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A Livšic type theorem for germs of analytic diffeomorphisms

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Abstract
We deal with the problem of the validity of the Livšic theorem for cocycles of diffeomorphisms over a hyperbolic dynamics that satisfy the periodic orbit condition. We give a result in the positive direction for cocycles of germs of analytic diffeomorphisms at the origin.

Mathematics Subject Classification: 37d99

1. Introduction

Given a map (dynamical system) $T : X \to X$ over a compact metric space X and a (topological) group $\mathcal{G}$, we consider a continuous $\mathcal{G}$-valued cocycle $A : \mathbb{N} \times X \to \mathcal{G}$, that is, a continuous map taking values in $\mathcal{G}$ and satisfying the cocycle relation

$$A(n + m, x) = A(n, T^m x)A(m, x)$$

for every m, n in $\mathbb{N}$ and every $x \in X$. This cocycle is completely determined by the continuous function $A(\cdot) := A(1, \cdot) : X \to \mathcal{G}$, and the cocycle relation yields

$$A(n, x) = A(T^{n-1}x)A(T^{n-2}x) \cdots A(x)$$

for every $n \geqslant 1$. A natural problem consists in determining sufficient conditions so that a given cocycle is conjugated to a cocycle taking values in a 'small' subgroup of $\mathcal{G}$. For the case of the trivial subgroup $\{e_{\mathcal{G}}\}$, the existence of the desired conjugacy is equivalent to the existence of a continuous function $B : X \to \mathcal{G}$ such that

$$A(x) = B(Tx)B(x)^{-1} \qquad \text{for all} \quad x \in X. \tag{1}$$

Whenever this *cohomological equation* associated to the cocycle A has a solution B, we say that A is a *coboundary*. The simplest obstruction for the existence of B is the *periodic orbit obstruction*: if $p \in X$ and $n \in \mathbb{N}$ satisfy $T^n p = p$, then

$$A(n, p) = \prod_{i=0}^{n-1} A(T^i x) = \prod_{i=0}^{n-1} B(T^{i+1}x)B(T^i x)^{-1} = B(T^n p)B(p)^{-1} = e_{\mathcal{G}}. \tag{2}$$

The *Livšic problem* consists in determining whether the condition (2) is not only necessary but also sufficient for A being a coboundary. This terminology originates in the seminal work of Livšic [4], who proved that this is the case whenever $\mathcal{G}$ is Abelian, A is Hölder-continuous and T is a topologically transitive hyperbolic diffeomorphism. Since then, many extensions of this classical result have been proposed. Perhaps the most relevant is Kalinin's recent version for $\mathcal{G} = \mathrm{GL}(d, \mathbb{C})$. In this paper, we address the Livšic problem for Hölder-continuous cocycles taking values in the group of germs of analytic diffeomorphisms. In the context of general diffeomorphisms, the answer to the Livšic problem is unclear, despite several results pointing in the positive direction whenever a certain localization property is satisfied. (See, for example, [2].)

To state our result, we denote by $\mathcal{G}erm_d$ the group of germs of local bi-holomorphisms of the complex space $\mathbb{C}^d$ fixing the origin. This may be identified to the group of holomorphic maps $F(Z) = A_1 Z + A_2 Z^2 + \ldots$ having positive convergence radius, with $A_1 \in \mathrm{GL}(d, \mathbb{C})$ (see section 1.2 for the details).

Main theorem. *Let $T : X \to X$ be a topologically transitive homeomorphism of a compact metric space X satisfying the closing property (see section 1.1 for the details). Let $F : X \to \mathcal{G}erm_d$ be a Hölder-continuous function/cocycle (see section 1.2 for a discussion about continuity). If F satisfies the condition (2), then there exists a Hölder-continuous function $H : X \to \mathcal{G}erm_d$ such that for all $x \in X$,*

$$F(x) = H(Tx) \circ H(x)^{-1}. \tag{3}$$

This theorem should be compared with [5], where the second-named author shows a KAM-type result for $\mathcal{G}erm_d$-valued cocycles over a minimal torus translation.

1.1. A reminder on Livšic's theorem for complex valued cocycles

Let X be a compact metric space with normalized diameter (*i.e.*, $\mathrm{diam}(X) = 1$). We say that a function $f : X \to \mathbb{C}$ is (C, α)-Hölder-continuous for $C > 0$ and $\alpha \in (0, 1]$ if for every pair of points x, y in X,

$$|f(x) - f(y)| \leqslant C\, \mathrm{dist}_X(x, y)^{\alpha}. \tag{4}$$

In the sequel, we will denote by $[f]_\alpha$ the smallest constant C for which f is (C, α)-Hölder-continuous. The next two results are straightforward.

Lemma 1. *If f vanishes at some point of X, then $\|f\| := \sup_{x \in X} |f(x)| \leqslant [f]_\alpha$.* □

Lemma 2. *Let $f, g : X \to \mathbb{C}$ be two α-Hölder-continuous functions. Then the functions $f + g$ and fg are α-Hölder-continuous, and*

(1) $[f + g]_\alpha \leqslant [f]_\alpha + [g]_\alpha$.
(2) $[fg]_\alpha \leqslant [f]_\alpha \|g\| + [g]_\alpha \|f\|$. □

Let $T: X \to X$ be a homeomorphism and let x, y be points of X. We say that the orbit segments $x, Tx, \ldots, T^k x$ and $y, Ty, \ldots, T^k y$ are *exponentially δ-close with exponent $\lambda > 0$* if for every $j = 0, \ldots, k$,

$$\mathrm{dist}_X(T^j x, T^j y) \leqslant \delta \mathrm{e}^{-\lambda \min\{j, k-j\}}.$$

We say that T satisfies the *closing property* if there exist $c, \lambda, \delta_0 > 0$ such that for every $x \in X$ and $k \in \mathbb{N}$ so that $\mathrm{dist}_X(x, T^k x) < \delta_0$, there exists a point $p \in X$ with $T^k p = p$ so that letting $\delta := c\, \mathrm{dist}_X(x, T^k x)$, the orbit segments $x, Tx, \ldots, T^k x$ and $p, Tp, \ldots, T^k p$ are exponentially δ-close with exponent λ and there exists a point $y \in X$ such that for every $j = 0, \ldots, k$,

$$\mathrm{dist}_X(T^j p, T^j y) \leqslant \delta \mathrm{e}^{-\lambda j} \qquad \text{and} \qquad \mathrm{dist}_X(T^j y, T^j x) \leqslant \delta \mathrm{e}^{-\lambda(n-j)}.$$

Important examples of maps satisfying the closing property are hyperbolic diffeomorphisms of compact manifolds.

In this work, we will use two versions of the Livšic result. The first of these (see theorem 3) corresponds to the original Livšic theorem for complex valued cocycles. This theorem will be used as the main ingredient for an iterative scheme. In this procedure, we will require certain good estimates for the solutions of cohomological equations (see corollary 4). For this reason, in the next paragraph, we review the proof of the Livšic theorem and we record some key estimates. The second version (extension) of the Livšic result we will use (see theorem 5) corresponds to a recent and remarkable theorem by B Kalinin, who proves the Livšic theorem for matrix-valued cocycles (satisfying no localization property).

Theorem 3 (Livšic, see [4]). *Let $T: X \to X$ be a topologically transitive homeomorphism of a compact metric space X satisfying the closing property. Let $\psi: X \to \mathbb{C}$ be an α-Hölder-continuous function for which the condition (2) holds, that is, for every point $p \in X$ and $k \geqslant 1$ such that $T^k p = p$, one has $\sum_{j=0}^{k-1} \psi(T^j p) = 0$. Then there exists an α-Hölder-continuous function $\phi: X \to \mathbb{C}$ that is a solution of the cohomological equation*

$$\phi \circ T - \phi = \psi.$$

Proof. Let $x_0 \in X$ be such that $\overline{\{T^n x_0\}_{n \in \mathbb{N}}} = X$. We define ϕ by letting $\phi(x_0) := 0$ and $\phi(T^n x_0) := \sum_{j=0}^{n-1} \psi(T^j x_0)$. We next check that ϕ is α-Hölder-continuous on $\{T^n x_0\}_{n \in \mathbb{N}}$. Let $n > m$. There are two cases to consider:

- Assume that $\mathrm{dist}_X(T^m x_0, T^n x_0) < \delta_0$. Then there exists a point $p \in X$ satisfying $T^{n-m} p = p$ and such that for every $j = 0, \ldots, n-m$,

$$\mathrm{dist}_X(T^j(T^m x_0), T^j p) \leqslant c\, \mathrm{dist}_X(T^n x_0, T^m x_0) \mathrm{e}^{-\lambda \min\{j, n-m-j\}}.$$

  This yields

$$\begin{aligned} |\phi(T^n x_0) - \phi(T^m x_0)| &= \left| \sum_{j=0}^{n-m-1} \psi(T^{m+j} x_0) \right| \\ &= \left| \sum_{j=0}^{n-m-1} \big(\psi(T^{m+j} x_0) - \psi(T^j p)\big) + \sum_{j=0}^{n-m-1} \psi(T^j p) \right| \\ &\leqslant \sum_{j=0}^{n-m-1} \left| \psi(T^{m+j} x_0) - \psi(T^j p) \right| \end{aligned}$$

$$
\begin{aligned}
&\leqslant \sum_{j=0}^{n-m-1} [\psi]_\alpha \operatorname{dist}_X(T^{m+j}x_0, T^j p)^\alpha \\
&\leqslant \sum_{j=0}^{n-m-1} c^\alpha [\psi]_\alpha \operatorname{dist}_X(T^n x_0, T^m x_0)^\alpha \mathrm{e}^{-\lambda\alpha \min\{j,n-m-j\}} \\
&\leqslant \frac{2c^\alpha[\psi]_\alpha}{1-\mathrm{e}^{-\lambda\alpha}} \operatorname{dist}_X(T^n x_0, T^m x_0)^\alpha.
\end{aligned}
$$

- Assume that $\operatorname{dist}_X(T^n x_0, T^m x_0) \geqslant \delta_0$. Since x_0 has dense orbit and X is compact, there exists $N \in \mathbb{N}$, depending only on X, T, and δ_0, such that $\{x_0, Tx_0, \ldots, T^N x_0\}$ is a δ_0-dense set in X. For $n - m \leqslant N$, one easily shows that

$$|\phi(T^n x_0) - \phi(T^m x_0)| \leqslant N\|\psi\|.$$

For $n - m > N$, there exist r, s in $\{0, 1, \ldots, N\}$ such that $\operatorname{dist}_X(T^s x_0, T^n x_0) \leqslant \delta_0$ and $\operatorname{dist}_X(T^r x_0, T^m x_0) \leqslant \delta_0$. Using the preceding case, this yields

$$
\begin{aligned}
|\phi(T^n x_0) - \phi(T^m x_0)| &\leqslant |\phi(T^n x_0) - \phi(T^s x_0)| + |\phi(T^m x_0) - \phi(T^r x_0)| + |\phi(T^s x_0) \\
&\quad - \phi(T^r x_0)| \\
&\leqslant \frac{4[\psi]_\alpha c^\alpha}{1-\mathrm{e}^{-\lambda\alpha}}\delta_0^\alpha + N\|\psi\| \\
&\leqslant \left(\frac{4[\psi]_\alpha c^\alpha}{1-\mathrm{e}^{-\lambda\alpha}} + \frac{N\|\psi\|}{\delta_0^\alpha}\right) \operatorname{dist}_X(T^n x_0, T^m x_0)^\alpha. \qquad \square
\end{aligned}
$$

A careful reading of the proof above yields useful estimates enclosed in the following.

Corollary 4. *The solution ϕ to the cohomological equation is α-Hölder continuous, and there exists K depending only on T, X, and α such that $[\phi]_\alpha \leqslant K([\psi]_\alpha + \|\psi\|)$.* $\square$

Theorem 5 (Kalinin, see [3]). *Let T be a topologically transitive homeomorphism of a compact metric space X satisfying the closing property. Let $A : X \to GL(d, \mathbb{C})$ be an α-Hölder function for which the condition (2) holds. Then there exists an α-Hölder function $B : X \to GL(d, \mathbb{C})$ such that for all $x \in X$,*

$$A(x) = B(Tx)B(x)^{-1}.$$

1.2. The group $\mathcal{G}erm_d$

For $d \geqslant 1$, we introduce the following notation:

- $\boldsymbol{j} := (j_1, \ldots, j_d)$ is a point of non-negative integer entries, with $j_i \geqslant 0$ for every $1 \leqslant i \leqslant d$.
- $|\boldsymbol{j}| := j_1 + \ldots + j_d$.
- $\boldsymbol{j} \preceq \boldsymbol{k}$ if $j_i \leqslant k_i$ for every $1 \leqslant i \leqslant d$.
- $\boldsymbol{j} \prec \boldsymbol{k}$ if $\boldsymbol{j} \preceq \boldsymbol{k}$ and $j_{i_*} < k_{i_*}$ for some i_*.
- $Z = (z_1, z_2, \ldots, z_d)$ is a point in $\mathbb{C}^d$.
- $Z^{\boldsymbol{j}} := z_1^{j_1} z_2^{j_2} \cdots z_d^{j_d}$.

Then we can define a formal power series on $\mathbb{C}^d$ as $F(Z) := (F_1(Z), F_2(Z), \ldots, F_d(Z))$, where each $F_i(Z)$ has the form

$$F_i(Z) = \sum_{|\boldsymbol{j}| \geqslant 0} t_{\boldsymbol{j}}^i Z^{\boldsymbol{j}}$$

for some coefficients $t^i_j \in \mathbb{C}$. This formal power series becomes an analytic map if there exists $R > 0$ such that $\limsup_j |t^i_j|^{\frac{1}{|j|}} \leqslant \frac{1}{R}$ for every i. Indeed, in this case, each F_i is a convergent series on $D(0, R)^d$ (this is the set of points $Z = (z_1, \ldots, z_d)$ such that $|z_s| < R$ holds for every s).

Let $\mathcal{H}(d, R)$ be the set of continuous functions $F : \overline{D(0, R)^d} \to \mathbb{C}^d$ that may be written as a convergent power series on $D(0, R)^d$ and satisfy $F'(0) \in GL(d, \mathbb{C})$. This space is a subset of a natural complex vector space which can be endowed with the inner product

$$\langle F, G\rangle_R := \sum_i \left(\int_{\partial D(0,R)^d} F_i \overline{G_i} \, \mathrm{d}Z \right).$$

The L^2-norm of an element $F \in \mathcal{H}(d, R)$ of the form $F_i(Z) = \sum_{|j| \geqslant 0} t^i_j Z^j$ is

$$\|F\|_{2,R} := \langle F, F\rangle_R^{1/2} = \left(\sum_i \sum_{|j| \geqslant 1} |t^i_j|^2 R^{2|j|} \right)^{1/2}.$$

We let $\mathcal{H}_0(d, R)$ be the subset of $\mathcal{H}(d, R)$ formed by those F satisfying $F(0) = 0$, and we define the set of *local holomorphic diffeomorphisms* of $\mathbb{C}^d$ as

$$\mathcal{G}_d := \bigcup_{R>0} \mathcal{H}_0(d, R).$$

On this set, we introduce the following equivalence relation: we say that F, G in $\mathcal{G}_d$ are equivalent if there exists a neighbourhood of the origin on which F and G coincide. With this identification, the set $\mathcal{G}_d$ becomes a group, that we call the *group of germs of analytic diffeomorphisms* of $\mathbb{C}^d$ and we denote by $\mathcal{G}erm_d$.

Although we will not worry about providing a precise topology for $\mathcal{G}erm_d$, we will certainly need to consider maps from X to $\mathcal{G}erm_d$ that are 'continuous' in a precise sense. Since X is compact, any reasonable definition should lead to functions that factor throughout an space $\mathcal{H}_0(d, R)$ for some positive R. Accordingly, given $C > 0$, $\alpha \in (0, 1]$, and $R > 0$, a map $\Psi : X \to \mathcal{H}_0(d, R)$ will be said to be (C, α, R)-Hölder-continuous if $\Psi(x)$ belongs to $\mathcal{H}_0(d, R)$ for every $x \in X$, and for every pair of points x, y in X,

$$\|\Psi(x) - \Psi(y)\|_{2,R} \leqslant C \, \mathrm{dist}_X(x, y)^\alpha.$$

In terms of the coefficients of the power series, this condition reads as follows:

Lemma 6. *If $\Psi : X \to \mathcal{H}_0(d, R)$ is (C, α, R)-Hölder and writes as*

$$\Psi_i(x)(Z) = \sum_{|j|>0} t^i_j(x) Z^j,$$

then each coefficient $t^i_j : X \to \mathbb{C}$ is a $\left(\frac{C}{R^{|j|}}, \alpha\right)$-Hölder-continuous function.

Proof. The Hölder condition for Ψ yields

$$\left(\sum_i \sum_{|j| \geqslant 1} |t^i_j(x) - t^i_j(y)|^2 R^{2|j|} \right)^{1/2} \leqslant C \, \mathrm{dist}_X(x, y)^\alpha,$$

which implies that

$$|t^i_j(x) - t^i_j(y)|^2 \leqslant \frac{C^2}{R^{2|j|}} \, \mathrm{dist}_X(x, y)^{2\alpha}.$$

□

In an opposite direction, given a list $\{t^i_{\boldsymbol{j}} : X \to \mathbb{C}, \boldsymbol{j} \succeq 0, 1 \leqslant i \leqslant d\}$ of continuous functions, we are interested in finding conditions that ensure that $F := (F_1, \ldots, F_d)$, formally defined by $F_i(x)(Z) := \sum_{\boldsymbol{j}} t^i_{\boldsymbol{j}}(x) Z^{\boldsymbol{j}}$, represents a convergent power series lying in $\mathcal{H}_0(d, R)$ for some $R > 0$.

Lemma 7. *Assume that each function $t^i_{\boldsymbol{j}}$ is a $(\frac{C}{R^{|\boldsymbol{j}|}}, \alpha)$-Hölder-continuous function for some positive constants C, R. Assume also that each $t^i_{\boldsymbol{j}}$ vanishes at some point of X. Then for all $\delta < 1$, the formal power series F_i is convergent on $D(0, R)^d$, and $x \mapsto F(x) = (F_1(x), \ldots, F_d(x))$ is a $(O(\frac{\delta}{1-\delta})^{1/2}, \alpha)$-Hölder continuous map from X to $\mathcal{H}_0(d, \delta R)$.*

Proof. Since each $t^i_{\boldsymbol{j}}$ vanishes at some point of X, lemma 1 gives $\|t^i_{\boldsymbol{j}}\| \leqslant \frac{C}{R^{|\boldsymbol{j}|}}$ for every $i, \boldsymbol{j}$. This implies that each F_i is a convergent power series on $D(0, R)^d$. Moreover, for all x, y in X,

$$\begin{aligned}
\|F(x) - F(y)\|^2_{2,\delta R} &= \sum_i \sum_{\boldsymbol{j}} |t^i_{\boldsymbol{j}}(x) - t^i_{\boldsymbol{j}}(y)|^2 (\delta R)^{2|\boldsymbol{j}|} \\
&\leqslant \sum_i \sum_{\boldsymbol{j}} C^2 \operatorname{dist}_X(x, y)^{2\alpha} \delta^{2|\boldsymbol{j}|} \\
&= dC^2 \operatorname{dist}_X(x, y)^{2\alpha} \sum_{s=1}^{\infty} \sum_{|\boldsymbol{j}|=s} \delta^{2s} \\
&= dC^2 \operatorname{dist}_X(x, y)^{2\alpha} \sum_{s=1}^{\infty} \frac{(s+d-1)!}{s!(d-1)!} \delta^{2s} \\
&= dC^2 O\left(\frac{\delta}{1-\delta}\right) \operatorname{dist}_X(x, y)^{2\alpha}.
\end{aligned}$$

□

The Faà di Bruno formula. We will need to consider compositions of power series in several complex variables. The following is a simplified formulation of the multivariate version by Constantine and Savits [1] of the well known Faà di Bruno formula:

Theorem 8 (see [1]). *Let $A(Z) = \sum_{|\boldsymbol{j}| \geqslant 1} a_{\boldsymbol{j}} Z^{\boldsymbol{j}}$ and $B_i(Z) = \sum_{|\boldsymbol{j}| \geqslant 1} b^i_{\boldsymbol{j}} Z^{\boldsymbol{j}}$, $1 \leqslant i \leqslant d$, be formal power series in d variables. Then the power series*

$$C(Z) = A\,(B_1(Z), B_2(Z), \ldots, B_d(Z)) = \sum_{|\boldsymbol{j}| \geqslant 1} c_{\boldsymbol{j}} Z^{\boldsymbol{j}}$$

has coefficients

$$c_{\boldsymbol{j}_*} = \sum_{|\boldsymbol{j}|=1} a_{\boldsymbol{j}} b^{\boldsymbol{j}}_{\boldsymbol{j}_*} + \sum_{1 < |\boldsymbol{j}|,\ \boldsymbol{j} \leqslant \boldsymbol{j}_*} a_{\boldsymbol{j}} P(\boldsymbol{j}_*, \boldsymbol{j})\{B\}, \tag{5}$$

where $P(\boldsymbol{j}_*, \boldsymbol{j})\{B\}$ is a polynomial in the variables $\{b^i_{\bar{\boldsymbol{j}}}\}^{1 \leqslant i \leqslant d}_{\bar{\boldsymbol{j}} < \boldsymbol{j}_*}$ that is homogeneous of degree $|\boldsymbol{j}|$ and has positive integer coefficients.

The Faà di Bruno formula is actually much more precise and requires more complex notation. For instance, in the case $d = 1$, one has

$$P(j_*, j)\{B\} = \sum_{r_1 + \ldots + r_j = j_*} B_{r_1} \cdots B_{r_j}.$$

A generating function. Let us define $J : D(0,1)^d \to \mathbb{C}^d$ by the convergent power series

$$J_i(Z) = z_i - \sum_{|j|>1} Z^j.$$

Since $DJ(0) = id_{\mathbb{C}^d}$, there exists an analytic map G defined in a neighbourhood of the origin in $\mathbb{C}^d$ such that $G(0) = 0$ and

$$J \circ G(Z) = Z \text{ for every } Z \text{ in that neighbourhood.} \tag{6}$$

In terms of power series, one can write

$$G_i(Z) = z_i + \sum_{|j|>1} g^i_j Z^j,$$

where the coefficients verify $|g^i_j| < K^{|j|-1}$ for some $K > 0$ and every $|j| \succ 1$. Moreover, these coefficients satisfy a fundamental recurrence relation. Indeed, using $J \circ G(Z) = Z$ and the Faà di Bruno formula (5), one obtains

$$0 = g^i_{j_*} - \sum_{1<|j|,\ j \leqslant j_*} P(j_*, j)\{G\}. \tag{7}$$

Recall that $P(j_*, j)\{G\}$ depends only on the values of $g^s_{\tilde{j}}$ for $\tilde{j} \prec j_*$ and every s. Hence, one can recursively compute $g^i_{j_*}$ in terms of the previously defined $g^s_{\tilde{j}}$.

For any $S > 0$, we consider $J_S : D(0, S^{-1})^d \to \mathbb{C}^d$ defined by $J_S(Z) := \frac{1}{S} J(SZ)$. When solving the equation $J_S \circ G_S(Z) = Z$, one gets a map $G_S = (G_{S,1}, \ldots, G_{S,d})$, where each $G_{S,i}(Z)$ has the form $G_{S,i}(Z) = z_i + \sum_{|j|>1} g^i_{S,j} Z^j$ for certain coefficients $g^i_{S,j}$ satisfying

$$g^i_{S,j_*} = \sum_{1<|j|,\ j \leqslant j_*} S^{|j|-1} P(j_*, j)\{G_S\}. \tag{8}$$

Lemma 9. *Each coefficient $g^i_{S,j}$ is a positive real number. Moreover, there exists a constant $\mathcal{R} = \mathcal{R}(S) > 0$ such that $g^i_{S,j} \leqslant \mathcal{R}^{|j|-1}$ for every j.* □

1.3. Proof of the main theorem

A first reduction. Let $F(x)(Z) = A_1(x)Z + (\sum_{|j|>1} a^i_j(x) Z^j)_{1 \leqslant i \leqslant d}$ be the power series expansion of the cocycle viewed as a (C, α, R)-Hölder-continuous function $\Psi : X \to \mathcal{H}_0(d, R)$. The map $x \mapsto A_1(x) \in GL(d, \mathbb{C})$ is an α-Hölder-continuous function. Since the condition (2) holds for F, we must have

$$\prod_{j=0}^{n-1} A_1(T^j p) = \frac{\partial}{\partial Z} F(T^{n-1}p) \circ \ldots \circ F(p)\Big|_{Z=0} = id_{\mathbb{C}^d}$$

for every $p \in X$ and $n \in \mathbb{N}$ such that $T^n p = p$. In other words, the $GL(d, \mathbb{C})$-valued cocycle A_1 satisfies the condition (2). By Kalinin's version of the Livšic theorem, there exists an α-Hölder-continuous function $H_1 : X \to GL(d, \mathbb{C})$ such that $A_1(x) = H_1(Tx) H_1(x)^{-1}$ for all $x \in X$. Consequently, the $\mathcal{G}erm_d$-valued cocycle $H_1(x)(Z) := H_1(x)Z$ conjugates F to a cocycle of the form

$$(x, Z) \longmapsto \left(Tx, Z + \left(\sum_{|j|>1} a^i_j(x) Z^j \right)_{1 \leqslant i \leqslant d} \right).$$

Thus, we can assume that $A_1(x) = id_{\mathbb{C}^d}$ for all $x \in X$.

An iterative procedure. We look for a map $H : X \to \mathcal{G}erm_d$ solving the cohomological equation (3) and having the form $H(x)(Z) = Z + (\sum_{|j|>1} h^i_j(x)Z^j)_{1\leqslant i\leqslant d}$. Notice that this equation may be written as $F(x) \circ H(x) = H(Tx)$. Applying the Faà di Bruno formula (5) to the left-side expression, one concludes that each coefficient h^i_j can be defined recursively as the solution of a cohomological equation for $\mathbb{C}$-valued data:

$$(ec^i_{j_*}) \quad h^i_{j_*}(Tx) - h^i_{j_*}(x) = \sum_{1<|j|,\ j\leqslant j_*} a^i_j(x) P(j_*, j)\{H\}(x).$$

A necessary condition for the existence of the coefficient $h^i_{j_*}$ is that the condition (2) holds for the function

$$R^i_{j_*} := \sum_{1<|j|,\ j\leqslant j_*} a^i_j P(j_*, j)\{H\}. \tag{9}$$

Lemma 10. *Each $R^i_{j_*}$, with $i, |j_*| \succ 1$, is a well-defined α-Hölder-continuous function for which the condition (2) holds. As a consequence, given any $x_0 \in X$, the equation $(ec^i_{j_*})$ has an α-Hölder-continuous solution $h^i_{j_*}$ vanishing at x_0.*

Proof. Suppose that the conclusion of the lemma holds for every j such that $|j| < k$, and let us consider the case where $|j| = k$. Using the explicit formula (9), lemma 2 shows that the function $R^i_{j_*}$ is α-Hölder-continuous. Consider the continuous $\mathcal{G}erm_d$-valued function

$$H_{<k} : x \mapsto Z + \left(\sum_{|j|<k} h^i_j(x) Z^j \right)_{1\leqslant i\leqslant d}.$$

An easy computation shows that $\tilde{F}(x) := H_{<k}(Tx) \circ F(x) \circ H_{<k}(x)^{-1}$ has the form

$$\tilde{F}(x)(Z) = Z + \left(\sum_{|j|=k} R^i_j(x) Z^j + \sum_{|j|>k} \tilde{a}^i_j(x) Z^j \right)_{1\leqslant i\leqslant d}$$

for some Hölder-continuous functions $\tilde{a}^i_j : X \to \mathbb{C}$. Moreover, for any $x \in X$ and $m \in \mathbb{N}$, one has

$$\tilde{F}(T^{m-1}x) \circ \ldots \circ \tilde{F}(x)(Z) = Z + \left(\sum_{|j|=k} \left(\sum_{v=0}^{m-1} R^i_j(T^v x) \right) Z^j + \mathcal{O}(|Z|^{k+1}) \right)_{1\leqslant i\leqslant d}.$$

Since $\tilde{F}$ is conjugated to F, the condition (2) holds for $\tilde{F}$. By the previous equality, this implies that for all $p \in X$ and $n \in \mathbb{N}$ such that $T^n p = p$, one has $\sum_{v=0}^{n-1} R^i_j(T^v x) = 0$. Therefore, the condition (2) holds for R^i_j, and we can apply the Livsic's theorem to establish the existence of an α-Hölder-continuous solution to $(ec^i_{j_*})$. Finally, by adding a constant if necessary, we may assume that this solution vanishes at x_0. □

To prove that the (up to now) formal map H is a genuine local diffeomorphism (that is, each formal power series $Z \mapsto z_i + \sum_{|j|>1} h^i_j(x) Z^j$ is convergent in a certain (uniform) neighbourhood of the origin), we will need to estimate the growth of the α-Hölder constant of the coefficients h^i_j. Indeed, if we show that this growth is at most exponential, then lemma 7 will apply, thus concluding the proof of the main theorem. To get the desired control, we will use the *majorant series method* introduced by Siegel for his work [6] on the linearization theorem for holomorphic germs with Diophantine rotation number (see also [7] for the higher-dimensional case).

Lemma 11. *There exists $S > 0$ such that*

$$[h^i_{\boldsymbol{j}}]_\alpha \leqslant g^i_{S,\boldsymbol{j}}$$

for every $\boldsymbol{j}$, i, where $h^i_{S,\boldsymbol{j}}$ is defined as in (8). Consequently, $\|h^i_{\boldsymbol{j}}\|$ grows at most exponentially.

Proof. Since F takes values on some $\mathcal{H}_0(d, R)$ and is an α-Hölder function, there exists $\kappa > 0$ such that

$$\|a^i_{\boldsymbol{j}}\| \leqslant \kappa^{|\boldsymbol{j}|} \qquad \text{and} \qquad [a^i_{\boldsymbol{j}}]_\alpha \leqslant \kappa^{|\boldsymbol{j}|}.$$

Assume that $[h^i_{\boldsymbol{j}}]_\alpha \leqslant g^i_{S,\boldsymbol{j}}$ for every $\boldsymbol{j} \preceq \boldsymbol{j}_*$. Since $h^i_{\boldsymbol{j}}$ vanishes at x_0 (except for $|\boldsymbol{j}| = 1$, for which $h^i_{\boldsymbol{j}} \equiv 1$), we also have $\|h^i_{\boldsymbol{j}}\| \leqslant g^i_{S,\boldsymbol{j}}$ for every $\boldsymbol{j} \preceq \boldsymbol{j}_*$. Moreover, since $P(\boldsymbol{j}_*, \boldsymbol{j})\{H\}$ is an homogeneous polynomial in $\{h^s_{\tilde{\boldsymbol{j}}}\}^{1\leqslant s\leqslant d}_{\tilde{\boldsymbol{j}}<\boldsymbol{j}_*}$ with positive coefficients,

$$\|P(\boldsymbol{j}_*, \boldsymbol{j})\{H\}\| \leqslant P(\boldsymbol{j}_*, \boldsymbol{j})\{\|H\|\} \leqslant P(\boldsymbol{j}_*, \boldsymbol{j})\{G_S\}.$$

Except for $|\boldsymbol{j}| = 1$ (for which $h^i_{\boldsymbol{j}} \equiv 1$), every $h^i_{\boldsymbol{j}}$ vanishes at x_0. Therefore, by lemma 2,

$$[P(\boldsymbol{j}_*, \boldsymbol{j})\{H\}]_\alpha \leqslant 2^{|\boldsymbol{j}|-1} P(\boldsymbol{j}_*, \boldsymbol{j})\{G_S\}.$$

The fundamental estimate of corollary 4 then yields

$$\begin{aligned}
[h^i_{\boldsymbol{j}_*}]_\alpha &\leqslant K\left(\left[\sum_{\boldsymbol{j}\leqslant\boldsymbol{j}_*} a^i_{\boldsymbol{j}} P(\boldsymbol{j}_*, \boldsymbol{j})\{H\}\right]_\alpha + \left\|\sum_{\boldsymbol{j}\leqslant\boldsymbol{j}_*} a^i_{\boldsymbol{j}} P(\boldsymbol{j}_*, \boldsymbol{j})\{H\}\right\|\right)\\
&\leqslant K\left(\sum_{\boldsymbol{j}\leqslant\boldsymbol{j}_*} \|a^i_{\boldsymbol{j}}\| [P(\boldsymbol{j}_*,\boldsymbol{j})\{H\}]_\alpha + \sum_{\boldsymbol{j}\leqslant\boldsymbol{j}_*} [a^i_{\boldsymbol{j}}]_\alpha \|P(\boldsymbol{j}_*,\boldsymbol{j})\{H\}\| + \sum_{\boldsymbol{j}\leqslant\boldsymbol{j}_*} \|a^i_{\boldsymbol{j}}\| \|P(\boldsymbol{j}_*, \boldsymbol{j})\{H\}\|\right)\\
&\leqslant \sum_{\boldsymbol{j}\leqslant\boldsymbol{j}_*} K\left((2\kappa)^{|\boldsymbol{j}|} + 2\kappa^{|\boldsymbol{j}|}\right) P(\boldsymbol{j}_*, \boldsymbol{j})\{G_S\}\\
&< g^i_{S,\boldsymbol{j}_*},
\end{aligned}$$

where the last inequality holds by taking $S \gg 2K\kappa$. □

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