 0$.

Then we can finish the proof of Lemma 2 by letting

$$g_1 := f, \;\; g_2 \in \langle a, b \rangle \text{ such that } \tau_{\mu_1}(g_2) \neq 0, \quad \text{and } g_3 := c.$$

Subcase 1.2: $\tau_{\mu_1}(\langle a, b \rangle) = 0$.

Let u, v be nontrivial elements in $\langle e, d \rangle$ such that $\tau_{\mu_1}(u) = 0$, $\tau_{\mu_1}(v) \neq 0$, and $K_2 := Ker(\tau_{\mu_1}) = \langle u, a, b, c \rangle$. It follows from (6) that there is $h \in \langle a, b \rangle$ such that $id \neq [u, h] \in \langle c \rangle$. In particular, K_2 is non-Abelian, and c is in the kernel of τ_{μ_2}. Besides, by Theorem 2, $\tau_2(K_2)$ is isomorphic to $\mathbb{Z}$. There are two possibilities:

- If $\tau_{\mu_2}(u) \neq 0$, then we finish the proof of Lemma 2 by letting

$$g_1 := v, \;\; g_2 := u, \;\; g_3 := c.$$

- If $\tau_{\mu_2}(u) = 0$, then there is $h' \in \langle a, b \rangle$ with nontrivial image under τ_{μ_2}. We hence finish the proof of Lemma 2 by letting

$$g_1 := f, \;\; g_2 := h', \;\; g_3 := c.$$

Case 2: The elements h_1 and h_2 do not commute modulo $\langle c \rangle$ and $\tau_{\mu_0}(f) \neq 0$.

Notice that some element in $\langle d, e \rangle$ lies in $Ker(\tau_{\mu_0})$, hence we can take

$$h_1 = f^{m_1} e^{m_2} d^{m_3}\,, \text{ with } m_1 \neq 0\,, \quad \text{and} \quad h_2 = e^{\ell} d^{n}.$$

Now, since h_1 and h_2 do not commute modulo $\langle c \rangle$, the subgroup $\langle e, d \rangle$ cannot be fully contained in $Ker(\tau_{\mu_0})$. Therefore, if we let p, q to be integers such that $\ell q + pn = 1$, then

$$\tau_{\mu_0}(e^{-p} d^{q}) \neq 0.$$

Now since $c \in K_1$ does not fix x_0, we have that the action of K_1 on J_1 has no global fixed point. Further, (6) implies that $\tau_{\mu_1}(c) = 0$, which together with (7) implies that $\tau_{\mu_1}(a^{\ell} b^{-n}) = 0$.

Notice that, as $K_1 = \langle f^{m_1} e^{m_2} d^{m_3}, e^{\ell} d^{n}, a, b, c \rangle$ is non-Abelian, the image $\tau_{\mu_1}(K_1)$ has rank 1. Besides, it is determined by the image of the set $\{f^{m_1} e^{m_2} d^{m_3}, e^{\ell} d^{n}, a^{p} b^{q}\}$, where $\ell q + pn = 1$.

Subcase 2.1: $\tau_{\mu_1}(f^{m_1} e^{m_2} d^{m_3}) = 0$.

In this case, we have that either $e^{\ell} d^{n}$ or $a^{p} b^{q}$ has nontrivial image under τ_{μ_1}. In the first case, we finish the proof by letting

$$g_1 := e^{-p} d^{q}, \;\; g_2 := e^{\ell} d^{n}, \;\; g_3 := c,$$

and in the second case, by letting

$$g_1 := f, \;\; g_2 := a^{q} b^{q}, \;\; g_3 := c.$$

Subcase 2.2: $\tau_{\mu_1}(f^{m_1} e^{m_2} d^{m_3}) \neq 0$.

If either $e^{\ell} d^{n}$ or $a^{p} b^{q}$ has nontrivial image under τ_{μ_1}, then we can repeat the argument of Subcase 2.1. So, we assume that

$$\tau_{\mu_1}(f^{m_1} e^{m_2} d^{m_3}) \neq 0\,, \; \tau_{\mu_1}(e^{\ell} d^{n}) = 0\,, \; \tau_{\mu_1}(a^{p} b^{q}) = 0.$$

In this case we, $K_2 = Ker(\tau_{\mu_1})$ is generated by the set $\{e^{\ell} d^{n}, a, b, c\}$. It then follows from (6) that K_2 is non-Abelian, and $\tau_{\mu_2}(c) = 0$. By Theorem 2, the image $\tau_{\mu_2}(K_2)$ has rank 1. There are two possibilities:

- If $\tau_{\mu_2}(e^{\ell} d^{n}) \neq 0$, then we finish the proof of Lemma 2 by letting

$$g_1 := e^{-p} d^{q}, \;\; g_2 := e^{\ell} d^{n}, \;\; g_3 := c.$$

- If $\tau_{\mu_2}(e^{\ell} d^{n}) = 0$, then some nontrivial $w \in \langle a, b \rangle$ must have a nonzero image under τ_{μ_2}. We can hence finish the proof by letting

$$g_1 := f, \;\; g_2 := w, \;\; g_3 := c.$$

Case 3: The elements h_1 and h_2 do not commute modulo $\langle c \rangle$ and $\tau_{\mu_0}(f) = 0$.

Again, we can take

$$h_1 = f \quad \text{and} \quad h_2 = e^{\ell} d^{n},$$

and if we let p, q to be integers such that $\ell q + pn = 1$, then

$$\tau_{\mu_0}(e^{-p} d^{q}) \neq 0.$$

Moreover, since $c \in K_1$ does not fix x_0, the action of K_1 on J_1 has no global fixed point. Further, by (6), we have $\tau_{\mu_1}(c) = 0$, which together with (7) implies that $\tau_{\mu_1}(a^\ell b^{-n}) = 0$. Besides all of this, we have that the image $\tau_{\mu_1}(K_1)$ has rank 1, and it is determined by the image of the set $\{f, e^\ell d^n, a^p b^q\}$.

Subcase 3.1: $\tau_{\mu_1}(f) = 0$ and $\tau_{\mu_1}(e^\ell d^n) \neq 0$.

In this case, we can finish the proof of Lemma 2 by letting

$$g_1 := e^{-p} d^q, \;\; g_2 := e^\ell d^n, \;\; g_3 := c.$$

Subcase 3.2: $\tau_{\mu_1}(f) = 0$, $\tau_{\mu_1}(e^\ell d^n) = 0$.

There are three possibilities:

- If $\ell = 0$, then we may take $p = 1$ and $q = 0$. Hence in this case the image $\tau_{\mu_1}(K_1)$ is determined by the image of the set $\{f, d, a\}$. Since $\tau_{\mu_1}(f) = \tau_{\mu_1}(e^\ell d^n) = 0$, it follows that $\tau_{\mu_1}(a) \neq 0$. We can hence finish the proof by letting

$$g_1 := e\,, \;\; g_2 := a\,, \;\; g_3 := c.$$

- If $n = 0$, then we may proceed in an analogous way as above.

- Finally, assume that $\ell n \neq 0$. Since $\tau_{\mu_1}(a^p b^q) \neq 0$, we have $K_2 := Ker(\tau_{\mu_1}) = \langle f, e^\ell d^n, a^\ell b^{-n}, c\rangle$. Moreover, by (6), we have $[e^\ell d^n, a^\ell b^{-n}] = c^{2\ell n}$, so $\tau_{\mu_2}(c) = 0$. By (7), this implies that $\tau_{\mu_2}(a^\ell b^{-n}) = 0$. As a consequence, the image of τ_{μ_2} is determined by the image of the set $\{f, e^\ell b^n\}$. If, on the one hand, $\tau_{\mu_2}(f) \neq 0$, then we can finish the proof by letting

$$g_1 := a^p b^q\,, \;\; g_2 := f\,, \;\; g_3 := c.$$

  If, on the other hand, $\tau_{\mu_2}(e^\ell b^n) \neq 0$, then we can finish the proof by letting

$$g_1 := e^{-p} d^q\,, \;\; g_2 := e^\ell d^n, g_3 := c.$$

Subcase 3.3: $\tau_{\mu_1}(f) \neq 0$ and $\tau_{\mu_1}(e^\ell d^n) \neq 0$.

In this case, we can finish the proof by letting

$$g_1 := e^{-p} d^q\,, \;\; g_2 := e^\ell d^n\,, \;\; g_3 := c.$$

Subcase 3.4: $\tau_{\mu_1}(f) \neq 0$ and $\tau_{\mu_1}(e^\ell d^n) = 0$.

In this case, $\langle e^\ell d^n, a^\ell b^{-n}, c\rangle$ is contained in K_2. Let i be the smallest integer such that τ_{μ_i} is defined on $\langle e^\ell d^n, a^\ell b^{-n}, c\rangle$ and that this restriction is a nontrivial morphism into $\mathbb{R}$. By (6), we have $\tau_{\mu_i}(c) = 0$, hence either $e^\ell d^n$ or $a^\ell b^{-n}$ has nontrivial image under τ_{μ_i}.

- If $\tau_{\mu_i}(e^\ell d^n) \neq 0$, we can finish the proof of Lemma 2 by letting

$$g_1 := e^{-p} d^q\,, \;\; g_2 := e^\ell d^n\,, \;\; g_3 := c.$$

- If $\tau_{\mu_i}(a^\ell b^{-n}) \neq 0$, we can finish the proof of Lemma 2 by letting

$$g_1 := f\,, \;\; g_2 := a^\ell b^{-n}\,, \;\; g_3 := c.$$

This finishes the proof of Lemma 2, hence that of Theorem B. □

3 The embedding

We next prove Theorem A. For the rest of this work, we fix α such that $0 < \alpha < 1/2$. In order to produce an embedding of N_4 into $\mathrm{Diff}_+^{1+\alpha}([0,1])$, we will project to the interval the action provided by Proposition 1 using the so-called Pixton-Tsuboi maps [13, 15]. This technique is summarized in the next

Lemma 4. *There exists a family of C^∞ diffeomorphisms $\varphi_{I',I}^{J',J} : I \to J$ between intervals I, J, where I' (resp. J') is an interval contiguous to I (resp. J) by the left, such that:*
– (Equivariance) For all I, I', J, J', K, K' as above,

$$\varphi_{J',J}^{K',K} \circ \varphi_{I',I}^{J',J} = \varphi_{I',I}^{K',K};$$

– (Derivatives at the endpoints) For all I, I', J, J' we have

$$D\varphi_{I,I'}^{J,J'}(x_-) = \frac{|J'|}{|I'|}, \quad D\varphi_{I,I'}^{J,J'}(x_+) = \frac{|J|}{|I|}.$$

where x_- (resp. x_+) is the left (resp. right) endpoint of I.
– (Regularity) There is a constant M such that for all $x \in I$:

$$D\log(D\varphi_{I',I}^{J',J})(x) \leq \frac{M}{|I|} \cdot \left| \frac{|I|}{|J|} \frac{|J'|}{|I'|} - 1 \right|.$$

provided that $\max\{|I'||I|, |J'|, |J|\} \leq 2\min\{|I'||I|, |J'|, |J|\}$.

To produce our action, we let $I_{i,j,k}$ be a collection of intervals indexed by $\mathbb{Z}^3$ whose union is dense in $[0,1]$ and that are disposed preserving the lexicographic order of $\mathbb{Z}^3$. We then define the homeomorphisms d, e, f of $[0,1]$ as those whose restrictions to $I_{i,j,k}$ coincide, respectively, with

$$\varphi_{I_{i,j,k-1}, I_{i,j,k}}^{I_{i+1,j,k-1}, I_{i+1,j,k}}, \quad \varphi_{I_{i,j,k-1}, I_{i,j,k}}^{I_{i,j+1,k-1}, I_{i,j+1,k}}, \quad \varphi_{I_{i,j,k-1}, I_{i,j,k}}^{I_{i,j,k+ij-1}, I_{i,j,k+ij}}.$$

By (*Equivariance*), this produces a faithful action of N_4 by homeomorphisms of $[0,1]$.

Proposition 2. *For an appropriate choice of the lengths $|I_{i,j,k}|$, the homeomorphisms e, f, d are simultaneously of class $C^{1+\alpha}$.*

The rest of this work is devoted to the proof of this result. To begin with, we let p, q, r be positive reals for which the following conditions hold:
(i) $r < 2$,
(ii) $4r \leq p$,
(iii) $4r \leq q$,
(iv) $\alpha \leq \frac{2}{r} - 1$,
(v) $4 \leq p(1-\alpha)$,
(vi) $4 \leq q(1-\alpha)$,
(vii) $1/p + 1/q + 1/r < 1$.
For example, we can take $p = q := 4/\alpha$ and $r := 4/3$.

Now, let $I_{i,j,k}$ be an interval such that

$$|I_{i,j,k}| := \frac{1}{|i|^p + |j|^q + |k|^r + 1}.$$

Condition (vii) ensures that

$$\sum_{(i,j,k)\in\mathbb{Z}^3} \big|I_{i,j,k}\big| < \infty,$$

hence the $I_{i,j,k}$'s can be disposed on a finite interval respecting the lexicographic order. This interval can be though of as $[0,1]$ after renormalization.

It is proved in [2] that, with any choice of lengths as above, the maps e and d are $C^{1+\alpha}$ diffeomorphisms. Thus, in order to finish the proof, we need to show

Lemma 5. *For any choice of lengths of intervals satisfying properties* (i),...,(vii) *above, the homeomorphism f is a $C^{1+\alpha}$ diffeomorphism.*

Notice that this lemma is equivalent to that the expression

$$\frac{\big|\log Df(x) - \log Df(y)\big|}{|x-y|^{\alpha}}$$

is uniformly bounded (independently of x and y). To check this, due to property (*Derivatives at the endpoints*) above, it suffices to consider points x,y in intervals $I_{i,j,k}$ and $I_{i,j,k'}$, respectively; this means that the first "two levels" i and j coincide (compare [2, §3.3, III]). We first treat points x,y in the same interval $I_{i,j,k}$, and then points in intervals with different indices k,k'.

Let us consider points x and y in the same interval $I_{i,j,k}$. By (*Regularity*) in Lemma 4 and the Mean Value Theorem,

$$\begin{aligned}\frac{|\log Df(x)-\log Df(y)|}{|x-y|^{\alpha}} &\leq \frac{M}{|I_{i,j,k}|^{\alpha}}\left|\frac{|I_{i,j,k}|}{|I_{i,j,k+ij}|}\frac{|I_{i,j,k+ij-1}|}{|I_{i,j,k-1}|}-1\right| \\ &= M\left|\frac{(|i|^p{+}|j|^q{+}|k+ij|^r{+}1)(|i|^p{+}|j|^q{+}|k-1|^r{+}1)}{(|i|^p{+}|j|^q{+}|k|^r{+}1)(|i|^p{+}|j|^q{+}|k+ij-1|^r{+}1)}-1\right|(|i|^p{+}|j|^q{+}|k|^r{+}1)^{\alpha}.\end{aligned}$$

Using the Mean Value Theorem, the last expression is easily seen to be smaller than or equal to

$$\frac{MC}{(|i|^p{+}|j|^q{+}|k|^r{+}1)^{1-\alpha}(|i|^p{+}|j|^q{+}|k+ij-1|^r{+}1)}, \tag{9}$$

where

$$\begin{aligned}C \;:=\; &|i|^p r(|k|{+}1)^{r-1} + |j|^q r(|k|{+}1)^{r-1} + |i|^p r(|k+ij|{+}1)^{r-1} + |j|^q r(|k+ij|{+}1)^{r-1} + \\ &+ r(|k|{+}1)^{r-1} + r(|k+ij|{+}1)^{r-1} + |k+ij|^r r(|k|{+}1)^{r-1} + |k|^r r(|k+ij|{+}1)^{r-1}.\end{aligned}$$

We need to check that the expression in (9) is bounded (uniformly) by some constant depending on p, q, r and α (but not on i,j,k). To do this, we will bound the leading terms in (9), which are the following:

$$(I)\quad \frac{|i|^p r(|k+ij|{+}1)^{r-1}}{(|i|^p{+}|j|^q{+}|k|^r{+}1)^{1-\alpha}(|i|^p{+}|j|^q{+}|k+ij-1|^r{+}1)},$$

$$(II)\quad \frac{|j|^q r(|k+ij|{+}1)^{r-1}}{(|i|^p{+}|j|^q{+}|k|^r{+}1)^{1-\alpha}(|i|^p{+}|j|^q{+}|k+ij-1|^r{+}1)},$$

$$(III)\quad \frac{|k+ij|^r r(|k|{+}1)^{r-1}}{(|i|^p{+}|j|^q{+}|k|^r{+}1)^{1-\alpha}(|i|^p{+}|j|^q{+}|k+ij-1|^r{+}1)},$$

$$(IV)\quad \frac{|k|^r r(|k+ij|+1)^{r-1}}{(|i|^p+|j|^q+|k|^r+1)^{1-\alpha}(|i|^p+|j|^q+|k+ij-1|^r+1)}.$$

The term (I) is smaller than

$$\frac{r(|k|+|ij|+1)^{r-1}}{(|i|^p+|j|^q+|k|^r+1)^{1-\alpha}}.$$

We have two cases:

- If $|k|\leq |ij|$, then $\frac{r(|k|+|ij|+1)^{r-1}}{(|i|^p+|j|^q+|k|^r+1)^{1-\alpha}} \leq \frac{r(2|ij|+1)^{r-1}}{(|i|^p+|j|^q+1)^{1-\alpha}}$. There are two possibilities:
  - If $|i| \leq |j|$, then $\frac{r(2|ij|+1)^{r-1}}{(|i|^p+|j|^q+1)^{1-\alpha}} \leq \frac{r(2|j|^2+1)^{r-1}}{(|j|^q+1)^{1-\alpha}}$, which is bounded since $2(r-1) \leq q(1-\alpha)$, which follows from conditions (i) and (iv) above.
  - If $|i| \geq |j|$, then $\frac{r(2|ij|+1)^{r-1}}{(|i|^p+|j|^q+1)^{1-\alpha}} \leq \frac{r(2|i|^2+1)^{r-1}}{(|i|^p+1)^{1-\alpha}}$, which is again bounded since $2(r-1) \leq p(1-\alpha)$ (conditions (i) and (iii)).
- If $|ij|\leq |k|$, then $\frac{r(|k|+|ij|+1)^{r-1}}{(|i|^p+|j|^q+|k|^r+1)^{1-\alpha}} \leq \frac{r(2|k|+1)^{r-1}}{(|k|^r+1)^{1-\alpha}}$, and this expression is bounded because $\alpha \leq 1/r$, which follows from (i) together with $\alpha < 1/2$.

The term (II) is similar to (I), and it can be ruled out by the same procedure.

We next deal with (III). Since

$$\frac{|k+ij|^r}{|i|^p+|j|^q+|k+ij-1|^r+1}$$

is obviously bounded and

$$\frac{(|k|+1)^{r-1}}{(|i|^p+|j|^q+|k|^r+1)^{1-\alpha}} \leq \frac{(|k|+1)^{r-1}}{(|k|^r+1)^{1-\alpha}},$$

we have that the term (III) is bounded because $\alpha \leq 1/r$.

Finally, we deal with (IV). Observe that this expression equals

$$\frac{r|k|^{r-1}}{(|i|^p+|j|^q+|k|^r+1)^{1-\alpha}} \cdot \frac{|k|(|k+ij|+1)^{r-1}}{|i|^p+|j|^q+|k+ij-1|^r+1}.$$

The first factor is bounded again because $\alpha \leq 1/r$. The second factor is smaller than or equal to

$$\frac{(|k+ij|+|ij|)(|k+ij|+1)^{r-1}}{|i|^p+|j|^q+|k+ij-1|^r+1}.$$

We consider two cases:

- If $|ij|\leq |k+ij|$, then

  $$\frac{(|k+ij|+|ij|)(|k+ij|+1)^{r-1}}{|i|^p+|j|^q+|k+ij-1|^r+1} \leq \frac{2|k+ij|(|k+ij|+1)^{r-1}}{|k+ij-1|^r+1},$$

  which is obviously bounded.

- If $|k+ij|\leq |ij|$, then

$$\frac{(|k+ij|+|ij|)(|k+ij|+1)^{r-1}}{|i|^p+|j|^q+|k+ij-1|^r+1} \leq \frac{2|ij|(|ij|+1)^{r-1}}{|i|^p+|j|^q+1},$$

  which is easily seen to be bounded because $2r \leq q$ and $2r \leq p$, which follow from conditions (ii) and (iii).

Finally, we consider points $x \in I_{i,j,k}$ and $y \in I_{i,j,k'}$, whith $k' > k$. For simplicity, we assume that $k'-k \geq 2$. (The case $k'=k+1$ follows from the previous one using property (*Derivatives at the endpoints*) just comparing to the right endpoint of $I_{i,j,k}$.) Besides, we assume k,k' to be positive (the negative situation follows by symmetry). In this case, using [2, (20)], it readily follows that $|\log Df(x) - \log Df(y)|$ is smaller than or equal to

$$\left|\log\frac{|I_{i,j,k+ij}|}{|I_{i,j,k}|} - \log\frac{|I_{i,j,k'+ij}|}{|I_{i,j,k'}|}\right| + \left|\log\frac{|I_{i,j,k+ij-1}|}{|I_{i,j,k-1}|} - \log\frac{|I_{i,j,k+ij}|}{|I_{i,j,k}|}\right| + \left|\log\frac{|I_{i,j,k'+ij-1}|}{|I_{i,j,k'-1}|} - \log\frac{|I_{i,j,k'+ij}|}{|I_{i,j,k'}|}\right|.$$

The last two terms are easy to estimate, as the indices k,k' do not mix in none of these. Hence, we need to estimate the first term; more precisely, we need to find an upper bound for

$$\frac{1}{|x-y|^\alpha}\left|\log\frac{|I_{i,j,k+ij}|}{|I_{i,j,k}|} - \log\frac{|I_{i,j,k'+ij}|}{|I_{i,j,k'}|}\right|.$$

Notice that

$$\left|\log\frac{|I_{i,j,k+ij}|}{|I_{i,j,k}|} - \log\frac{|I_{i,j,k'+ij}|}{|I_{i,j,k'}|}\right| = \left|\log\left(\frac{|i|^p+|j|^q+|k|^r+1}{|i|^p+|j|^q+|k+ij|^r+1}\cdot\frac{|i|^p+|j|^q+|k'+ij|^r+1}{|i|^p+|j|^q+|k'|^r+1}\right)\right|.$$

The last expression equals

$$\left|\log\left(1+\frac{(|i|^p+|j|^q+|k|^r+1)(|i|^p+|j|^q+|k'+ij|^r+1)-(|i|^p+|j|^q+|k+ij|^r+1)(|i|^p+|j|^q+|k'|^r+1)}{(|i|^p+|j|^q+|k+ij|^r+1)(|i|^p+|j|^q+|k'|^r+1)}\right)\right|,$$

which is smaller than or equal to

$$M\left|\frac{(|i|^p+|j|^q+|k|^r+1)(|i|^p+|j|^q+|k'+ij|^r+1)-(|i|^p+|j|^q+|k+ij|^r+1)(|i|^p+|j|^q+|k'|^r+1)}{(|i|^p+|j|^q+|k+ij|^r+1)(|i|^p+|j|^q+|k'|^r+1)}\right|,$$

where M is a universal constant. By the Mean Value Theorem, the last expression is smaller than or equal to

$$M\left|\frac{C}{(|i|^p+|j|^q+|k+ij|^r+1)(|i|^p+|j|^q+|k'|^r+1)}\right|,$$

where C equals

$$\begin{aligned} C \;:=\; & |i|^p r(k'+|ij|)^{r-1}|ij| + |j|^q r(k'+|ij|)^{r-1}|ij| + |i|^p r(k+|ij|)^{r-1}|ij| + |j|^q r(k+|ij|)^{r-1}|ij| + \\ & + r(k'+|ij|)^{r-1}|ij| + r(k+|ij|)^{r-1}|ij| + |k|^r r(k'+|ij|)^{r-1}|ij| + |k'|^r r(k+|ij|)^{r-1}|ij|. \end{aligned}$$

Therefore, it is enough to obtain an upper bound for

$$\frac{|i|^p r(k'+|ij|)^{r-1}|ij| + |j|^q r(k'+|ij|)^{r-1}|ij| + |k|^r r(k'+|ij|)^{r-1}|ij| + |k'|^r r(k+|ij|)^{r-1}|ij|}{(|i|^p+|j|^q+|k+ij|^r+1)(|i|^p+|j|^q+|k'|^r+1)|x-y|^\alpha}. \tag{10}$$

To estimate this expression, we consider four different cases:

Case 1: $k' \leq 2k+1$,

Case 2: $k'^r \leq |i|^p + |j|^q$,
Case 3: $k' \geq 2k+2$, $k'^r \geq |i|^p + |j|^q$ and $k^r \geq |i|^p + |j|^q$,
Case 4: $k' \geq 2k+2$, $k'^r \geq |i|^p + |j|^q$ and $k^r \leq |i|^p + |j|^q$.

Case 1: Using $|x-y| \geq (k'-k-1)|I_{i,j,k'}|$, we see that we need to estimate the value of

$$\frac{|i|^p r(k'+|ij|)^{r-1}|ij| + |j|^q r(k'+|ij|)^{r-1}|ij| + |k|^r r(k'+|ij|)^{r-1}|ij| + |k'|^r r(k+|ij|)^{r-1}|ij|}{(|i|^p+|j|^q+|k+ij|^r+1)(|i|^p+|j|^q+|k'|^r+1)^{1-\alpha}(k'-k-1)^\alpha}.$$

To do this, we will show that the each of the terms

$$(I) \quad \frac{|i|^p r(k'+|ij|)^{r-1}|ij|}{(|i|^p+|j|^q+|k+ij|^r+1)(|i|^p+|j|^q+|k'|^r+1)^{1-\alpha}},$$

$$(II) \quad \frac{|j|^q r(k'+|ij|)^{r-1}|ij|}{(|i|^p+|j|^q+|k+ij|^r+1)(|i|^p+|j|^q+|k'|^r+1)^{1-\alpha}},$$

$$(III) \quad \frac{|k|^r r(k'+|ij|)^{r-1}|ij|}{(|i|^p+|j|^q+|k+ij|^r+1)(|i|^p+|j|^q+|k'|^r+1)^{1-\alpha}},$$

$$(IV) \quad \frac{|k'|^r r(k+|ij|)^{r-1}|ij|}{(|i|^p+|j|^q+|k+ij|^r+1)(|i|^p+|j|^q+|k'|^r+1)^{1-\alpha}}$$

is bounded.

Notice that the terms (I) and (II) are smaller than or equal to

$$\frac{r(k'+|ij|)^{r-1}|ij|}{(|i|^p+|j|^q+|k'|^r+1)^{1-\alpha}} = r\frac{(k'+|ij|)^{r-1}}{(|i|^p+|j|^q+|k'|^r+1)^{\frac{1-\alpha}{2}}}\frac{|ij|}{(|i|^p+|j|^q+|k'|^r+1)^{\frac{1-\alpha}{2}}}. \tag{11}$$

In the last expression, the second factor is estimated by

$$\frac{|ij|}{(|i|^p+|j|^q+|k'|^r+1)^{\frac{1-\alpha}{2}}} \leq \frac{|ij|}{(|i|^p+|j|^q+1)^{\frac{1-\alpha}{2}}},$$

which is bounded because $4 \leq p(1-\alpha)$ and $4 \leq q(1-\alpha)$ (conditions (v) and (vi)). To check that the first factor

$$\frac{(k'+|ij|)^{r-1}}{(|i|^p+|j|^q+|k'|^r+1)^{\frac{1-\alpha}{2}}}$$

is bounded as well, notice that:

- If $k' \leq |ij|$, this factor is smaller than or equal to

$$\frac{(2|ij|)^{r-1}}{(|i|^p+|j|^q+1)^{\frac{1-\alpha}{2}}},$$

which is bounded because $r<2$, $4 \leq p(1-\alpha)$ and $4 \leq q(1-\alpha)$ (conditions (i), (v) and (vi)).

- If $|ij| \leq k'$, this factor is smaller than or equal to

$$\frac{(2k')^{r-1}}{(|k'|^r+1)^{\frac{1-\alpha}{2}}},$$

which is bounded because $\alpha \leq 2/r-1$ (condition (iv)).

Next, we show that the expressions (III) and (IV) above are bounded. To do this, notice that from the previous estimates, we know that the expression

$$\frac{(k+|ij|)^{r-1}|ij|}{(|i|^p+|j|^q+|k|^r+1)^{1-\alpha}}$$

is bounded as $r<2$, $4\leq p(1-\alpha)$, $4\leq q(1-\alpha)$ and $\alpha\leq 2/r-1$ (conditions (i), (v), (vi) and (iv)). Therefore, it suffices to estimate the factors

$$\frac{|k|^r}{|i|^p+|j|^q+|k+ij|^r+1} \quad \text{and} \quad \frac{|k'|^r}{|i|^p+|j|^q+|k+ij|^r+1}. \tag{12}$$

The first factor is analyzed in two cases:

- If $|k+ij|\geq |ij|$, then
$$\frac{|k|^r}{|i|^p+|j|^q+|k+ij|^r+1}\leq \frac{(2|k+ij|)^r}{|k+ij|^r+1},$$
which is obviously bounded.
- If $|ij|\geq |k+ij|$, then
$$\frac{|k|^r}{|i|^p+|j|^q+|k+ij|^r+1}\leq \frac{(2|ij|)^r}{|i|^p+|j|^q+1},$$
which is bounded because $2r\leq p$ and $2r\leq q$ (these follow from conditions (ii) and (iii)).

For the second factor, we have

$$\frac{|k'|^r}{|i|^p+|j|^q+|k+ij|^r+1}\leq \frac{|2k+1|^r}{|i|^p+|j|^q+|k+ij|^r+1},$$

which is bounded since $2r\leq p$ and $2r\leq q$ (conditions (ii) and (iii)).

Case 2: This is similar to Case 1, except for that the very last expression $\frac{|k'|^r}{|i|^p+|j|^q+|k+ij|^r+1}$ above is now estimated by

$$\frac{|k'|^r}{|i|^p+|j|^q+|k+ij|^r+1}\leq \frac{|i|^p+|j|^q}{|i|^p+|j|^q+|k+ij|^r+1},$$

which is obviously bounded.

Case 3: In this case, we have the estimate

$$|x-y|\geq \frac{M}{(k+1)^{r-1}},$$

where M is a universal constant (see [2, §3.3, item (c)]). Thus, in order to estimate (10), it is enough to estimate the expression

$$(k+1)^{\alpha(r-1)}\frac{(|i|^p+|j|^q)r(k'+|ij|)^{r-1}|ij|+|k|^r r(k'+|ij|)^{r-1}|ij|+|k'|^r r(k+|ij|)^{r-1}|ij|}{(|i|^p+|j|^q+|k+ij|^r+1)(|i|^p+|j|^q+|k'|^r+1)}.$$

To do this, we will separately deal with the expressions below:

$$(I)\quad (k+1)^{\alpha(r-1)}\frac{(|i|^p+|j|^q)r(k'+|ij|)^{r-1}|ij|}{(|i|^p+|j|^q+|k+ij|^r+1)(|i|^p+|j|^q+|k'|^r+1)},$$

$$(II)\quad (k+1)^{\alpha(r-1)}\frac{|k|^r r(k'+|ij|)^{r-1}|ij|}{(|i|^p+|j|^q+|k+ij|^r+1)(|i|^p+|j|^q+|k'|^r+1)},$$

$$(III)\quad (k+1)^{\alpha(r-1)}\frac{|k'|^r r(k+|ij|)^{r-1}|ij|}{(|i|^p+|j|^q+|k+ij|^r+1)(|i|^p+|j|^q+|k'|^r+1)}.$$

For expression (I), it is enough to estimate

$$\frac{(k'+|ij|)^{r-1}|ij|k^{\alpha(r-1)}}{(|i|^p+|j|^q+|k'|^r+1)}=\frac{(k'+|ij|)^{r-1}|ij|}{(|i|^p+|j|^q+|k'|^r+1)^{1-\alpha}}\cdot\frac{k^{\alpha(r-1)}}{(|i|^p+|j|^q+|k'|^r+1)^{\alpha}}.$$

The second factor above is obviously bounded. For the first one, notice that it coincides with (11), and this is bounded because of the corresponding estimate in Case 1 (this estimate still applies).

For expression (II), it is enough to obtain an upper bound for

$$\frac{|k|^r(k'+|ij|)^{r-1}|ij|k^{\alpha(r-1)}}{(|i|^p+|j|^q+|k+ij|^r+1)(|i|^p+|j|^q+|k'|^r+1)},$$

which may be rewritten as

$$\frac{|k|^r}{(|i|^p+|j|^q+|k+ij|^r+1)}\cdot\frac{k^{\alpha(r-1)}}{(|i|^p+|j|^q+|k'|^r+1)^{\alpha}}\cdot\frac{(k'+|ij|)^{r-1}|ij|}{(|i|^p+|j|^q+|k'|^r+1)^{1-\alpha}}.$$

To do this, notice that the second factor is obviously bounded, and the third and the first one were already considered (see (11) and (12), respectively), and the corresponding estimates still apply.

Finally, for expression (III), it suffices to provide an upper bound for

$$\frac{|k'|^r(k+|ij|)^{r-1}|ij|k^{\alpha(r-1)}}{(|i|^p+|j|^q+|k+ij|^r+1)(|i|^p+|j|^q+|k'|^r+1)},$$

which reduces to estimate the expression

$$\frac{(k+|ij|)^{r-1}|ij|k^{\alpha(r-1)}}{|i|^p+|j|^q+|k+ij|^r+1}.$$

However, this equals

$$\frac{|i|^p+|j|^q+|k|^r+1}{|i|^p+|j|^q+|k+ij|^r+1}\cdot\frac{(k+|ij|)^{r-1}|ij|}{(|i|^p+|j|^q+|k|^r+1)^{1-\alpha}}\cdot\frac{k^{\alpha(r-1)}}{(|i|^p+|j|^q+|k|^r+1)^{\alpha}},$$

and these expressions were all considered when dealing with expression (II).

Case 4: In this case, we have the estimate

$$|x-y|\geq\frac{k'-k-1}{(k+1+S^{1/r})^{r-1}(k'+S^{1/r})},$$

where $S:=1+|i|^p+|j|^q$ (see [2, §3.3, item (d)]). Thus, in order to estimate (10), we need to obtain an upper bound for the expression

$$\frac{(k+1+S^{1/r})^{\alpha(r-1)}(k'+S^{1/r})^{\alpha}}{(k'-k-1)^{\alpha}}\cdot\frac{(|i|^p+|j|^q)r(k'+|ij|)^{r-1}|ij|+|k|^r r(k'+|ij|)^{r-1}|ij|+|k'|^r r(k+|ij|)^{r-1}|ij|}{(|i|^p+|j|^q+|k+ij|^r+1)(|i|^p+|j|^q+|k'|^r+1)}.$$

To do this, first notice that the term

$$\frac{(k'+S^{1/r})^{\alpha}}{(k'-k-1)^{\alpha}}$$

is bounded, as it readily follows from the hypothesis of this case. Hence, we need to estimate the expression

$$(k+1+S^{1/r})^{\alpha(r-1)} \frac{(|i|^p{+}|j|^q)r(k'+|ij|)^{r-1}|ij|{+}|k|^r r(k'+|ij|)^{r-1}|ij|{+}|k'|^r r(k+|ij|)^{r-1}|ij|}{(|i|^p+|j|^q+|k+ij|^r+1)(|i|^p+|j|^q+|k'|^r+1)}.$$

For this, we will separately consider the terms

$$(I) \quad (k+1+S^{1/r})^{\alpha(r-1)} \frac{(|i|^p{+}|j|^q{+}|k|^r)r(k'+|ij|)^{r-1}|ij|}{(|i|^p+|j|^q+|k+ij|^r+1)(|i|^p+|j|^q+|k'|^r+1)},$$

$$(II) \quad (k+1+S^{1/r})^{\alpha(r-1)} \frac{|k'|^r r(k+|ij|)^{r-1}|ij|}{(|i|^p+|j|^q+|k+ij|^r+1)(|i|^p+|j|^q+|k'|^r+1)}.$$

For (I), the factor

$$\frac{|i|^p{+}|j|^q{+}|k|^r}{|i|^p+|j|^q+|k+ij|^r+1}$$

is uniformly bounded because $2r \leq p$ and $2r \leq q$ (these follow from conditions (iii) and (iv)). Thus, we need to control the factor

$$\frac{(k'+|ij|)^{r-1}|ij|(k+1+S^{1/r})^{\alpha(r-1)}}{(|i|^p+|j|^q+|k'|^r+1)} = \frac{(k'+|ij|)^{r-1}|ij|}{(|i|^p+|j|^q+|k'|^r+1)^{1-\alpha}} \cdot \frac{(k+1+S^{1/r})^{\alpha(r-1)}}{(|i|^p+|j|^q+|k'|^r+1)^{\alpha}}.$$

From the previous computations (see (11)), we know that

$$\frac{(k'+|ij|)^{r-1}|ij|}{(|i|^p+|j|^q+|k'|^r+1)^{1-\alpha}}$$

is bounded. Moreover, due to the conditions $1+k^r \leq S \leq 1+k'^r$ (which follow from the hypothesis of this case), we have

$$\frac{(k+1+S^{1/r})^{\alpha(r-1)}}{(|i|^p+|j|^q+|k'|^r+1)^{\alpha}} \leq \frac{(S^{1/r}+1+S^{1/r})^{\alpha(r-1)}}{S^{\alpha}},$$

and the right-hand expression is obviously bounded.

Finally, to deal with expression (II), it suffices to estimate the expression

$$\frac{(k+1+S^{1/r})^{\alpha(r-1)}(k+|ij|)^{r-1}|ij|}{|i|^p+|j|^q+|k+ij|^r+1}.$$

However, this may be rewritten as the product

$$\frac{(k+1+S^{1/r})^{\alpha(r-1)}}{(|i|^p+|j|^q+|k|^r+1)^{\alpha}} \cdot \frac{(k+|ij|)^{r-1}|ij|}{(|i|^p+|j|^q+|k|^r+1)^{1-\alpha}} \cdot \frac{|i|^p+|j|^q+|k|^r+1}{|i|^p+|j|^q+|k+ij|^r+1},$$

and as in the case of (I), it is easy to see that each factor therein is bounded.

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