Advanced Studies in Pure Mathematics 72, 2017
Geometry, Dynamics, and Foliations 2013
pp. 59–69

A paradigm for codimension one foliations

Bertrand Deroin, Dmitry Filimonov, Victor Kleptsyn and Andrés Navas

Abstract.

We summarize some of the recent works, devoted to the study of one-dimensional (pseudo)group actions and codimension one foliations. We state a conjectural alternative for such actions (generalizing the already obtained results) and describe the properties in both alternative cases. We also discuss the generalizations for holomorphic one-dimensional actions. Finally, we state some open questions that seem to be already within the reach.

§1. Introduction: the paradigm and its corollaries

The purpose of this text is to summarize some recent works ([4, 10, 6, 11]) of a joint project of the authors, devoted to the study of one-dimensional (pseudo)group actions and codimension one foliations, and to suggest a paradigm, that seems to arise from these works. This paradigm, partially already proven, describes the properties of different classes of such actions; establishing it in full generality, at least in the analytic case, seems to be already within the reach.

Our study was motivated by the following questions, going back to 1980's, that were asked by D. Sullivan, É. Ghys, and G. Hector; we're stating them both in the group actions and foliations setting:

Received September 28, 2014.
Revised October 8, 2014.
2010 *Mathematics Subject Classification.* Primary 37C85; Secondary 37E10, 57S05.
Key words and phrases. Foliations, pseudogroups, group actions, local discreteness, Markov partitions.
B. Deroin was partially supported by ANR-13-BS01-0002. D. Filimonov and V. Kleptsyn were partially supported by the RFBR project 13-01-00969_a. A. Navas acknowledges the support of DySyRF (PIA Research project ACT 1103).

Question 1.1 (Ghys, Sullivan)**.** Let G be a finitely generated group of (C^2-)smooth circle diffeomorphisms, acting on the circle minimally. Is this action necessarily ergodic with respect to the Lebesgue measure?

Let $\mathcal{F}$ be a transversely (C^2-)smooth codimension one foliation of a compact manifold, which is minimal. Is it necessarily ergodic with respect to the Lebesgue measure?

Question 1.2 (Ghys, Sullivan)**.** Let G be a finitely generated group of (C^2-)smooth circle diffeomorphisms, acting on the circle with a Cantor miminal set K. Is K necessarily of zero Lebesgue measure?

Let $\mathcal{F}$ be a transversely (C^2-)smooth codimension one foliation of a compact manifold, having an exceptional minimal set $\mathcal{K}$. Is it necessarily of zero Lebesgue measure?

Question 1.3 (Hector)**.** Let G be a finitely generated group of (C^2-)smooth circle diffeomorphisms, acting on the circle with a Cantor miminal set K. Does the action of G on the set of the connected components of $S^1 \setminus K$ necessarily have but a finite number of orbits?

Let $\mathcal{F}$ be a transversely (C^2-)smooth codimension one foliation of a compact manifold M, having an exceptional minimal set $\mathcal{K}$. Does the complement $M \setminus \mathcal{K}$ necessarily have at most finite number of connected components?

Our results partially answer these questions; what is even more important, some general paradigm seems to turn up. Namely, it *seems* that the correct way of distinguishing different types of behavior of (non-measure-preserving) actions is not by their minimality *vs* exceptional minimal set presence, but by the (C^1-)local discreteness *vs* non-discreteness (properly understood for the case of foliations or pseudogroups; we will give precise definitions below). A short way of stating this paradigm (once again, with the precise definitions to be given below) is the following one:

Paradigm. *For a finitely-generated subgroup of* $\mathrm{Diff}^2(\mathbf{S}^1)$, *as well as for a* C^2*-codimension one foliation of a compact manifold*, *if there is no* (transverse) *invariant measure*, *the following dichotomy holds*:

- *Either the action has local flows in its local closure*,
- *Or it admits a Markov partition* (of the minimal set).

This is closely related to what was done and suggested as a generic behavior for the case of an exceptional minimal set by Cantwell and Conlon in [2, 3]. Though, for the case of an exceptional minimal set we expect that Markov partition always exists, as the sense in which we understand the Markov partition is slightly weaker than the one of

Cantwell–Conlon (and this covers also the type of behavior mentioned in [3, §7] that did not fit in their definition).

This paradigm implies the following dichotomy for the properties of a non-measure preserving action:

C^1-locally discrete	C^1-locally non-discrete
admits a Markov partition (on a minimal set)	has a local flow in local closure, is topologically rigid
either possesses non-expandable points, or is non-minimal, or satisfies EMD	is minimal and has no non-expandable points

Table 1. Paradigm: alternative properties

The last possibility in the left column is a locally-discrete minimal dynamics without non-expandable points (EMD = Expanding, Minimal, Discrete); the full classification of such actions seem to be within the reach, as we will discuss in Section 3.

Note also that one can consider the questions analogous to the questions asked above for the actions of general pseudo-groups in *complex* dimension one. In the foliations setting, it corresponds to the study of dynamics in a neighborhood of nonsingular minimal sets of complex codimension one foliations (or simply transversely conformal foliations: there is no need to assume a complex structure on the manifold or on the leaves). Once again, for the case if there are no non-expandable points on a minimal set, there is a natural full classification conjecture that seems within the reach. Though, it is yet unclear to us what should be the correct statement in full generality. Again, we will discuss all of this below, in Section 3.

To conclude this section, we would like to mention that the minimal actions that are however locally discrete *seem* to form a thin boundary between two “generic” types of behavior: the actions with an exceptional minimal set and “rich” non-discrete dynamics (having local flows in the local closure).

Acknowledgements. We are very grateful to É. Ghys, J. Rebelo, L. Conlon, S. Matsumoto, Y. Mitsumatsu and M. Triestino for the interesting and fruitful discussions.

We would also like to thank the organizers of the “Geometry and Foliations” conference in Tokyo, 2013, thanks to which this text was born.

§2. Definitions and the expansion procedures

Throughout this text, we will assume without specifying it explicitly each time that the action under consideration is at least C^2-smooth (and often that it is analytic), and that for this action there is no common invariant measure. The former assumption implies that the well-developed distortion control technique can be applied to long composition of "basic" maps (we refer the reader, for instance, to [4, Section 4] or to [10, Section 3] for a short introduction to this technique). At the same time, the latter assumption implies (due to an extension of Sacksteder's theorem, see [5]) that there are maps having hyperbolic fixed points belonging to the minimal set.

A promising way of studying the dynamics of an action is to apply a local expansion procedure: conjugate the action by maps that expand small intervals into large ones, while keeping the distortion bounded. This idea goes back to Sullivan's exponential expansion procedure (see, e.g., [19], as well as [16]). Namely, assume that for any point x of a minimal set there exists an element g of the (pseudo)group such that $g'(x) > 1$. Due to the compactness arguments then there exist a finite set of such elements $g_1, \dots, g_k$ and a constant $\lambda > 1$, such that for any point x of this minimal set for some $i \in \{1, \dots, k\}$ one has $g_i'(x) > \lambda$. Hence, arbitrarily small intervals intersecting the minimal set can be exponentially expanded by a composition of g_i's till they become of size bounded away from zero, and the distortion of such composition stays uniformly bounded. This rather easily allows to obtain, under this assumption, the positive answers to the Questions 1.1–1.3.

Though, this strategy does not work in all the situations. An obstacle to the application of this strategy is the presense of non-expandable points:

Definition 2.1. *A point x of a minimal set is* non-expandable *for the action of a (pseudo)group $\mathcal{G}$, if for any $g \in \mathcal{G}$ (defined in x) one has* $|g'(x)| \leq 1$.

In the analytic setting, a presence of non-expandable points immediately implies that the action is locally discrete in the sense of the following (slightly weaker, than the usual one) definition:

Definition 2.2. *A group G is C^1-locally discrete if for any interval I intersecting the minimal set the identity is the isolated point in $C^1(I, \mathbf{S}^1)$ of the set of the restrictions on I of the maps of the group. For the case of the foliation, one considers the holonomy maps that are defined on I, with the intermediate lengths that stay bounded. For the* *case of finitely generated pseudogroups, one considers the composition of generators, and does not allow the "gluing" of different pieces.*

(We prefer to use this agreement in Definition 2.2, as it guarantees that the pseudogroup action, corresponding to the doubling map $x \mapsto 2x$ mod 1, is also considered as locally discrete even though with gluing one can obtain all the maps $x \mapsto x + \frac{1}{2^n}$.)

Indeed, if a (pseudo)group is not locally discrete and its action does not preserve a measure, the arguments of Scherbakov–Nakai–Loray–Rebelo ([8, 15, 14, 17]) imply that it contains local flows in its local closure. Roughly speaking, due to the absence of a preserved measure there is a map with a hyperbolic fixed point; expanding the sequence of maps C^1-convergent to the identity with help of this map, one finds local flows in the local closure of the (pseudo)group. That is, then there exist two intervals, $I \subset J$, and a change of coordinates, after which the C^1-closure of the set of maps of the (pseudo)group that send I into J contains all sufficiently small translations (see, e.g., [6, Proposition 2.8]). We refer the reader to the works [8, 15, 14, 17], as well as to the short reminder in [6, Section 2.2], for the details. We conclude this by mentioning that even though the cited papers use the analyticity assumption, it seems that this assumption can be weakened. Namely, the main case of the above arguments, that is, the hyperbolic fixed point being not fixed by the maps that locally converge to the identity, can be handled even for the C^2-smooth dynamics. For the other cases, it seems that a well-chosen weaker assumption would suffice, too.

On the other hand, the presence of non-expandable points implies that these points cannot be sent near hyperbolic repelling ones without too much contraction in the process (otherwise, the composition with a power of the corresponding repelling map would be expanding at such a point). And as having local flows in the local closure implies that a point of a minimal set can be sent close to another point of this set with a uniform bound on the derivative, the above arguments imply that in presence of non-expandable points the action should be (C^1-)locally discrete.

In [4], it was noticed that in all the known examples possessing non-expandable points belonging to a minimal set satisfy an additional assumption: any such point is an isolated fixed point for one of the maps of the group. This condition was called condition ($\star$) for the case of a minimal action and condition ($\Lambda\star$) for the action with an exceptional minimal set Λ. For the case of C^2-actions, a slightly weaker definiton was given:

Definition 2.3 ([4])**.** *A minimal action of a finitely generated (pseudo)group $\mathcal{G}$ has property ($\star$), if any non-expandable fixed point is right- and left- isolated fixed point for some maps $g_+, g_- \in \mathcal{G}$.*

An action of a finitely generated (pseudo)group $\mathcal{G}$ with a Cantor minimal set K has property ($\Lambda\star$), if any non-expandable fixed point $x \in K$ is right- and left- isolated fixed point for some maps $g_+, g_- \in \mathcal{G}$.

Though the non-expandable points are an obstacle to the "fast" expansion procedure, when this assumption is satisfied, one can modify the Sullivan's exponential expansion strategy to obtain a "slow" one, that still allows a uniform control on the distortion. Namely, if a point is sufficiently near a non-expandable one, one iterates the corresponding $g_\pm$ (or their inverses) till the point leaves the neighborhood of a non-expandable point. Such a modification has allowed us in [4] to obtain under this assumption the positive answers to Questions 1.1–1.3.

The following (courageous) conjecture (going back to [4, Question 3.1]), says that such property holds for *all* the C^2 (finitely generated, no measure preserving) actions:

Conjecture 2.4. *Any finitely generated C^2-(pseudo)group of one-dimensional transformations (for instance, a finitely generated subgroup of $\mathrm{Diff}^2(\mathbf{S}^1)$ or a holonomy pseudogroup corresponding to a codimension one foliation of a compact manifold) that does not have an invariant measure, satisfies property ($\star$) or ($\Lambda\star$).*

§3. Results: obtained and nearby goals

As we have already mentioned, under the assumption of the property ($\star$) or ($\Lambda\star$), one can construct a bounded-distortion expansion procedure, and hence obtain positive answers to Questions 1.1–1.3; it was done in [4]. Moreover, the same "slow" expansion procedure allows to establish all the properties listed (in the respective cases) in Table 1.

Indeed, if there is at least one non-expandable point, it was shown in [10, Theorem 1] that the action admits a Markov partition (on a minimal set). The proof is based on the following idea: one can use as endpoints of the partition the non-expandable points and a finite sufficiently dense set of their images. Then, outside the non-expandable point works the expansion procedure, thus on the complement to this set one can provide an expansion, and any image of a non-expandable point after a finite number of iterations is sent to a non-expandable point.

Moreover, it was shown in [10, Theorem 2] that using the expansion procedure associated to this Markov partition to "magnify" any map from the group, under a sufficiently strong magnification one sees but a finite number of elementary "bricks". This is a corollary of a local discreteness of the group: the expansion procedure has a bounded distortion, and the distortion coming from the map under magnification tends to zero, because it is applied on smaller and smaller intervals. This justifies the application of a term "Markov partition" (in a sense that is weaker, than in [2], but only up to a "finite index"). Finally, if the dynamics is locally discrete, but there are no non-expandable points, the Markov partition can be constructed with help of the (sufficiently dense set of) repelling fixed points: again, expanding them and their neighborhoods, one ends up with a finite set of endpoints due to the local discreteness.

On the other hand, for a C^1-locally non-discrete action, as we have already discussed earlier, there are vector fields in the local closure (what is a theorem in the analytic setting and plausible statement in sufficiently smooth one). Their presence implies the minimality of the action (as a Cantor minimal set cannot be preserved by a flow), absence of non-expandable points. It also implies topological rigidity: very roughly speaking, a topological conjugacy between two such actions should send one local vector field to the other one, and thus would be smooth. This idea goes back to the study of polynomial foliations of $\mathbb{C}^2$ (see, e.g., [8, 15, 14])—and in this setting was realized by A. Eskif and J. Rebelo in [9].

The last line of Table 1 requires a short comment: it is formally a triviality (a locally discrete action either possesses a non-expandable point, or is non-minimal, or is a EMD: a minimal, locally discrete action without non-expandable points). Though, the last case is quite close to be fully classifiable. That is, for the case of analytic group actions on the circle an unpublished result of B. Deroin states that such an action is analytically conjugate to a finite cover of a Fuchsian group, corresponding to a compact Riemannian surface. For the case of pseudogroups or foliations, it is quite natural to conjecture that a similar result will hold, with one more opportunity: the action can be conjugate to the pseudogroup corresponding to an expanding circle endomorphism. (For the case of foliations, such holonomy is realized by a Hirsch foliation; see, e.g., [7].)

The situation of absence of non-expandable points in a minimal set also looks very promising for the study of the holomorphic dynamics. For this case, one can state an analogue of the real one-dimensional action conjecture: a locally discrete action either comes out of a Kleinian group action, or corresponds to a Julia set of a holomorphic map. At the same time, if an action is locally non-discrete, the minimal set is (due to the local flows argument) real-analytic.

This conjecture is partically proven: in an unpublished work by B. Deroin, it is shown that in the locally discrete case the orbits of the action, equipped with the metric coming from the generators of the action, are Gromov-hyperbolic. Also, their boundaries at infinity are locally homeomorphic, hence the topological dimension of the boundary does not depend on the orbit. If this dimension is positive, in the same work it is shown that the minimal set comes out of a Kleinian group after an analytic change of coordinates. On the other hand, if it is zero, namely if the orbits are quasi-isometric to a tree, it is quite natural to expect that the minimal set comes out of a Julia set for a holomorphic map, though this is not yet proven.

We conclude the discussion on the holomorphic pseudo-groups acting by noticing that there are much more examples of actions in the complex dimension one (for instance, those constructed using the mating procedure [1]), than in the real dimension one. Due to all of this, it is not even yet clear to us how the characterization conjecture should be stated in this case.

Returning back to the real dimension one and adding up what was said, we see that to establish the main paradigm, as well as Conjecture 2.4, it suffices to get a contradiction in the case on which a lot is known already. Namely, one assumes that there is a non-expandable point that is not an isolated fixed point for any acting element, and that the action is C^1-locally discrete. Then, a next—though a bit technical—idea is to consider the possible "close return" of such a non-expandable point. If the image is sufficiently close to the initial point, the non-expansivity property implies that such composition is close to a translation. Taking a chain of commutators starting with two such almost-translations, one can eventually obtain local non-discreteness (and hence the desired contradiction).

This idea was realized in [6] for the case of a (virtually) free group of analytic circle diffeomorphisms:

Theorem 3.1 ([6])**.** *Let G be a* (*virtually*) *free finitely generated subgroup of the group of analytic circle diffeomorphisms, such that the action G does not have finite orbits. Then, G satisfies property* ($\star$) *or* ($\Lambda\star$) (*depending on whether the action is minimal or possesses an exceptional minimal set*).

Remark 3.2. Recall, that due to a result by Ghys [13], a finitely generated group of analytic circle diffeomorphisms, acting with a Cantor minimal set, is *always* virtually free. Hence, Theorem 3.1 implies positive answers for Questions 1.2 and 1.3 for the case of an analytic group action on the circle.

On the other hand, finitely generated groups acting on the circle can have one, two or a Cantor set of ends. For the latter case, it looks quite promising to generalize the arguments of Theorem 3.1. The second case is of no interest to us: in this case, the group would be quasi-isometric to $\mathbb{Z}$, in particular would be amenable, hence for its action there would be an invariant measure. Finally, the former case with two additional assumptions can be also handled using the same idea:

Theorem 3.3 ([11]). *Let G be a finitely generated subgroup of the group of analytic circle diffeomorphisms, acting minimally, that has one end, is finitely presented, and in which one cannot find elements of arbitrarily large finite order. Then, G satisfies property* ($\star$).

This is the up-to-date state of the project. There are different directions in which an attack can be continued:

- It is tempting to generalize Theorem 3.1 to the case of the transversely analytic foliations. Though, to do so one would require the appropriate generalization of the freeness of the group (as there is no more group, but only a pseudogroup of holonomies). Also, in a general setting of an exceptional minimal set in a foliation there is Duminy's theorem, saying that semiproper leaves have infinitely many ends, but there is no Ghys' theorem on the freeness of the group (as there is no more group).
- It is very interesting to handle the case of minimal analytic actions on the circle completely. The case of groups with infinitely many ends perhaps can be handled with the help of the same methods as the case of a free group acting. On the other hand, perhaps one can get rid of additional assumptions in Thm. 3.3.
- It is also very interesting to find out the correct generalization of the above descriptions to the case of holomorphic one-dimensional actions.
- Most of the technique used in the cited works does not use too much the analyticity; almost all of it can be generalized to the C^2 setting (perhaps, with some technical difficulties). Though, passing to the smooth maps one will have to consider germs instead of global maps (as maps can coincide locally, but not globally), and thus perhaps use the understanding from the generalization to the foliations.

References

[1] S. Bullett, Matings in holomorphic dynamics, in *Geometry of Riemann surfaces*, London Math. Soc. Lecture Note Ser. **368**, Cambridge Univ. Press, Cambridge, 2010, 88–119.

[2] J. Cantwell and L. Conlon, Leaves of Markov local minimal sets in foliations of codimension one, *Publ. Mat.* **33** (1989), 461–484.

[3] J. Cantwell and L. Conlon, Foliations and subshifts, *Tohoku Math. J. (2)* **40** (1988), 165–187.

[4] B. Deroin, V. Kleptsyn and A. Navas, On the question of ergodicity for minimal group actions on the circle, *Mosc. Math. J.* **9** (2009), 263–303.

[5] B. Deroin, V. Kleptsyn and A. Navas, Sur la dynamique unidimensionnelle en régularité intermédiaire, *Acta Math.* **199** (2007), 199–262.

[6] B. Deroin, V. Kleptsyn and A. Navas, *Towards the solution of some fundamental questions concerning group actions on the circle and codimension-one foliations*, preprint, arXiv:1312.4133.

[7] B. Deroin and C. Vernicos, Feuilletage de Hirsch, mesures harmoniques et g-mesures, *Publ. Mat. Urug.* **12** (2011), 79–85.

[8] P. M. Elizarov, Yu. S. Il'yashenko,A. A. Shcherbakov and S. M. Voronin, Finitely generated groups of germs of one-dimensional conformal mappings, and invariants for complex singular points of analytic foliations of the complex plane, in *Nonlinear Stokes phenomena*, Adv. Soviet Math. **14**, Amer. Math. Soc., Providence, RI, 1993, 57–105.

[9] A. Eskif and J. Rebelo, *Absolute topological rigidity for locally non-discrete subgroups of* $\mathrm{Diff}^w(\mathbf{S}^1)$, manuscript.

[10] D. A. Filimonov and V. A. Kleptsyn, The structure of groups of circle diffeomorphisms with the property of nonexpandable points being fixed, *Funct. Anal. Appl.* **46** (2012), 191–209.

[11] D. A. Filimonov and V. A. Kleptsyn, One-end finitely presented groups acting on the circle, *Nonlinearity* **27** (2014), 1205–1223.

[12] É. Ghys, Sur les groupes engendrés par des difféomorphismes proches de l'identité, *Bol. Soc. Brasil. Mat. (N.S.)* **24** (1993), 137–178.

[13] É. Ghys, Classe d'Euler et minimal exceptionnel, *Topology* **26** (1987), 93–105.

[14] F. Loray and J. C. Rebelo, Minimal, rigid foliations by curves on $\mathbb{CP}^n$, *J. Eur. Math. Soc.* (JEMS) **5** (2003), 147–201.

[15] I. Nakai, Separatrices for nonsolvable dynamics on $(\mathbf{C}, 0)$, *Ann. Inst. Fourier (Grenoble)* **44** (1994), 569–599.

[16] A. Navas, Sur les groupes de difféomorphismes du cercle engendrés par des éléments proches des rotations, *Enseign. Math. (2)* **50** (2004), 29–68.

[17] J. C. Rebelo, Subgroups of $\mathrm{Diff}_+^\infty(\mathbb{S}^1)$ acting transitively on 4-tuples, *Trans. Amer. Math. Soc.* **356** (2004), 4543–4557.

[18] J. C. Rebelo, Ergodicity and rigidity for certain subgroups of $\mathrm{Diff}^\omega(S^1)$, *Ann. Sci. École Norm. Sup. (4)* **32** (1999), 433–453.

[19] M. Shub and D. Sullivan, Expanding endomorphisms of the circle revisited, *Ergodic Theory Dynam. Systems* **5** (1985), 285–289.

Bertrand Deroin
CNRS UMR 8553
Departement de Mathématiques et Applications, École Normale Supérieure
45, rue d'Ulm
75012 Paris, France
E-mail address: bertrand.deroin@ens.fr

Dmitry Filimonov
National Research University Higher School of Economics (HSE)
20 Myasnitskaya ulitsa,
101000 Moscow, Russia
E-mail address: mityafil@gmail.com

Victor Kleptsyn
CNRS, Institut de Recherche Mathématique de Rennes (IRMAR, UMR 6625),
Bât. 22-23, Campus Beaulieu,
263 avenue du Général Leclerc,
35042 Rennes, France
E-mail address: victor.kleptsyn@univ-rennes1.fr

Andrés Navas
Departamento de Matemáticas, Facultad de Ciencia, Universidad de Santiago,
Alameda 3363, Estación Central,
Santiago, Chile
E-mail address: andres.navas@usach.cl