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ON BOUNDED COCYCLES OF ISOMETRIES OVER MINIMAL DYNAMICS

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ABSTRACT. We show the following geometric generalization of a classical theorem of W. H. Gottschalk and G. A. Hedlund: a skew action induced by a cocycle of (affine) isometries of a Hilbert space over a minimal dynamical system has a continuous invariant section if and only if the cocycle is bounded. Equivalently, the associated twisted cohomological equation has a continuous solution if and only if the cocycle is bounded. We interpret this as a version of the Bruhat–Tits Center Lemma in the space of continuous functions. Our result also holds when the fiber is a proper CAT(0) space. One of the applications concerns matrix cocycles. Using the action of $\mathrm{GL}(n,\mathbb{R})$ on the (nonpositively curved) space of positively definite matrices, we show that every bounded linear cocycle over a minimal dynamical system is cohomologous to a cocycle taking values in the orthogonal group.

1. INTRODUCTION

Over the last years, the study of cocycles has been a central subject in many branches of mathematics including not only dynamical systems and group theory, but also geometry, foliations and mathematical physics. This work uses ideas and techniques coming from the former two areas to deal with cocycles over minimal dynamics and taking values in the group of isometries of a nonpositively curved space.

In a general form, a *cocycle* associated to a dynamical system on a base space is a map into a group G that is equivariant with respect to this system. These data naturally induce a skew action on a (perhaps nontrivial) fiber bundle, where the fibers are isomorphic to the phase space of the action of G. The possibility of "reducing" this fibered system is related to a central problem, namely solving an associated *cohomological equation*. Since we are interested in the possibility of reducing our cocycles into cocycles taking values in some compact group, we concentrate on skew actions satisfying a natural geometric counterpart, namely, a boundedness property. Before stating our main (somewhat

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technical) result, we prefer to illustrate its consequences giving several applications.

A matrix version of the Gottschalk–Hedlund Theorem. Let Γ be a semigroup acting minimally by homeomorphisms of a compact metric space X. Let A be a linear cocycle over this action, that is, a continuous map $A\colon \Gamma \times X \to \mathrm{GL}(n,\mathbb{R})$ satisfying $A(fg,x) = A(f,g(x))A(g,x)$ for all $x \in X$ and all f,g in the acting semigroup Γ.

THEOREM A. *Assume that there is a point $x_0 \in X$ and a constant $C > 0$ such that for all $f \in \Gamma$,*

$$\max\left\{\left\| A(f,x_0)\right\|, \left\| A(f,x_0)^{-1}\right\|\right\} \leq C.$$

Then A is cohomologous to a cocycle $\tilde{A}\colon \Gamma \times X \to \mathrm{O}(n,\mathbb{R})$, that is, for a certain continuous map $B\colon X \to \mathrm{GL}(n,\mathbb{R})$, one has $B(f(x))^{-1}A(f,x)B(x) = \tilde{A}(f,x) \in O(n,\mathbb{R})$ for all $x \in X$ and all $f \in \Gamma$.

This theorem generalizes a classical result of W. H. Gottschalk and G. A. Hedlund [10], which essentially corresponds to the case $n=1$. Indeed, Gottschalk and Hedlund considered cocycles into the (commutative) group $\mathbb{R}$, which fits in our framework by looking at a real number λ as the 1-dimensional linear map given by multiplication by e^{λ}; see §3 for more details.

Theorem A should also be compared with Kalinin's recent remarkable extension of Livšic's theorem to matrix cocycles [14]. In this setting, the base dynamical system[1] is given by that of an Anosov diffeomorphism T. Given a Hölder-continuous cocycle A over this dynamical system,[2] the condition for its cohomological triviality, that is, for the existence of a Hölder-continuous $B\colon X \to \mathrm{GL}(n,\mathbb{R})$ such that $A(T,x) = B(Tx)B(x)^{-1}$ for all $x \in X$, is that the products of A along periodic orbits are trivial:

1: OK as is?

2: OK as is?

$$T^n(x) = x \quad \Longrightarrow \quad \prod_{i=0}^{n-1} A\big(T, T^i(x)\big) = \mathrm{Id}.$$

In view of the method of proof of our Theorem A (see §4.1), it is natural to ask whether the Kalinin–Livšic theorem admits a version for cocycles taking values in the group of isometries of a nonpositively curved space.

A criterion of conformality à la Sullivan–Tukia. Let again Γ be a semigroup acting minimally on a compact metric space X, and $A\colon X \to \mathrm{GL}(n,\mathbb{R})$ a cocycle over this action. Recall that the *quasiconformal distortion* of the linear map $A(f,x)$ is defined as

$$K_A(f,x) := \left\| A(f,x)^{-1}\right\| \cdot \left\| A(f,x)\right\|.$$

Roughly, this measures how distorted is the image under $A(f,x)$ of a ball centered at the origin.

THEOREM B. *If there exist a point $x_0 \in X$ and a constant $C > 0$ with $K_A(f,x_0) \leq C$ for all $f \in \Gamma$, then there is a continuous invariant conformal structure on the bundle $X \times \mathbb{R}^n$. More precisely, the cocycle A is cohomologous to a cocycle taking values in the subgroup of conformal linear maps.*

This result should be compared with a theorem independently due to Sullivan [31] and Tukia [32] (compare also [20]), according to which every uniformly quasiconformal group of diffeomorphisms of a 2-manifold is quasiconformally conjugate to a group of conformal maps. Indeed, the first step for the proof of this theorem consists of finding an invariant conformal structure; the Ahlfors–Bers integrability theorem then allows one to obtain the conjugacy. It should be pointed out that B. Kalinin and V. Sadovskaya obtained in [15] an analogous result for linear cocycles over a hyperbolic dynamical system[3] in the spirit of Livšic's theorem. 3: OK as is?

A Bruhat–Tits Lemma in the space of continuous and bounded functions. A useful lemma due to Bruhat and Tits states that every action by isometries of either a proper CAT(0) space or of a Hilbert space and with a bounded orbit must have a fixed point. Although this still holds for actions on $\mathscr{L}^p$ spaces for $1 < p < \infty$, this is no longer true for actions on spaces of continuous functions (see Example 2) and subspaces of $\mathscr{L}^1$ spaces (see [3, Example 2.23]). We next concentrate on the former case in a more general situation.

Let X be a compact metric space, $\mathscr{H}$ a (real) separable Hilbert space, and $C(X,\mathscr{H})$ the space of continuous functions on X with values in $\mathscr{H}$. In order to discuss affine isometric actions on $C(X,\mathscr{H})$, we need to recall a classical result [6].

THEOREM (Banach–Stone)**.** *If π is a linear surjective isometry of $C(X,\mathbb{R})$, then there exist a unique homeomorphism $T\colon X \mapsto X$ and a unique continuous function* $\operatorname{sgn}\colon X \mapsto \{-1,+1\}$ *such that for every $\varphi \in C(X,\mathbb{R})$, one has*

$$\pi(\varphi)(x) = \operatorname{sgn}(x)\varphi(T^{-1}(x)).$$

An almost direct consequence of this theorem is that every action of a group Γ by linear isometries of $C(X,\mathbb{R})$ comes from an action on the basis X together with a cocycle $\operatorname{sgn}\colon \Gamma \times X \mapsto \{-1,+1\}$:

$$f\colon \varphi(\cdot) \mapsto \operatorname{sgn}\big(f, f^{-1}(\cdot)\big)\varphi\big(f^{-1}(\cdot)\big).$$

Here, the cocycle equality is $\operatorname{sgn}(fg,x) = \operatorname{sgn}(f,g(x))\operatorname{sgn}(g,x)$. Moreover, the function sgn must be continuous on the variable x.

An analogous statement holds in the space $C(X,\mathscr{H})$ (the corresponding version of the Banach–Stone Theorem is provided by [12]). Thus, every action π by linear isometries of $C(X,\mathscr{H})$ comes from an action (by homeomorphisms) on the basis X together with a cocycle $\Psi\colon \Gamma \times X \to O(\mathscr{H})$. More precisely,

$$\pi(f)\varphi(x) := \Psi\big(f, f^{-1}(x)\big)\varphi\big(f^{-1}(x)\big), \tag{1}$$

where Ψ satisfies

$$\Psi(fg,x) = \Psi\big(f, g(x)\big)\Psi(g,x).$$

Now let $I\colon \Gamma \to \operatorname{Isom}(C(X,\mathscr{H}))$ be an isometric action. By the Mazur–Ulam Theorem [6], I is the sum of a linear isometric action π with translations given by a cocycle $\rho\colon \Gamma \to C(X,\mathscr{H})$, where the cocycle relation is

$$\rho(fg) = \rho(f) + \pi(f)\big(\rho(g)\big). \tag{2}$$

THEOREM C. *In the context above, assume that the action on the basis is minimal. Then the existence of a bounded orbit for the affine isometric action $\pi + \rho$ on $C(X, \mathscr{H})$ implies that of a fixed point (function).*

The minimality of the action on X is necessary, as Example 2 in §2 shows. However, for spaces of bounded measurable functions, there is no need to treat any continuity issue, and an analogous (and much simpler!) version holds with no hypothesis on this action. For simplicity, we restrict ourselves to countable semigroups (this allows us to avoid tedious discussions concerning the measurability of certain naturally defined maps).

THEOREM D. *If an affine isometric action $I\colon \Gamma \to \mathrm{Isom}(\mathscr{L}^{\infty}_{\mu}(X, \mathscr{H}))$ has a bounded orbit, then it has a fixed point.*

Having our Theorem D as a partial motivation, U. Bader, T. Gelander and N. Monod have recently shown an analogous result for $\mathscr{L}^1$ spaces [2]. Their clever proof is mostly geometric, hence completely different from ours. Quite surprisingly, it applies more generally to isometries of preduals of von Neumann algebras.

Despite the intrinsic interest of Theorems C and D, their possible applications in Rigidity Theory are quite limited. Indeed, every countable group acts affinely on an $\mathscr{L}^{\infty}$ space and on a space of continuous functions without bounded orbits. For instance, one may consider the action of Γ on $\ell^{\infty}(\Gamma)$ with regular linear part and translation part given by $\rho(g)(h) := d(h, g) - d(h, id)$.

2. STATEMENT OF THE MAIN THEOREM AND PROOF OF THEOREMS A, B AND C

As we have already announced, Theorems A, B and C above are almost direct consequences of a general principle that is captured by our Main Theorem below. Roughly, for every skew action by isometries of a CAT(0) space [7] over a minimal dynamical system,[4] the existence of a bounded orbit is equivalent to the existence of a *continuous* invariant section. The proof of Theorem D uses a baby form of this principle; see §4.1.

4: OK as is?

Consider a minimal action by continuous maps of a semigroup Γ on a compact metric space X. Let $\mathscr{H}$ be either a proper CAT(0) space or a Hilbert space. We consider a skew action by isometries of $\mathscr{H}$:

$$f\colon (x, v) \mapsto \big(f(x), I(f, x)v\big).$$

Here, for each $f \in \Gamma$, the map $I(f, \cdot)\colon X \to \mathrm{Isom}(\mathscr{H})$ is continuous and satisfies the cocycle relation

$$I(fg, x) = I\big(f, g(x)\big) I(g, x).$$

MAIN THEOREM. *In the setting above, assume that for some $x_0 \in X$ and $v_0 \in \mathscr{H}$ there is a bounded subset $B \subset \mathscr{H}$ such that $I(f, x_0)v_0$ belongs to B for every $f \in \Gamma$. Then there exists a continuous section $x \mapsto (x, \varphi(x)) \in X \times \mathscr{H}$ that is invariant under the skew action of Γ, that is, it satisfies $I(f, x)\varphi(x) = \varphi(f(x))$ for all $f \in \Gamma$ and all $x \in X$.*

Note that the CAT(0) hypothesis is necessary, as the simple example of an irrational rotation on the torus shows. Indeed, existence of an invariant continuous section is forbidden in this case due to the minimality; however, all the orbits are bounded because the underlying product space—namely, the torus—is compact.

In §4.2, we give four independent proofs of the Main Theorem in the case of proper CAT(0) spaces. Letting $\mathscr{H}$ be the hyperbolic plane, this covers a case already considered in [34, Proposition 1]. (Actually, our fourth proof is strongly motivated by that of [34].)

The proof of the Main Theorem for infinite-dimensional Hilbert spaces is given in §4.3. This proof is much more subtle than the four proofs in §4.2. The necessity of a different argument is explained by means of a clarifying example in §4.3.1 of a cocycle whose linear part is induced by the shift on an orthonormal basis. (These cocycles are extensively studied in Appendix B.) Let us mention that the argument still applies (with minor modifications that we leave to the reader) to the case where the fiber is a (separable) uniformly-convex Banach space, thus leading to an analogous theorem in this more general situation. The eventual extension to $\mathscr{L}^1$ spaces seems to be an interesting problem. Finally, we should point out that, although stated for semigroup actions, the Main Theorem extends (with slight modifications in the proof) to *pseudogroups*, and would also extend to *groupoids*, thus yielding potential applications for foliations.

In what follows, we assume the validity of the Main Theorem, and we proceed to give proofs for Theorems A, B and C, together with a corollary and an example for the last of these theorems.

Proof of Theorem A. The space $\mathrm{Pos}(n)$ of positive-definite symmetric matrices of order $n \times n$ is a locally-symmetric space of nonpositive curvature, hence a proper CAT(0)-space. The distance between $P \in \mathrm{Pos}(n)$ and the identity is given by the square root of the sum of the squares of the logarithms of its eigenvalues. In particular, there exists $\tilde{C} > 0$ such that $\max\{\|P\|, \|P^{-1}\|\} \leq C$ implies that the distance between P and $\mathrm{Id} \in \mathrm{Pos}(n)$ is smaller than or equal to $\tilde{C}$. (See [19, Chapter XII] for more details.)

The linear group $\mathrm{GL}(n,\mathbb{R})$ acts by isometries of $\mathrm{Pos}(n)$, with g sending P into $g \cdot P := gPg^T$. The condition $\max\{\|A(f,x_0)\|, \|A(f,x_0)^{-1}\|\} \leq C$ implies that the orbit of the point (x_0, Id) under the associated skew action is bounded. By the Main Theorem, there exists an invariant continuous section $\varphi\colon X \to \mathrm{Pos}(n)$.

The exponential map at the identity $\exp_{\mathrm{Id}}\colon Sym(n) \to \mathrm{Pos}(n)$ is a diffeomorphism between the space of symmetric matrices of order $n \times n$ and $\mathrm{Pos}(n)$. Hence, there is a continuous map $v\colon X \to Sym(n)$ such that for each $x \in X$,

$$\varphi(x) = \exp_{\mathrm{Id}}(v(x)) \exp_{\mathrm{Id}}(v(x))^T.$$

We define the continuous map $B\colon X \to \mathrm{GL}(n,\mathbb{R})$ by letting $B(x) := \exp_{\mathrm{Id}}(v(x))$. (Note that B takes values in $\mathrm{Pos}(n)$.) The equation of the invariance of φ yields

$$\begin{aligned}B(f(x))B(f(x))^T &= \varphi(f(x)) = A(f,x)\cdot\varphi(x)\\ &= A(f,x)\varphi(x)A(f,x)^T = A(f,x)B(x)B(x)^T A(f,x)^T,\end{aligned}$$

hence

$$B(f(x))^{-1}A(f,x)B(x)\big[B(f(x))^{-1}A(f,x)B(x)\big]^T = \mathrm{Id}.$$

Thus, the cocycle $B(f(x))^{-1}A(f,x)B(x)$ takes values in $\mathrm{O}(n,\mathbb{R})$, which ends the proof. □

Proof of Theorem B. The space $\mathrm{Conf}(n)$ of conformal structures on $\mathbb{R}^n$ identifies with the space of ellipsoids in $\mathbb{R}^n$ centered at the origin and having area 1. This can be identified with the space of positive-definite symmetric matrices of order $n\times n$ and determinant 1. Indeed, to each such matrix P, we can associate the ellipsoid $\{v\in\mathbb{R}^n : \langle P^{-1}v, v\rangle \le 1\}$. The latter is a Riemannian symmetric subspace of $\mathrm{Pos}(n)$ with nonpositive curvature, hence a proper CAT(0)-space. The linear group $\mathrm{GL}(n,\mathbb{R})$ acts by isometries on $\mathrm{Conf}(n)$, with g sending P into $g\cdot P := (\det g^T g)^{-1/n} gPg^T$. The condition $K_A(f,x_0)\le C$ implies that the orbit of the point (x_0,Id) under the associated skew action is bounded. By the Main Theorem, there exists an invariant continuous section $\varphi\colon X\to \mathrm{Conf}(n)$. As in the proof of Theorem A, using the exponential map at the identity, we can find a continuous map $B\colon X\to\mathrm{GL}(n,\mathbb{R})$ such that for every $x\in X$,

$$\varphi(x) = B(x)B(x)^T.$$

Denote $\lambda(x) = (\det A(f,x)^T A(f,x))^{-1/2n}$. The invariance of φ yields

$$\begin{aligned}B(f(x))B(f(x))^T &= \varphi(f(x)) = A(f,x)\cdot\varphi(x) = \lambda(x)^2 A(f,x)\varphi(x)A(f,x)^T\\ &= \lambda(x)A(f,x)B(x)\big[\lambda(x)A(f,x)B(x)\big]^T,\end{aligned}$$

hence

$$\lambda(x)B(f(x))^{-1}A(f,x)B(x)\big[\lambda(x)B(f(x))^{-1}A(f,x)B(x)\big]^T = \mathrm{Id}.$$

We thus conclude that the cocycle $\lambda(x)B(f(x))^{-1}A(f,x)B(x)$ takes values in $\mathrm{O}(n,\mathbb{R})$, and thus $B(f(x))^{-1}A(f,x)B(x)$ belongs to the conformal linear group of $\mathbb{R}^n$. □

REMARK 1. Note that if it is possible to solve the classical cohomological equation for the function $\log(\lambda)$, then this allows us to conjugate A into a cocycle taking values in $\mathrm{O}(n,\mathbb{R})$.

Proof of Theorem C. Writing $\rho(f,x) := \rho(f)(f^{-1}(x))$, so that the isometric action may be written as

$$I(f)\varphi(x) = \Psi\big(f,f^{-1}(x)\big)\varphi\big(f^{-1}(x)\big) + \rho\big(f,f^{-1}(x)\big),$$

we have the cocycle relations

$$\Psi(fg,x) = \Psi\big(f,g(x)\big)\Psi(g,x), \qquad \rho(fg,x) = \rho\big(f,g(x)\big) + \Psi\big(f,g(x)\big)\rho(g,x).$$

It is then easy to check that

$$f\colon (x,v)\mapsto \big(f(x),\Psi(f,x)v+\rho(f,x)\big)$$

defines a skew action on $X\times\mathcal{H}$ by isometries on the fibers. Since I is assumed to have a bounded orbit, all its orbits must be bounded. In particular, the orbit of the identically zero function is bounded, that is, there exists a constant C such that $\|\rho(f,f^{-1}(x))\|\le C$ for all $f\in\Gamma$ and all $x\in X$. This means that the orbit of the zero vector of $\mathcal{H}$ under the associated skew action on $X\times\mathcal{H}$ is bounded. By Theorem A, there exists a continuous function $\varphi_0\colon X\to\mathcal{H}$ satisfying, for all $f\in\Gamma$ and all $x\in X$,

$$\varphi_0\big(f(x)\big)=\Psi(f,x)\varphi_0(x)+\rho(f,x).$$

Replacing x by $f^{-1}(x)$, this equality becomes

$$\varphi_0(x)=\Psi\big(f,f^{-1}(x)\big)\varphi_0\big(f^{-1}(x)\big)+\rho(f,f^{-1}(x))=I(f)\varphi_0(x),$$

thus showing that $\varphi_0\in C(X,\mathcal{H})$ is a fixed point of I. □

The next corollary to Theorem C was kindly suggested to the second-named author by P. Py, and should be compared with the results of Appendix A. For the statement, note that if a linear representation π on $C(X,\mathcal{H})$ has the form (1), then it also induces a linear representation on $\mathcal{L}^\infty_\mu(X,\mathcal{H})$. The cohomology space $H^1(\pi,C(X,\mathcal{H}))$ (resp. $H^1(\pi,\mathcal{L}^\infty_\mu(X,\mathcal{H}))$) is the quotient of the space of *cocycles*, i.e., continuous maps ρ from Γ into $C(X,\mathcal{H})$ (resp. $\mathcal{L}^\infty_\mu(X,\mathcal{H})$) satisfying the cocycle relation (2), by the subspace of *coboundaries*, i.e., those cocycles for which the associated affine action on $\mathcal{H}$ has a fixed point.

COROLLARY. *If the Γ-action on X is minimal, then for any quasi-invariant probability measure μ on X the natural map $H^1\big(\pi,C(X,\mathcal{H})\big)\to H^1\big(\pi,\mathcal{L}^\infty_\mu(X,\mathcal{H})\big)$ is injective.*

Proof. Suppose $\rho\colon\Gamma\to C(X,\mathcal{H})$ is a cocycle that is cohomologically trivial in $\mathcal{L}^\infty_\mu(X,\mathcal{H})$. Due to Theorem C, we need to show that $\rho(g)$ is bounded as a function in $C(X,\mathcal{H})$ independently of g. To do this, we may assume that Γ is countable. Indeed, if ρ is not bounded, then there exists a sequence $g_n\in\Gamma$ such that $\|\rho(g_n)\|_{C(X,\mathcal{H})}\ge n$, for each $n\in\mathbb{N}$. Thus, the cocycle ρ is unbounded when restricted to the countably generated subgroup $\langle g_1,g_2,\ldots\rangle$.

Now, since ρ is cohomologically trivial in $\mathcal{L}^\infty_\mu(X,\mathcal{H})$, it may be written in the form

$$\rho\big(g,g^{-1}(x)\big)=\rho(g)(x)=\Psi\big(g,g^{-1}(x)\big)\varphi\big(g^{-1}(x)\big)-\varphi(x) \tag{3}$$

for a certain function $\varphi\in\mathcal{L}^\infty_\mu(X,\mathcal{H})$, where the second equality above holds μ-a.e. Let X_0 be the set of points $x\in X$ for which equality (3) does not hold for some $g\in\Gamma$. Since Γ is assumed to be countable, X_0 has zero μ-measure. Let C be the essential supremum of $\|\varphi\|$. Then the μ-measure of $X^*:=\{x\colon\|\varphi(x)\|>C\}$ is zero, as well as that of $X_1:=\bigcup_{g\in\Gamma}g^{-1}(X^*)$. Let x_0 be a point in the full μ-measure set $X\smallsetminus(X_0\cup X_1)$. Then equality (3) holds at x_0 for all $g\in\Gamma$. Moreover,

$\|\varphi\big(g(x_0)\big)\| \le C$ also holds for all $g \in \Gamma$. This allows us to conclude that, for all $g \in \Gamma$,

$$\big\|\rho\big(g, g^{-1}(x_0)\big)\big\| \le 2C. \tag{4}$$

We claim that for all $h \in \Gamma$, we have $\|\rho(h)\| \le 4C$. Indeed, the cocycle identity yields

$$\rho\big(gh, (gh)^{-1}(x_0)\big) = \Psi\big(g, g^{-1}(x_0)\big)\rho\big(h, (gh)^{-1}(x_0)\big) + \rho\big(g, g^{-1}(x_0)\big).$$

Thus, by (4),

$$\big\|\rho\big(h, h^{-1}g^{-1}(x_0)\big)\big\| \le \big\|\rho\big(gh, (gh)^{-1}(x_0)\big)\big\| + \big\|\rho\big(g, g^{-1}(x_0)\big)\big\| \le 4C.$$

Fix $x \in X$. Taking a sequence (g_n) in Γ such that $g_n^{-1}(x_0) \to x$ as $n \to \infty$, we obtain

$$\big\|\rho(h)(x)\big\| = \big\|\rho\big(h, h^{-1}(x)\big)\big\| = \lim_{n\to\infty} \big\|\rho\big(h, h^{-1}g_n^{-1}(x_0)\big)\big\| \le 4C,$$

which shows our claim and hence the Corollary. □

We close this section with an example showing that the hypothesis of minimality for the action on X above is necessary. (A more interesting example in that the action on the basis is topologically transitive can be derived from [16, Exercise 2.9.2]; see also [30].)

EXAMPLE 2. Consider a parabolic element $T \in \mathrm{PSL}(2,\mathbb{R})$ acting on $X := \mathrm{S}^1$. Denoting by x_0 the unique fixed point of T, we let $\psi\colon \mathrm{S}^1 \mapsto \mathbb{R}$ be a function having a single discontinuity at x_0, so that $T(x_0)$ equals $\lim_{x\to x_0^+} T(x)$ and is different from $\lim_{x\to x_0^-} T(x)$. Then the function $x \mapsto \psi(x) - \psi(T(x))$ is continuous (it vanishes at x_0). Therefore, we may consider the affine isometric action of $\Gamma \sim \mathbb{Z}$ on $C(\mathrm{S}^1,\mathbb{R})$ generated by

$$I(T)\varphi(x) := \varphi\big(T^{-1}(x)\big) + \psi(x) - \psi\big(T^{-1}(x)\big).$$

Since, for every $n \in \mathbb{Z}$,

$$I(T^n)\varphi(x) = \varphi\big(T^{-n}(x)\big) + \psi(x) - \psi\big(T^{-n}(x)\big),$$

the orbit of any $\varphi \in C(\mathrm{S}^1,\mathbb{R})$ is bounded in norm by $\|\varphi\|_{C(X,\mathbb{R})} + 2\|\psi\|_{\mathscr{L}^\infty}$. We claim that, however, there is no fixed point in $C(\mathrm{S}^1,\mathbb{R})$ for this action, so that the cocycle $\psi - \psi\circ T$ is trivial in $H^1\big(\pi, \mathscr{L}^\infty_\mu(X,\mathscr{H})\big)$ but nontrivial in $H^1\big(\pi, C(X,\mathscr{H})\big)$. Indeed, the equality $I(T^{-1})\varphi = \varphi$ yields, for every $x \in \mathrm{S}^1$ and all $n \in \mathbb{N}$,

$$\varphi - \psi = (\varphi - \psi)\circ T = \ldots = (\varphi - \psi)\circ T^n.$$

Since the (forward) T-orbit of any $x \in \mathrm{S}^1 \setminus \{x_0\}$ converges to x_0, say from the right, this implies that the value of $\varphi - \psi$ is constant and equals $\varphi(x_0) - \lim_{x\to x_0^+}\psi(x)$. Clearly, this implies that φ cannot be continuous.

3. FURTHER APPLICATIONS: COHOMOLOGICAL EQUATIONS

Several problems in dynamical systems reduce to solving a linear functional (or *cohomological*) equation. For example, the (linearized version of the) conjugacy problem for circle diffeomorphism (see [11]), the study of interval-exchange maps (see [21]), the existence of eigenvalues of the Koopman operator associated with a dynamical system (see [17]), time changes for flows (see [16]), etc. One of the most basic results about the existence of continuous solutions for these equations is the classical Gottschalk–Hedlund Theorem that we next recall (see [10, Chapter 14] for more details). Note that the converse of this result is also true but much more elementary.

THEOREM (Gottschalk–Hedlund)**.** *Let X be a compact metric space, $T\colon X \to X$ a minimal continuous map and $\rho\colon X \to \mathbb{R}$ a continuous function. If there exists a point $x_0 \in X$ such that*

$$\sup_{n\in\mathbb{N}} \Big| \sum_{j=0}^{n-1} \rho\big(T^j(x_0)\big) \Big| < \infty, \tag{5}$$

then the cohomological equation

$$\varphi \circ T - \varphi = \rho \tag{6}$$

has a continuous solution $\varphi\colon X \to \mathbb{R}$.

The origin of the Gottschalk–Hedlund Theorem was the study of a special 2-dimensional system, nowadays known as *cylindrical cascade*. Let X, T and ρ be as before. The cylindrical cascade associated to these data is the map

$$\begin{aligned} F\colon X \times \mathbb{R} &\to X \times \mathbb{R} \\ (x,t) &\mapsto \big(T(x), t + \rho(x)\big). \end{aligned}$$

Gottschalk and Hedlund observed that F is topologically conjugate to the map $(x,t) \mapsto (T(x), t)$ if and only if the cohomological equation (6) has a continuous solution.

The map F above can be thought of as the skew action induced by a minimal $\mathbb{N}$-action on X and a cocycle of isometries (translations) of $\mathbb{R}$. Moreover, the hypothesis (5) corresponds to the orbit of the point $(x_0, 0)$ under this skew action being bounded. This fits into both the framework and the hypothesis of our Main Theorem for the case of a cocycle I into the group of isometries of a Hilbert space $\mathscr{H}$. Indeed, writing $I = \Psi + \rho$, with Ψ being the linear part of I and ρ being the translation part, the cocycle relations become

$$\Psi(fg,x) = \Psi\big(f, g(x)\big)\Psi(g,x), \qquad \rho(fg,x) = \rho\big(f,g(x)\big) + \Psi\big(f,g(x)\big)\rho(g,x). \tag{7}$$

Whenever this is satisfied, we have an associated skew action on $X \times \mathscr{H}$:

$$f\colon (x,v) \mapsto \big(f(x), I(f,x)v\big).$$

The Main Theorem asserts that the existence of a bounded orbit for this skew action implies the existence of a continuous invariant section φ. Since this means that $I(f,x)\varphi(x) = \varphi(f(x))$, we have that φ satisfies the twisted cohomological equation

$$\varphi\big(f(x)\big) - \Psi(f,x)\varphi(x) = \rho(f,x). \tag{8}$$

Moreover, conjugation by the homeomorphism $S\colon (x,v) \mapsto \big(x, v - \varphi(x)\big)$ yields, for each $f \in \Gamma$,

$$\begin{aligned} SfS^{-1}(x,v) &= Sf\big(x, v+\varphi(x)\big) \\ &= S\big(f(x), I(f,x)(v+\varphi(x))\big) \\ &= S\big(f(x), \Psi(f,x)v + I(f,x)\varphi(x)\big) \\ &= \big(f(x), \Psi(f,x)v + I(f,x)\varphi(x) - \varphi(f(x))\big) \\ &= \big(f(x), \Psi(f,x)v\big). \end{aligned}$$

In other words, conjugation by S reduces the cocycle I to its linear part Ψ.

EXAMPLE 3. If $\mathscr{H} = \mathbb{R}$ and $\Psi(f,x) = \mathrm{Id}$ for every (f,x), then the Main Theorem is the version for semigroups of the "equivariant Gottschalk–Hedlund Lemma" of [25] (also contained in [26, Section 3.6.2]), which - as the second-named author discovered while writing this article - was originally obtained by J. Moulin Ollagnier and D. Pinchon in [22] (compare [18]). Note that for $\Gamma \sim \mathbb{N}$, this corresponds to the classical Gottschalk–Hedlund Theorem. Nevertheless, even in this particular case, the proof we will provide for the Main Theorem differs from the classical ones in a key geometric argument. For $\Gamma \sim \mathbb{R}^+$, this is an equivalent form of the main result of [23].

EXAMPLE 4. Again in dimension 1, let $\Gamma \sim \mathbb{N}$ act on X by powers of a continuous, minimal map T. Letting $\Psi(n,x) := (-\mathrm{Id})^n$, the Main Theorem yields the following statement: if, for a continuous function $\rho\colon X \to \mathbb{R}$, the values of the alternating sums

$$\sum_{k=0}^{n-1} (-1)^k \rho\big(T^k(x_0)\big),$$

are uniformly bounded (independently of n) for some $x_0 \in X$, then the cohomological equation

$$\varphi\big(T(x)\big) + \varphi(x) = \rho(x)$$

has a continuous solution φ. The interest on this equation comes from the problem of extracting a square root of the associated cylindrical cascade. More precisely, if $T^{1/2}$ is a square root of T, then $(x,v) \mapsto (T^{1/2}(x), v+\varphi(x))$ is a square root of $(x,v) \mapsto (T(x), v+\rho(x))$ if and only of φ satisfies the cohomological equation

$$\varphi\big(T^{1/2}(x)\big) + \varphi(x) = \rho(x).$$

EXAMPLE 5. Let $\mathscr{H} = \mathbb{R}^2 \sim \mathbb{C}$, and consider an action of $\Gamma \sim \mathbb{N}$ on X by powers of a continuous, minimal map T. Assume that $\Psi(n,x) = \Psi(n)$ does not depend on x and preserves orientation. Then it coincides with the rotation of angle $n\beta$, where $e^{i\beta} = \Psi(1,x)$ for any x. Given $z \in \mathbb{C}$, the cocycle relation yields

$$I(n,x)z = \Psi(n)z + \rho(n,x) = e^{in\beta}z + \sum_{k=0}^{n-1} e^{i(n-k-1)\beta}\rho\big(1, T^k(x)\big).$$

In this case, the boundedness hypothesis means that for $\rho(x) := \rho(1,x)$ and some $x_0 \in X$, the norm of

$$\sum_{k=0}^{n-1} e^{-ik\beta}\rho\big(T^k(x_0)\big) \tag{9}$$

is uniformly bounded (independently of n). Moreover, equation (8) becomes

$$\varphi\big(T(x)\big) - e^{i\beta}\varphi(x) = \rho(x). \tag{10}$$

As a consequence of the Main Theorem, if the sums (9) are uniformly bounded, then the cohomological equation (10) has a continuous solution φ. (An alternative, less geometric proof of this fact can be derived from the results of [28].) We should point out that this equation corresponds to the linearized version of that encoding the stability of a closed orbit under perturbation in a toy model of a planetary system; see [29]. We refer to [9] for an accurate study of the dynamics of the associated map $(x,z) \mapsto (T(x), e^{i\beta}z + \rho(x))$.

EXAMPLE 6. Given an integer $q \geq 1$, an irrational angle α, an arbitrary angle β, and a continuous function $\rho \colon \mathrm{S}^1 \to \mathbb{C}$, we consider the skew map $(\theta, z) \mapsto (\theta+\alpha, e^{i\beta}z + \rho(\theta))$ from $\mathrm{S}^1 \times \mathbb{C}$ into itself. (This corresponds to a particular case of Example 5.) One easily checks that finding a q^{th} root of the form $(\theta, z) \mapsto (\theta + \alpha/q, e^{i\beta/q}z + \varphi(\theta))$ for this map is equivalent to solving the *cyclotonic equation* 5: OK as is?

$$\sum_{k=0}^{q-1} e^{\frac{ik\beta}{q}} \varphi\Big(\theta + \frac{(q-k-1)\alpha}{q}\Big) = \rho(\theta). \tag{11}$$

Note that, for $\beta := 0$ and $q := 2$, we retrieve an equation similar to that of Example 4.

Despite the strange form of equation (11), we claim that if α and β are independent over the rationals, then it is equivalent to the twisted cohomological equation

$$\varphi(\theta+\alpha) - e^{i\beta}\varphi(\theta) = \rho\Big(\theta + \frac{\alpha}{q}\Big) - e^{\frac{i\beta}{q}}\rho(\theta). \tag{12}$$

Indeed, if φ solves (11), then replacing θ by $\theta - \alpha/q$ and multiplying both sides by $e^{\frac{i\beta}{q}}$, we obtain

$$\sum_{k=0}^{q-1} e^{\frac{i(k+1)\beta}{q}} \varphi\Big(\theta + \frac{(q-1-(k+1))\alpha}{q}\Big) = e^{\frac{i\beta}{q}}\rho\Big(\theta - \frac{\alpha}{q}\Big),$$

that is,

$$\sum_{k=1}^{q} e^{\frac{ik\beta}{q}} \varphi\Big(\theta + \frac{(q-k-1)\alpha}{q}\Big) = e^{\frac{i\beta}{q}}\rho\Big(\theta - \frac{\alpha}{q}\Big). \tag{13}$$

Subtracting (13) from (11) yields

$$\varphi\Big(\theta+\frac{(q-1)\alpha}{q}\Big)-e^{\frac{iq\beta}{q}}\varphi\Big(\theta-\frac{\alpha}{q}\Big)=\rho(\theta)-e^{\frac{i\beta}{q}}\rho\Big(\theta-\frac{\alpha}{q}\Big).$$

Finally, replacing θ by $\theta+\alpha/q$ yields (12).

Conversely, assume that φ solves (12). Then

$$\begin{aligned}\sum_{k=0}^{q-1}e^{\frac{ik\beta}{q}}\varphi\Big(\theta+\frac{(q-k-1)\alpha}{q}\Big)\\ &=\sum_{k=0}^{q-1}e^{\frac{ik\beta}{q}}\left[e^{i\beta}\varphi\Big(\theta-\frac{(k+1)\alpha}{q}\Big)+\rho\Big(\theta-\frac{k\alpha}{q}\Big)-e^{\frac{i\beta}{q}}\rho\Big(\theta-\frac{(k+1)\alpha}{q}\Big)\right]\\ &=\rho(\theta)-e^{i\beta}\rho(\theta-\alpha)+e^{i\beta}\sum_{k=0}^{q-1}e^{\frac{ik\beta}{q}}\varphi\Big(\theta-\frac{(k+1)\alpha}{q}\Big).\end{aligned}$$

Letting

$$\psi(\theta):=\sum_{k=0}^{q-1}e^{\frac{ik\beta}{q}}\varphi\Big(\theta+\frac{(q-k-1)\alpha}{q}\Big),$$

this equality may be rewritten as

$$\psi(\theta)=e^{i\beta}\psi(\theta-\alpha)+\rho(\theta)-e^{i\beta}\rho(\theta-\alpha),$$

that is,

$$\psi(\theta)-\rho(\theta)=e^{i\beta}\big[\psi(\theta-\alpha)-\rho(\theta-\alpha)\big].$$

Thus, $\theta\mapsto\psi(\theta)-\rho(\theta)$ is an eigenfunction for the Koopman operator associated to the rotation of angle $-\alpha$, with eigenvalue $e^{i\beta}$. If $\psi-\rho$ were nonzero, then β should be a rational multiple (mod. 1) of α (see, for instance, [33, Theorem 3.5]), which is contrary to our hypothesis.

4. PROOF OF THE MAIN THEOREM

4.1. **A general strategy and proof of Theorem D.** An important case covered by the Main Theorem corresponds to that where X is a single point. In this case, our result reduces to the version for semigroups of the Bruhat–Tits Lemma. To better discuss this link, we recall the general framework (see [5, Proposition 5.10] for more details). Let $\mathcal{H}$ be either a proper CAT(0) space or a (real and separable) Hilbert space. Given a bounded subset of $\mathcal{H}$, for each $v\in\mathcal{H}$ we let

$$r_B(v):=\inf\big\{r>0\colon B\subset \mathrm{Ball}(v,r)\big\}=\sup_{w\in B}d(v,w).$$

The *radius* of B is defined as $r_B:=\inf\big\{r_B(v)\colon v\in\mathcal{H}\big\}$.

The following facts hold:

- The infimum of $r_B(\cdot)$ is attained. Indeed, in case of a proper CAT(0) space, this follows from the compactness of the closed (bounded) balls. In case of a Hilbert space, this follows from the relative compactness of bounded subsets of $\mathcal{H}$ when endowed with the weak topology, and the fact that the distance function is lower-semicontinuous.

– Actually, it is attained at a unique point. Indeed, this follows from the "convexity properties" of the distance function on $\mathscr{H}$, that is, the CAT(0) property.

The unique point realizing the infimum is called the geometric (or Chebyshev) *center* of B. This point $w := \mathrm{ctr}(B)$ is thus characterized as being the unique one satisfying $B \subset \overline{\mathrm{Ball}(w, r_B)}$.

By construction, if $I \colon \mathscr{H} \to \mathscr{H}$ is an isometry, then $r_B = r_{I(B)}$ and $I(\mathrm{ctr}(B)) = \mathrm{ctr}(I(B))$. Moreover, we have the following fact (a proof is given further on).

PROPOSITION 7. *The map $B \mapsto \mathrm{ctr}(B)$ is continuous with respect to the Hausdorff topology on closed bounded subsets of $\mathscr{H}$.*

Let us again recall the statement of the Bruhat–Tits Center Lemma [8].

LEMMA (Bruhat–Tits)**.** *Let Γ be a group acting by isometries of $\mathscr{H}$. If the action has a bounded orbit, then there is a point in $\mathscr{H}$ that is fixed by every element of Γ.*

Indeed, the center of the bounded orbit must remain fixed. If we conjugate by the translation sending this fixed point to the origin, then the action of every element of Γ becomes an isometry fixing the origin, that is, a linear isometry.

It is worth mentioning that Bruhat–Tits Lemma still holds for *semigroup* actions, but the proof needs an extra argument. Indeed, if B is a bounded forward-invariant set (as for example a bounded orbit of the semigroup), it is not completely obvious that its center is invariant under every $f \in \Gamma$. To see that this is the case, note that, letting $r := r_B$, from $B \subset \overline{\mathrm{Ball}(\mathrm{ctr}(B), r)}$ we obtain $f(B) \subset \overline{\mathrm{Ball}(f(\mathrm{ctr}(B)), r)}$. Now, as $f(B) \subset B$, we also have $f(B) \subset \overline{\mathrm{Ball}(\mathrm{ctr}(B), r)}$. Since $r = r_{f(B)}$, this necessarily implies that $f(\mathrm{ctr}(B)) = \mathrm{ctr}(B)$, as desired.

The main idea. The construction above provides us with a basic strategy of proof for the Main Theorem. Indeed, according to the hypothesis, the Γ-orbit of a certain point $(x_0, v_0) \in X \times \mathscr{H}$ remains in a bounded subset of $X \times \mathscr{H}$. By continuity, its closure $M := \overline{\mathrm{orb}(x_0, v)}$ is a closed, forward-invariant set. Note that since the Γ-action on X is assumed to be minimal, the projection of M to X is the whole space. As a consequence, the Γ-orbit of *any* point (x, v) remains in a bounded subset of $X \times \mathscr{H}$ (which depends on (x, v)). Indeed, if $v^* \in \mathscr{H}$ is such that (x, v^*) belongs to M, then the Γ-orbit of (x, v^*) is contained in M. Since for each $v \in \mathscr{H}$ and all $f \in \Gamma$,

$$d\big(I(f,x)v, I(f,x)v^*\big) = d(v, v^*),$$

this implies that the Γ-orbit of (x, v) is also bounded.

For each $x \in X$, let $M_x := \{v \in \mathscr{H} : (x, v) \in M\}$. Note that $I(f,x)M_x = M_{f(x)}$ for all $f \in \Gamma$ and all $x \in X$. The (nonempty) set M_x is bounded, hence we may consider its center $\varphi(x) := \mathrm{ctr}(M_x)$. Since the center map commutes with isometries, the curve $x \mapsto (x, \varphi(x))$ is invariant under the skew action. However, it is not evident at all that the thus-obtained map φ is continuous (*a priori*, it is just *measurable*). Indeed, we will need to elaborate a little bit to show that this is always the case for proper spaces. For infinite-dimensional Hilbert space fibers, this may fail to happen, hence we will need to slightly modify our approach. The proof for this case is strongly motivated by the main argument of Namioka–Asplund's proof of the Ryll-Nardzewski Fixed-Point Theorem [24]. Let us point out that a slight modification allows us to apply this argument also for CAT(0) proper spaces. More importantly, it easily applies to cocycles of noncontracting maps of $\mathscr{H}$, thus extending our Main Theorem to this framework.

As a first illustration of the preceding idea, we next give a

Proof of Theorem D. The proof is similar to that of Theorem C though much simpler since we do not need to take care of continuity issues. Let $\rho\colon \Gamma \to \mathscr{L}^{\infty}_{\mu}(X,\mathscr{H})$ be the translation part associated to the representation I, so that

$$I(f)\varphi(x) = \Psi\big(f, f^{-1}(x)\big)\varphi\big(f^{-1}(x)\big) + \rho\big(f, f^{-1}(x)\big). \tag{14}$$

Assume that the I-orbit $\mathrm{orb}(\varphi_0)$ of φ_0 is bounded so that the norm of each point therein is bounded from above by a constant C. For each $x \in X$, we let $N_x := \{\varphi(x) : \varphi \in \mathrm{orb}(\varphi_0)\}$. Then for μ-almost-every $x \in X$, this set N_x is bounded in norm by C. We may thus consider the function $\varphi\colon X \to \mathscr{H}$ defined by $\varphi(x) := \mathrm{ctr}(N_x)$. One can check that this is a measurable function. (This is an easy exercise if $\mathscr{H}$ has finite dimension, but a little bit harder in the infinite-dimensional case.) Moreover, it clearly belongs to $\mathscr{L}^{\infty}_{\mu}(X,\mathscr{H})$. We claim that φ is a fixed point of the isometric action. Indeed, due to (14), for μ-almost-every $x \in X$ and every $g \in \Gamma$, we have

$$\begin{aligned}
N_{g^{-1}(x)} &= \big\{\varphi(g^{-1}(x)) : \varphi \in \mathrm{orb}(\varphi_0)\big\}\\
&= \big\{\Psi\big(g, g^{-1}(x)\big)^{-1}\big[I(g)\varphi(x) - \rho\big(g, g^{-1}(x)\big)\big]\varphi \in \mathrm{orb}(\varphi_0)\big\}\\
&= \Psi\big(g, g^{-1}(x)\big)^{-1}\big\{I(g)\varphi(x) : \varphi \in \mathrm{orb}(\varphi_0)\big\} - \Psi\big(g, g^{-1}(x)\big)^{-1}\rho\big(g, g^{-1}(x)\big)\\
&= \Psi\big(g, g^{-1}(x)\big)^{-1}(N_x) - \Psi\big(g, g^{-1}(x)\big)^{-1}\rho\big(g, g^{-1}(x)\big),
\end{aligned}$$

hence

$$\Psi\big(g, g^{-1}(x)\big)(N_{g^{-1}(x)}) + \rho\big(g, g^{-1}(x)\big) = N_x.$$

Taking the center at both sides we obtain, for μ-almost-every $x \in X$,

$$\Psi\big(g, g^{-1}(x)\big)\varphi\big(g^{-1}(x)\big) + \rho\big(g, g^{-1}(x)\big) = \varphi(x),$$

which is equivalent to $I(g)\varphi = \varphi$. □

4.2. **The finite-dimensional case.**

4.2.1. *First proof.* In the context of the Main Theorem, assume that $\mathscr{H}$ is a proper CAT(0) space. Let M be any nonempty, compact invariant set for the skew action on $X \times \mathscr{H}$.

LEMMA 8. *Given $x \in X$, let (f_k) be a sequence of elements in Γ such that $f_k(x) \to x$. Then $I(f_k, x)(M_x)$ converges (in the Hausdorff topology) to M_x.*

Proof. If not, then one of the following two possibilities should arise.

1. There is a sequence of points $v_k \in I(f_k, x)(M_x)$ converging to a certain $v^* \notin M_x$.

This case is impossible. Otherwise, the sequence of points $(f_k(x), v_k) \in M$ would converge to the point $(x, v^*) \notin M$, thus contradicting the fact that M is closed.

2. There is a point $v^* \in M_x$ having a neighborhood $V \subset \mathscr{H}$ such that, for large-enough k, no point $w \in I(f_k, x)(M_x)$ belongs to V.

This case is impossible as well, but the argument is more subtle. First, note that since $I(f_k, x)(M_x)$ is uniformly bounded on k, the isometries $I(f_k, x)$ remain inside a compact subset of $\mathrm{Isom}(\mathscr{H})$ (endowed with the compact-open topology). Passing to a subsequence if necessary, we may assume that they converge to some $I \in \mathrm{Isom}(\mathscr{H})$. By our assumption, the vector v^* belongs to $M_x \setminus I(M_x)$. Moreover, we must have $I(M_x) \subset M_x$, because M is invariant and closed. Therefore, $I(M_x) \subsetneq M_x$. Since M_x is closed, this is impossible, because of the following

INDEPENDENT CLAIM. If C is a (nonempty) compact subset of $\mathscr{H}$ and J is an isometry such that $J(C) \subset C$, then $J(C) = C$.

To show this, first note that $J^n(C) \subset C$ for all $n \in \mathbb{N}$, which forces the subgroup $G = \overline{\langle J \rangle}$ of $\mathrm{Isom}(\mathscr{H})$ to be compact. Assume now that some vector v belongs to $C \setminus J(C)$. Then for all $n > m \geq 0$ we have $d(J^n(v), J^m(v)) = d(v, J^{m-n}(v)) > 0$.[6] 6: OK as is?
However, this contradicts the compactness of G. □

Now fix $x_0 \in X$, and let $v_0 := \mathrm{ctr}(M_{x_0})$.

LEMMA 9. *If (f_k) is a sequence of group elements such that $f_k(x_0) \to x_0$, then $I(f_k, x_0)\, v_0 \to v_0$.*

Proof. Since $I(f_k, x_0)(M_{x_0})$ converges to M_{x_0}, by Proposition 7, $\mathrm{ctr}(I(f_k, x_0)(M_{x_0}))$ converges to $\mathrm{ctr}(M_{x_0})$. Since the map ctr commutes with isometries,

$$I(f_k, x_0)\, v_0 = I(f_k, x_0)\big(\mathrm{ctr}(M_{x_0})\big) \longrightarrow \mathrm{ctr}(M_{x_0}) = v_0,$$

thus showing the lemma. □

Denote the closure of the orbit of (x_0, v_0) by $\hat{M}$. This is a compact invariant set. Moreover, by Lemma 9, the fiber $\hat{M}_{x_0}$ is reduced to v_0.

LEMMA 10. *For each $x \in X$, the set $\hat{M}_x$ is a single point.*

Proof. Assume for a contradiction that $\hat{M}_x$ contains at least two points, say $v \neq v^*$. Since the Γ-action on X is minimal, there exists a sequence (f_k) in Γ such that $f_k(x) \to x_0$. The points $(f_k(x), I(f_k, x)v)$ and $(f_k(x), I(f_k, x)v^*)$ belong to $\hat{M}$ for each k. Passing to a subsequence if necessary, we may assume that they converge to (x_0, w) and (x_0, w^*), respectively. Note that because $\hat{M}$ is invariant and closed, these two limit points are contained in $\hat{M}$, hence both w and w^* are in $\hat{M}_{x_0}$. Now for all k, we have

$$d\big(I(f_k, x)v, I(f_k, x)v^*\big) = d(v, v^*).$$

Therefore, $d(w, w^*) = d(v, v^*) > 0$. However, this contradicts the fact that $\hat{M}_{x_0} = \{v_0\}$. □

End of the proof. By Lemma 10, the set $\hat{M}$ is the graph of a well-defined function $\varphi\colon X \to \mathcal{H}$. Since $\hat{M}$ is compact, this function is continuous. Finally, because $\hat{M}$ is invariant, the curve $x \to (x, \varphi(x))$ satisfies all the desired properties. This concludes the proof of the Main Theorem provided we give a

Proof of Proposition 7. Given $\varepsilon > 0$, let B_ε be a set within Hausdorff distance $\mathrm{dist}_H(B, B_\varepsilon) \leq \varepsilon$ from B. The inclusions $B \subset \overline{\mathrm{Ball}(\mathrm{ctr}(B), r_B)}$ and $B_\varepsilon \subset \overline{\mathrm{Ball}(B, \varepsilon)}$ give $B_\varepsilon \subset \overline{\mathrm{Ball}(\mathrm{ctr}(B), r_B + \varepsilon)}$. Similarly, $B \subset \overline{\mathrm{Ball}(\mathrm{ctr}(B_\varepsilon), r_{B_\varepsilon} + \varepsilon)}$. As a consequence,

$$\left| r_B - r_{B_\varepsilon} \right| \leq \varepsilon. \tag{15}$$

Let m_ε be the midpoint between $\mathrm{ctr}(B)$ and $\mathrm{ctr}(B_\varepsilon)$. For each $w \in B_\varepsilon$, the median inequality (i.e., the CAT(0) property) yields

$$d(m_\varepsilon, w)^2 = \frac{d(\mathrm{ctr}(B_\varepsilon), w)^2}{2} + \frac{d(\mathrm{ctr}(B), w)^2}{2} - \frac{d(\mathrm{ctr}(B_\varepsilon), \mathrm{ctr}(B))^2}{4}.$$

Taking the supremum over all $w \in B_\varepsilon$ and using (15), we obtain

$$\begin{aligned}
r_{B_\varepsilon}^2 &\leq \sup_{w \in B_\varepsilon} d\big(m_\varepsilon, w\big)^2 \\
&\leq \frac{r_{B_\varepsilon}^2}{2} + \frac{\sup_{w \in B_\varepsilon} d(\mathrm{ctr}(B), w)^2}{2} - \frac{d(\mathrm{ctr}(B_\varepsilon), \mathrm{ctr}(B))^2}{4} \\
&\leq \frac{r_{B_\varepsilon}^2}{2} + \frac{1}{2}\left[\sup_{w \in B} d(\mathrm{ctr}(B), w) + \mathrm{dist}_H(B, B_\varepsilon)\right]^2 - \frac{d(\mathrm{ctr}(B_\varepsilon), \mathrm{ctr}(B))^2}{4} \\
&\leq \frac{r_{B_\varepsilon}^2}{2} + \frac{1}{2}[r_B + \varepsilon]^2 - \frac{d(\mathrm{ctr}(B_\varepsilon), \mathrm{ctr}(B))^2}{4},
\end{aligned}$$

hence

$$d\big(\mathrm{ctr}(B_\varepsilon), \mathrm{ctr}(B)\big)^2 \leq 4\left(\frac{[r_B + \varepsilon]^2}{2} - \frac{r_{B_\varepsilon}^2}{2}\right) \leq 2\big([r_B + \varepsilon]^2 - [r_B - \varepsilon]^2\big) = 8\varepsilon r_B.$$

Since the right-side expression converges to zero together with ε, this concludes the proof. □

4.2.2. *Second proof.* We next provide an even more geometric proof. Consider a compact, invariant set M for the skew action of Γ. Fix $x_0 \in X$, and write $v_0 := \mathrm{ctr}(M_{x_0})$. Denote also the closure of the orbit of (x_0, v_0) by $\hat{M}$. The main step in the first proof was to show that $\hat{M}_{x_0}$ is $\{v_0\}$. (Starting from this, Lemma 10 shows that $\hat{M}_x$ is a single point for each $x \in X$, which allows us to conclude the proof in the same way as before.)

Assume for contradiction that $\hat{M}_{x_0}$ contains a point v_0' distinct from v_0, and let $r_0 := r_{M_{x_0}}$ and $\varepsilon_0 := d(v_0', v_0) > 0$. There must be a sequence (f_k) in Γ such that $(f_k(x_0), I(f_k, x_0)v_0)$ converges to (x_0, v_0'). Since $M_{f_k(x_0)} \subset \overline{\mathrm{Ball}(I(f_k, x_0)v_0, r_0)}$, given $\varepsilon > 0$, we must have, for large-enough k,

$$M_{f_k(x_0)} \subset \overline{\mathrm{Ball}\big(v_0', r_0 + \varepsilon\big)}. \tag{16}$$

We now claim that for large-enough k, we also have

$$M_{f_k(x_0)} \subset \overline{\mathrm{Ball}(v_0, r_0+\varepsilon)}. \tag{17}$$

Indeed, if not, then there would be a sequence (v_n) such that v_{k_n} belongs to $M_{f_{k_n}(x_0)} \setminus \mathrm{Ball}(v_0, r_0+\varepsilon)$ for an increasing sequence of integers (k_n). Passing to a subsequence if necessary, this would yield a limit point $v^* \in M_{x_0} \setminus \mathrm{Ball}(v_0, r_0+\varepsilon)$, which is absurd.

Now, (16) and (17) yield (for a large-enough k depending on $\varepsilon > 0$)

$$M_{f_k(x_0)} \subset \overline{\mathrm{Ball}(v_0', r_0+\varepsilon)} \cap \overline{\mathrm{Ball}(v_0, r_0+\varepsilon)}.$$

The contradiction we seek comes from the fact that right-side set has radius at most $r_0 - \varepsilon$ provided that ε is at most 7 $\varepsilon_0^2/16r_0$. Indeed, letting v be the midpoint 7: OK as is? between v_0 and v_0', for each $w \in \overline{\mathrm{Ball}(v_0', r_0+\varepsilon)} \cap \overline{\mathrm{Ball}(v_0, r_0+\varepsilon)}$, the median inequality yields

$$d(w,v)^2 + \frac{d(v_0,v_0')^2}{4} \le \frac{d(w,v_0)^2}{2} + \frac{d(w,v_0')^2}{2}.$$

Thus,

$$d(w,v)^2 \le \frac{(r_0+\varepsilon)^2}{2} + \frac{(r_0+\varepsilon)^2}{2} - \frac{\varepsilon_0^2}{4} \le (r_0-\varepsilon)^2.$$

Therefore, the set $\overline{\mathrm{Ball}(v_0', r_0+\varepsilon)} \cap \overline{\mathrm{Ball}(v_0, r_0+\varepsilon)}$ is contained in $\overline{\mathrm{Ball}(v, r_0-\varepsilon)}$, which shows that its radius is at most $r_0 - \varepsilon$.

4.2.3. *Third proof.* This proof is restricted to the case of cocycles of isometries of $\mathbb{R}^\ell$, but likely it extends to general proper CAT(0) spaces. Its interest relies in that it relates the previous discussion to a classical notion.

The recurrence semigroup. Let us consider a general skew action of a semigroup Γ on $X \times \mathbb{R}^\ell$, namely $f\colon (x,v) \to (f(x), I(f,x)v)$, such that the Γ-action on X is minimal and each $I(f,x)$ is an isometry of $\mathbb{R}^\ell$. Given $x \in X$, we denote by R_x the set of isometries I of $\mathbb{R}^\ell$ such that $I = \lim_k I(f_k, x)$ for a sequence of elements $f_k \in \Gamma$ satisfying $f_k(x) \to x$. We begin with the following

LEMMA 11. *Assume that there is a compact subset K of* $\mathrm{Isom}(\mathbb{R}^\ell)$ *such that $I(f,x)$ lies in K for every $f \in \Gamma$ and all $x \in X$. (This is equivalent to the set $I(f,x)v$ being bounded for each $v \in \mathbb{R}^\ell$.) Then for every $x \in X$, the set R_x is a semigroup.*

Proof. Let d be the metric on X, and let dist be the left-invariant distance on the group of isometries of $\mathbb{R}^\ell$ induced by $\mathrm{dist}(\Psi+\rho, \mathrm{Id}) = \|\Psi - \mathrm{Id}\| + \|\rho\|$. One readily checks that there is a constant $C = C_K$ such that dist is perturbed under right-translation by a factor at most C, that is, $\mathrm{dist}(I_1 I, I_2 I) \le C\,\mathrm{dist}(I_1, I_2)$ for all $I \in K$ and all I_1, I_2 in $\mathrm{Isom}(\mathbb{R}^\ell)$.

Given I_1, I_2 in R_x, we need to show that $I_1 I_2$ also belongs to R_x. For $i \in \{1,2\}$, choose a sequence $(f_{i,k})_k$ such that $f_{i,k}(x) \to x$ and $I(f_{i,k}, x) \to I_i$. Given $\varepsilon > 0$, there is an integer $k_1 \in \mathbb{N}$ such that, for all $k \ge k_1$,

$$d\big(f_{1,k}(x), x\big) \le \varepsilon \quad \text{and} \quad \mathrm{dist}\big(I(f_{1,k}, x), I_1\big) \le \varepsilon.$$

By continuity, there exists $\delta \in]0, \varepsilon[$ such that if $d(x, y) \leq \delta$, then

$$d\big(f_{1,k_1}(x), f_{1,k_1}(y)\big) \leq \varepsilon \quad \text{and} \quad \operatorname{dist}\big(I(f_{1,k_1}, x), I(f_{1,k_1}, y)\big) \leq \varepsilon.$$

Fix $k_2 \in \mathbb{N}$ large enough that

$$d\big(f_{2,k_2}(x), x\big) \leq \delta \quad \text{and} \quad \operatorname{dist}\big(I(f_{2,k_2}, x), I_2\big) \leq \varepsilon.$$

We have

$$d\big(f_{1,k_1} f_{2,k_2}(x), x\big) \leq d\big(f_{1,k_1}(f_{2,k_2}(x)), f_{1,k_1}(x)\big) + d\big(f_{1,k_1}(x), x\big) \leq 2\varepsilon.$$

Moreover, using the almost invariance of dist, from

$$\operatorname{dist}(I(f_{1,k_1}, x), I(f_{1,k_1}, f_{2,k_2}(x))) \leq \varepsilon,$$

we get

$$\begin{aligned} &\operatorname{dist}\big(I(f_{1,k_1}, x) I(f_{2,k_2}, x), I(f_{1,k_1} f_{2,k_2}, x)\big) \\ &\qquad = \operatorname{dist}\big(I(f_{1,k_1}, x) I(f_{2,k_2}, x), I(f_{1,k_1}, f_{2,k_2}(x)) I(f_{2,k_2}, x)\big) \leq C\varepsilon. \end{aligned}$$

Therefore,

$$\begin{aligned} \operatorname{dist}(I_1 I_2, I(f_{1,k_1} f_{2,k_2}, x)) &\leq \operatorname{dist}(I_1 I_2, I(f_{1,k_1}, x) I(f_{2,k_2}, x)) \\ &\qquad + \operatorname{dist}(I(f_{1,k_1}, x) I(f_{2,k_2}, x), I(f_{1,k_1} f_{2,k_2}, x)) \\ &\leq \operatorname{dist}(I_1 I_2, I(f_{1,k_1}, x) I_2) \\ &\qquad + \operatorname{dist}(I(f_{1,k_1}, x) I_2, I(f_{1,k_1}, x) I(f_{2,k_2}, x)) + C\varepsilon \\ &= C \operatorname{dist}(I_1, I(f_{1,k_1}, x)) + \operatorname{dist}(I_2, I(f_{2,k_2}, x)) + C\varepsilon \\ &\leq (2C + 1)\varepsilon. \end{aligned}$$

Summarizing, for each $\varepsilon > 0$, we have found an element $f \in \Gamma$, namely, $f := f_{1,k_1} f_{2,k_2}$, such that

$$d\big(f(x), x\big) \leq 2\varepsilon \quad \text{and} \quad \operatorname{dist}\big(I_1 I_2, I(f, x)\big) \leq (2 + C)\varepsilon.$$

By definition, this shows that $I_1 I_2$ belongs to R_x. □

We will call R_x the *recurrence semigroup of* x. (A closely related notion was developed in [1].) We must emphasize that, in general, R_x is not a group, even if Γ has a group structure. Nevertheless, if Γ is a group and its action on the basis X is *equicontinuous*, then R_x is a group. Indeed, given $I \in R_x$, choose a sequence (f_k) in Γ so that $f_k(x) \to x$ and $I(f_k, x) \to I$. By equicontinuity, we also have $f_k^{-1}(x) \to x$, and passing to a subsequence if necessary, we may assume that $I(f_k^{-1}, x)$ converges to an isometry I'. Now, the proof of Lemma 11 yields that $I(f_n^{-1} f_m) \to I' I$ as n, m go to infinity. Letting $n = m$ go to infinity, this obviously implies $\mathrm{Id} = I' I$, that is, $I' \in R_x$ is the inverse of I.

LEMMA 12. *If M is the closure of the orbit of a point $(x, v) \in X \times \mathbb{R}^\ell$ under the skew action, then the set $M_x := \{w \in \mathbb{R}^\ell : (x, w) \in M\}$ coincides with $\{Iv : I \in R_x\}$.*

Proof. Each point in M is of the form $\lim_k (f_k(x), I(f_k,x)v)$ for a sequence of elements $f_k \in \Gamma$. Thus, each point of $w \in M_x$ has the form $\lim_k I(f_k,x)v$ for a sequence (f_k) in Γ such that $f_k(x) \to x$. The lemma follows from the fact that the set of isometries sending a prescribed vector $v \in \mathbb{R}^\ell$ into some fixed bounded neighborhood of another prescribed vector $w \in \mathbb{R}^\ell$ is compact. □

Assume now that the orbit of some point $(x,v) \in X \times \mathbb{R}^\ell$ is bounded, and let M be its closure. Fix a point $x_0 \in X$, and let $v_0 := \mathrm{ctr}(M_{x_0})$. Finally, let $\hat{M}$ be the closure of the orbit of the point (x_0, v_0).

LEMMA 13. *The set* $\hat{M}_{x_0} := \{w \in \mathbb{R}^\ell : (x_0, w) \in \hat{M}\}$ *reduces to* $\{v_0\}$.

Proof. By Lemma 12, the point v_0 may be written as $\mathrm{ctr}(\{Iv : I \in R_{x_0}\})$. By the semigroup version of the Bruhat–Tits Center Lemma, this point is fixed by every element of R_{x_0}. In other words, the set $\{Iv_0 : I \in R_{x_0}\}$ reduces to $\{v_0\}$. Finally, by Lemma 12 again, this set coincides with $\hat{M}_{x_0}$. □

The rest of the third proof works as the final part of the first one. Indeed, as in Lemma 10, one may show that for each $x \in X$, the set $\hat{M}_x$ is a single point. Hence, the set $\hat{M}$ is the graph of a well-defined function $\varphi\colon X \to \mathbb{R}^\ell$, and the curve $x \mapsto (x, \varphi(x))$ satisfies all the desired properties.

4.2.4. *Fourth proof.* Assume, once again, that there is a nonempty, compact, forward-invariant set M for the skew action of Γ. Let $D\colon X \to [0,\infty[$ be defined by letting $D(x)$ be the diameter of the fiber $M_x := \{w \in \mathscr{H} : (x,w) \in M\}$.

LEMMA 14. *The diameter of the fiber* M_x *does not depend on* x.

Proof. The function D is invariant under the Γ-action on X. Since this action is assumed to be minimal, in order to prove that D is constant, it suffices to show that it is upper-semicontinuous. To do this, let (x_n) be an arbitrary sequence of points converging to a certain $x \in X$. Let (x_{n_k}) be a subsequence such that $\lim_k D(x_{n_k}) = \limsup_n D(x_n) =: \Delta$. For each k, let (v_k, w_k) be a pair of points of $M_{x_{n_k}}$ at distance $D(x_{n_k})$. Passing to a subsequence if necessary, we may suppose that v_k (resp. w_k) converges to a certain $v \in M_x$ (resp. $w \in M_x$). Clearly, $d(v,w) \geq \lim_k d(v_k, w_k) = \Delta$. In particular, the diameter of M_x is greater than or equal to Δ. This shows the lemma. □

We will denote by $D(M)$ the common value of the diameter of the fibers M_x. Note that, denoting by $cv(M)$ the *convex closure* of M along the fibers, we have $D\big(cv(M)\big) = D(M)$. Moreover, straightforward arguments show that $cv(M)$ is also compact and invariant.

Now consider the set

$$M^* := \left\{ (x,v) : \begin{array}{l} \text{there exist } v_1, v_2 \text{ at distance } D \text{ in } M_x \text{ such that} \\ v \text{ is the midpoint of the segment } \overline{v_1 v_2} \end{array} \right\}.$$

Note [8] that $M^* = cv(M)^*$ and $cv(M^*)$ is contained in $cv(M)$. Moreover, M^* is invariant under the skew action. Furthermore, easy compactness-type arguments show that M^* is closed (hence compact) and nonempty. (*A priori*, the fibers

8: Is this ok? Originally, the text was all on one line and nearly as wide as the page. I figured the easiest thing was to put it in a minipage. It works, how well is up to you.

of M^* do not vary continuously.) Finally, the preceding lemma applied to M^* shows that all its fibers have the same diameter.

The next lemma is a direct consequence of [8, Lemma 3.2.3], and we reproduce the proof just for the reader's convenience.

LEMMA 15. *One has the inequality $D(M^*) \leq D(M)/\sqrt{2}$. Moreover, this estimate is sharp.*

Proof. Fix $x \in X$ and let v, w be points in M^*_x. By definition, there exist two pairs of points (v_1, v_2) and (w_1, w_2) at distance $D(M)$ in M_x such that v (resp. w) is the midpoint of $\overline{v_1 v_2}$ (resp. $\overline{w_1 w_2}$). The median inequality applied to the triangle $\Delta(v_1, w_1, w_2)$ yields

$$d(v_1, w)^2 + \frac{D(M)^2}{4} \leq \frac{d(v_1, w_1)^2}{2} + \frac{d(v_1, w_2)^2}{2} \leq \frac{D(M)^2}{2} + \frac{D(M)^2}{2} = D(M)^2,$$

hence, $d(v_1, w)^2 \leq 3D(M)^2/4$. Similarly, $d(v_2, w)^2 \leq 3D(M)^2/4$. Using this, the median inequality for $\Delta(v_1, v_2, w)$ yields

$$d(v, w)^2 + \frac{D(M)^2}{4} \leq \frac{d(v_1, w)^2}{2} + \frac{d(v_2, w)^2}{2} \leq \frac{3D(M)^2}{4}.$$

This easily leads to the estimate of the lemma. To see that this estimate is sharp, it suffices to consider the case where each M_x consists of four points that are the vertices of a tetrahedron. (In dimension 2, the constant $\sqrt{2}$ can be replaced by 2.) □

End of the proof. Let us define the sequence of nonempty, compact, invariant sets M_n by $M_1 := cv(M)$ and $M_n := cv(M^*_{n-1})$ for each $n \geq 2$. Since $M_n \subset M_{n-1}$ for each $n > 1$, the set $\hat{M} := \bigcap_{n\geq 1} M_n$ is also nonempty and compact, as well as invariant. Moreover, Lemma 15 implies that $D(\hat{M}) = 0$. In other words, each fiber $\hat{M}_x$ consists of a single point $\varphi(x)$, and the thus-defined function φ satisfies all the desired properties. (Its continuity follows from the fact that its graph is compact.)

REMARK 16. It is instructive to compare the technique of the preceding proof with the three previous ones. Given a bounded subset $B \subset \mathscr{H}$, we let $B_1 := B$, and having defined $B_2, \ldots, B_{n-1}$, we let B_n be the set of midpoints of segments between points of B_{n-1} situated at distance $\operatorname{diam}(B_{n-1})$. Finally, we call the point $\operatorname{ctr}^*(B) := \bigcap_{n\geq 1} B_n$ the Bruhat–Tits center of B.

In general, $\operatorname{ctr}^*(B)$ does not coincide with $\operatorname{ctr}(B)$. For example, if B consists of three points that are the vertices of a triangle Δ all of whose angles are $\leq \pi$, then $\operatorname{ctr}(B)$ coincides with the circumcenter of Δ. However, if the sides of Δ have different length, then $\operatorname{ctr}^*(B)$ is the midpoint of the largest side.

4.3. **The case of infinite-dimensional Hilbert space fibers.**

4.3.1. *The lack of continuity of invariant sections given by centers along the fibers.* For fibers that are infinite-dimensional Hilbert spaces, none of the strategies of proof proposed so far works. On the one hand, there is a serious technical problem in defining the recurrence semigroup (the group of linear isometries is not compact when endowed with the norm-topology). On the other hand, the center function is not continuous with respect to the Hausdorff topology inherited from the weak topology. Finally, the diameter of the fibers of an skew-invariant, weakly-compact set is not necessarily constant.

In a more concrete way, the example below showing that the center along the fibers of a weakly-compact invariant set may fail to be continuous illustrates all these technical problems.

EXAMPLE 17. Let us consider the Hilbert space $\mathscr{H} \sim \ell^2(\mathbb{Z})$, and let $\Gamma \sim \mathbb{Z}$ be acting on X by powers of a minimal homeomorphism T. Let us consider the skew action with linear part induced by $\Psi(1,x)v_n = v_{n+1}$ for every x, where $\{v_n\}$ is an orthonormal basis of $\mathscr{H}$, and with translation part $\rho\colon X \to \mathscr{H}$ vanishing everywhere. Fix $x_0 \in X$, and consider the two-points set $\{(x_0,0),(x_0,v_0)\}$. The closure (for the weak topology) of its orbit under the skew action is a set M whose fiber over $x \in X$ coincides with $\{0\}$ if x is not in the orbit of x_0, and with $\{0,v_n\}$ if $x = T^n(x_0)$. In the first case, we have $\mathrm{ctr}(M_x) = \{0\}$, while for $x = T^n(x_0)$ we have $\mathrm{ctr}(M_x) = \{v_n/2\}$. It is then easy to see that the function $\varphi\colon x \mapsto \mathrm{ctr}(M_x)$ is not weakly continuous (namely, it is discontinuous at every point).

4.3.2. *Existence of weakly-continuous invariant sections.* In this section, we deal with a skew action by isometries of a semigroup Γ such that the fibers are a Hilbert space $\mathscr{H}$ and the dynamics on the basis is minimal. We assume that, for all $f \in \Gamma$, the map $I(f,\cdot)\colon X \to \mathrm{Isom}(\mathscr{H})$ is continuous for the strong topology. Writing $I(f,\cdot) = \Psi(f,\cdot) + \rho(f,\cdot)$, this means that $\Psi(f,\cdot)\colon X \to U(\mathscr{H})$ is norm-continuous and $\rho(f,\cdot)\colon X \to \mathscr{H}$ is continuous for the strong topology on $\mathscr{H}$. We endow $X \times \mathscr{H}$ with the product topology, where the topology on the factor $\mathscr{H}$ is the weak one.

Suppose that there exists a bounded orbit for the skew action, and let us consider its convex closure M. By this we mean the smallest compact set that contains the given set and is convex along the fibers, in the sense that if $(x,v),(x,w)$ belong to M then $(x,\lambda v + (1-\lambda)w) \in M$ for all $0 \le \lambda \le 1$. The family $\mathscr{F}$ of nonempty, compact, invariant sets that are convex along the fibers is ordered by inclusion. A straightforward application of Zorn's Lemma shows that it contains a minimal element. The crucial step is the next

LEMMA 18. *For each minimal element M of the family $\mathscr{F}$, the fiber M_x above x consists of a single vector, for each $x \in X$.*

This lemma yields a weakly-continuous invariant section for the skew action. Indeed, the set M will be the graph of a function $\varphi\colon X \to \mathscr{H}$ which is weakly continuous, since its graph is compact. Moreover, since the set M is invariant, φ is an invariant function.

Proof of Lemma 18. Assume that for some $x_0 \in M$ the fiber M_{x_0} contains two vectors v_1, v_2 at a distance $\|v_1 - v_2\| =: \varepsilon > 0$. Let $r(M) > 0$ be the infimum of the radii r such that, for all $x \in X$, the fiber M_x is contained in $\mathrm{Ball}(0,r)$. Given $\kappa < 1$, there must exist $(y,w) \in M$ such that $\|w\| \geq \kappa r(M)$. Fix such a $w \in M_y$ and $\kappa < 1$ such that

$$\kappa > \sqrt[4]{1 - \frac{\varepsilon^2}{4r(M)^2}}.$$

For each $\lambda < 1$, let P_λ be the affine hyperplane $\lambda w + \langle w \rangle^\perp$. This hyperplane divides the whole fiber $\mathscr{H}$ above y into two closed hemispheres P_λ^+, P_λ^-, where w belongs to the interior of P_λ^+.

Let u be the midpoint between v_1 and v_2. By convexity, the point (x_0, u) must belong to M. We claim that the closure of its orbit must intersect the hemisphere $\{y\} \times P_\lambda^+$. Otherwise, the convex closure of its orbit would be a nonempty, compact, invariant set that is convex along the fibers and it is strictly contained in M (it does not contain (y,w)). However, this contradicts the fact that M is a minimal element of $\mathscr{F}$.

We thus conclude that for each $\lambda < 1$, there exists $f \in \Gamma$ such that $I(f,x_0)u \in P_\lambda^+$. Fixing such a λ so that

$$\lambda > \sqrt[4]{1 - \frac{\varepsilon^2}{4r(M)^2}},$$

we claim that this implies that

$$\text{either } I(f,x)v_1 \text{ or } I(f,x)v_2 \text{ lies outside } \overline{\mathrm{Ball}\big(0, r(M)\big)}. \tag{18}$$

Before proving this claim, note that it contradicts the definition of $r(M)$, thus concluding the proof.

The proof of (18) relies on the uniform convexity of $\mathscr{H}$. In a quantitative manner, since $I(f,x)u$ lies in P_λ^+, its norm is at least $\kappa\lambda r(M)$. By the median equality

$$\frac{\big\|I(f,x)v_1 - I(f,x)v_2\big\|^2}{4} = \frac{\big\|I(f,x)v_1\big\|^2}{2} + \frac{\big\|I(f,x)v_2\big\|^2}{2} - \big\|I(f,x)u\big\|^2,$$

this yields

$$\frac{\varepsilon^2}{4} \leq \frac{\big\|I(f,x)v_1\big\|^2}{2} + \frac{\big\|I(f,x)v_2\big\|^2}{2} - \kappa^2\lambda^2 r(M)^2.$$

Assuming that both $I(f,x)v_1$ and $I(f,x)v_2$ are in $\overline{\mathrm{Ball}(0, r(M))}$, this implies that

$$\frac{\varepsilon^2}{4} \leq r(M)^2 - \kappa^2\lambda^2 r(M)^2 = r(M)^2\big(1 - \kappa^2\lambda^2\big),$$

which contradicts our choice of the constants κ, λ. □

4.3.3. *Strong continuity of weakly-continuous invariant sections.* In order to complete the proof of the Main Theorem in the infinite-dimensional case, we need to show that in the context of §4.3.2, the following hods:

PROPOSITION 19. *A weakly-continuous solution of the cohomological equation (8) is strongly continuous. Equivalently, all weakly-continuous, skew-invariant sections are strongly continuous.*

To show this proposition, we begin by giving a geometrical criterion for strong continuity. To do this, given a function $\varphi\colon X \to \mathcal{H}$, we define its *oscillation* at a point x as

$$\operatorname{osc}(\varphi)(x) := \limsup_{\{y,z\}\to\{x\}} \left\|\varphi(y) - \varphi(z)\right\|.$$

Our first lemma should be clear from the definition.

LEMMA 20. *The map φ is strongly continuous at a point $x \in X$ if and only if* $\operatorname{osc}(\varphi)(x) = 0$.

Our second lemma involves the underlying dynamics of our setting.

LEMMA 21. *If the curve $x \mapsto (x, \varphi(x))$ is skew invariant, then the function $x \mapsto$ $\operatorname{osc}(\varphi)(x)$ is nondecreasing along the orbits of the action of Γ on X.*

Proof. Let $f \in \Gamma$ and $x \in X$ be given. It is enough to show that, given sequences (y_n), (z_n) converging to x so that $\|\varphi(y_n) - \varphi(z_n)\|$ converges to some value ε, then there exist $(\bar{y}_n)$ and $(\bar{z}_n)$ converging to $f(x)$ so that $\|\varphi(\bar{y}_n) - \varphi(\bar{z}_n)\|$ also converges to ε. We will show that this holds for $\bar{y}_n := f(y_n)$ and $\bar{z}_n := f(z_n)$. Indeed, the value of

$$\left\|\varphi\big(f(y_n)\big) - \varphi\big(f(z_n)\big)\right\|$$

may be written as

$$\left\|I(f, y_n)\varphi(y_n) - I(f, z_n)\varphi(z_n)\right\|,$$

and differs from

$$\left\|I(f, x)\varphi(y_n) - I(f, x)\varphi(z_n)\right\| = \|\varphi(y_n) - \varphi(z_n)\|$$

by no more than

$$\left\|I(f, y_n)\varphi(y_n) - I(f, x)\varphi(y_n)\right\| + \left\|I(f, z_n)\varphi(z_n) - I(f, x)\varphi(z_n)\right\|,$$

which equals

$$\left\|\big(\Psi(f, y_n) - \Psi(f, x)\big)\varphi(y_n) + \rho(y_n) - \rho(x)\right\| + \left\|\big(\Psi(f, z_n) - \Psi(f, x)\big)\varphi(z_n) + \rho(z_n) - \rho(x)\right\|.$$

Since φ is weakly continuous, it must be bounded, say by a constant $C > 0$. This implies that the last expression above is bounded from above by

$$C\big(\|\Psi(f, y_n) - \Psi(f, x)\| + \|\Psi(f, z_n) - \Psi(f, x)\|\big) + \|\rho(y_n) - \rho(x)\| + \|\rho(z_n) - \rho(x)\|.$$

By the norm-continuity of $\Psi(f, \cdot)$ and the strong continuity of $\rho(f, \cdot)$, this last expression converges to zero. This concludes the proof. □

The next lemma shows that the map $x \mapsto \operatorname{osc}(\varphi)(x)$ is upper-semicontinuous.

LEMMA 22. *For each $\varepsilon > 0$, the set $\{x : \operatorname{osc}(\varphi)(x) < \varepsilon\}$ is open in X.*

Proof. Given x_0 in this set, let $\varepsilon_0 := \operatorname{osc}(\varphi)(x_0) < \varepsilon$. Then there exists $\delta > 0$ such that, for all y, z at distance $< \delta$ from x_0, we have $\|\varphi(y) - \varphi(z)\| \le \frac{1}{2}(\varepsilon + \varepsilon_0)$. This clearly implies that, for all $x \in X$ such that $\operatorname{dist}(x, x_0) < \delta$, we have $\operatorname{osc}(\varphi)(x) \le \frac{1}{2}(\varepsilon + \varepsilon_0) < \varepsilon$. In other words, the δ-neighborhood of x_0 is contained in $\{x :$ $\operatorname{osc}(\varphi)(x) < \varepsilon\}$, thus showing the lemma. □

We are now ready to prove Proposition 19. Indeed, by Lemmata 21 and 22, for each $\varepsilon > 0$, the set $\{x : \mathrm{osc}(\varphi)(x) < \varepsilon\}$ is open and invariant under the Γ-action on X. Since this action is assumed to be minimal, such a set is either empty or coincides with the whole space X. If we are able to detect a point where φ is strongly continuous, then by Lemma 20 we will have a point in each of these sets. Hence, each of these sets will coincide with X, so that the oscillation of φ at every point will be zero. By Lemma 20 again, this will imply that φ is strongly continuous.

Thus, to conclude the proof, we need to ensure the existence of a point of strong continuity for φ. This follows from the following well-known

LEMMA 23. *Every weakly-continuous function $\varphi \colon X \to \mathscr{H}$ is strongly continuous on a G_δ-set.*

Proof. Let $\mathscr{H}_1 \subset \mathscr{H}_2 \subset \ldots$ be a sequence of finite-dimensional subspaces whose union $\bigcup_n \mathscr{H}_n$ is dense in $\mathscr{H}$. Since φ is weakly continuous, each of the functions $x \mapsto \|\pi_n(\varphi(x))\|$ is continuous, where $\pi_n \colon \mathscr{H} \to \mathscr{H}_n$ denotes the corresponding orthogonal projection. By a classical theorem of R. Baire [4] (see also [27]), the pointwise limit of these functions is continuous on a G_δ-set X_φ. But this pointwise limit is the function $x \mapsto \|\varphi(x)\|$. Recalling now that, in any Hilbert space, weak convergence plus convergence of the norm imply strong convergence, this yields the strong continuity of φ on X_φ. □

APPENDIX A. MEASURABLE VERSUS CONTINUOUS SOLUTIONS

We next give a rigidity result for measurable solutions of the cohomological equation (8) that corresponds to a dynamical version/extension of the Corollary to Theorem C. Given a probability measure μ on X that is quasi-invariant under the Γ-action, we will say that the linear part of a skew action on $X \times \mathscr{H}$ is *weakly ergodic* if the only measurable functions $\phi \colon X \to \mathscr{H}$ such that $\Psi(f,x)\phi(x) = \phi(f(x))$ for all $f \in \Gamma$ and μ-a.e. $x \in X$ are the constant ones.

EXAMPLE 24. If $\Psi(f,x) = \mathrm{Id}$ for all (f,x), then the linear part is weakly ergodic if and only if the Γ-action on X is ergodic with respect to μ.

EXAMPLE 25. In Example 5, assume that T is the rotation of angle $\alpha \notin \mathbb{Q}$ on the circle (endowed with the Lebesgue measure). If $\phi \colon \mathrm{S}^1 \to \mathbb{C}$ satisfies $\phi(\theta + \alpha) = e^{i\beta}\phi(\theta)$ for a.e. $\theta \in \mathrm{S}^1$, then ϕ must be constant unless α and β are rationally dependent. (See the final argument in Example 6.) We thus conclude that the linear part of the skew action is weakly ergodic provided α and β are independent over the rationals.

EXAMPLE 26. As in Example 17, assume that $\Gamma \sim \mathbb{Z}$ acts by powers of a minimal homeomorphism T and that the linear part of its skew action on an infinite-dimensional Hilbert space $\mathscr{H}$ is generated by

$$\Psi(1,x)(v_n) = \Psi(v_n) = v_{n+1},$$

where $\{v_n\}$ is an orthonormal basis of $\mathscr{H}$ and $x \in X$ is arbitrary. We claim that the weak ergodicity holds for any T-invariant probability measure μ. Indeed, let $\phi(x) = \sum_{n\in\mathbb{Z}} \phi_n(x) v_n$ be a measurable function from X to $\mathscr{H}$ such that $\Psi(\phi(x)) = \phi(T(x))$, for all $x \in X$. Then we have $\phi_{n+1}(T(x)) = \phi_n(x)$, for all $x \in X$. If ϕ is not μ-a.e. equal to zero, then for some $j \in \mathbb{Z}$ and $\delta > 0$ we have $\mu(C_{j,\delta}) > 0$, where $C_{j,\delta} := \{x \in X : |\phi_j(x)| \geq \delta\}$. By the Poincaré Recurrence Theorem, for μ-a.e. point $x \in C_{j,\delta}$ there exists an increasing infinite sequence (n_i) such that $T^{-n_i}(x) \in C_{j,\delta}$, hence $|\phi_{j+n_i}(x)| = |\phi_j(T^{-n_i}(x))| \geq \delta$. However, this is impossible, as $\phi(x)$ belongs to $\mathscr{H}$ for μ-a.e. $x \in X$.

EXAMPLE 27. Let Γ be a countable group provided with a spread-out, nondegenerate probability distribution p, and let $X := P(\Gamma, p)$ be the associate Poisson boundary endowed with the corresponding stationary measure μ. As a direct consequence of Kaimanovich's double ergodicity theorem [13], the linear part of every skew action by isometries of a Hilbert space over the natural action of Γ on X is weakly ergodic (we assume that X is metrizable and compact to fit in our general framework).

LEMMA 28. *Given a skew action on $X \times \mathscr{H}$ whose linear part is weakly ergodic with respect to μ, for any two skew-invariant* measurable *sections $x \mapsto (x, \varphi(x))$ and $x \mapsto (x, \overline{\varphi}(x))$, the difference $\varphi - \overline{\varphi}$ is a μ-a.e. constant vector. If, moreover, there is no common nonzero eigenvector for all the $\Psi(f, x)$, then there is at most one skew-invariant measurable solution of* (8).

Proof. For all $f \in \Gamma$, one has μ-a.e.

$$\varphi\big(f(x)\big) - \Psi(f, x)\varphi(x) = \rho(x) = \overline{\varphi}\big(f(x)\big) - \Psi(f, x)\overline{\varphi}(x),$$

hence,

$$\begin{aligned}\varphi\big(f(x)\big) - \overline{\varphi}\big(f(x)\big) &= \big[\Psi(f, x)\varphi(x) + \rho(x)\big] - \big[\Psi(f, x)\overline{\varphi}(x) + \rho(x)\big] \\ &= \Psi(f, x)\big(\varphi(x) - \overline{\varphi}(x)\big).\end{aligned}$$

Since the linear part of the skew action is assumed to be weakly ergodic, this implies that $\varphi - \overline{\varphi}$ is constant. In particular, $\varphi - \overline{\varphi}$ is a common eigenvector of all the $\Psi(f, x)$. □

PROPOSITION 29. *Given a skew action on $X \times \mathscr{H}$, assume that the cohomological equation* (8) *admits a solution $\varphi \in \mathscr{L}^\infty_\mu(X, \mathscr{H})$, where μ is such that the linear part is weakly ergodic. If the underlying semigroup Γ admits a topology with a countable, dense subset such that the skew action is continuous, then φ is continuous.*

Proof. For each f lying in a countable, dense subset Γ_0 of Γ, let $Y_f := \big\{x \in X : I(f, x)\varphi(x) \neq \varphi\big(f(x)\big)\big\}$. Then Y_f has null μ-measure, as well as $Y := \bigcup_{f\in\Gamma_0} Y_f$. Let C be the essential supremum of the function $x \mapsto \|\varphi(x)\|$, and let Z_0 be the preimage of $]C, \infty[$ under this function. Then Z_0 has null μ-measure, as well as $Z := \bigcup_{f\in\Gamma_0} f^{-1}(Z_0)$. Now, for each x_0 in the μ-full measure set $X \setminus (Y \cup Z)$ and all $f \in \Gamma_0$, we have

$$I(f, x_0)\varphi(x_0) = \varphi\big(f(x_0)\big) \quad \text{and} \quad \big\|\varphi\big(f(x_0)\big)\big\| \leq C.$$

Since Γ_0 is dense in Γ, this actually holds for all $f \in \Gamma$, by continuity. In other words, the Γ-orbit of the point $(x_0, v_0) := (x_0, \varphi(x_0))$ remains in a bounded subset of $\mathcal{H}$. The proposition then follows from the Main Theorem combined with Lemma 28. □

APPENDIX B. COCYCLES WHOSE LINEAR PART IS A SHIFT

Given a minimal homeomorphism $T\colon X \to X$, we consider the cocycle of isometries of a Hilbert space $\mathcal{H}$ induced by $I(1,x)v = \Psi(x)v + \rho(x)$, where $\rho\colon X \to \mathcal{H}$ and $\Psi\colon X \to O(\mathcal{H})$ are continuous. To simplify the notation, we write Tx instead of $T(x)$. For each $x \in X$ and $k \in \mathbb{N}$, define the k^{th} twisted Birkhoff-sum of the cocycle ρ as

$$S_k(\rho)(x) := \sum_{i=0}^{k-1} \Psi(T^{k-1}x)\Psi(T^{k-2}x)\cdots\Psi(T^{i+1}x)\rho(T^i x).$$

As is easy to check, the k^{th} iterate of $(x, v) \in X \times \mathcal{H}$ under the skew map $(x, v) \mapsto (Tx, \Psi(x)v + \rho(x))$ coincides with

$$\big(T^k x, I(k,x)v\big) = \big(T^k x, \Psi(T^{k-1}x)\cdots\Psi(Tx)v + S_k(\rho)(x)\big).$$

Assume that there is a bounded orbit for this map, hence a continuous solution φ to the cohomological equation

$$\varphi(Tx) = \Psi(x)\varphi(x) + \rho(x). \tag{19}$$

Then

$$\begin{aligned}
S_k(\rho)(x) &= \sum_{i=0}^{k-1} \Psi(T^{k-1}x)\cdots\Psi(T^{i+1}x)\rho(T^i x)\\
&= \sum_{i=0}^{k-1} \Psi(T^{k-1}x)\cdots\Psi(T^{i+1}x)\big[\varphi(T^{i+1}x) - \Psi(T^i x)\varphi(T^i x)\big]\\
&= \sum_{i=0}^{k-1} \Psi(T^{k-1}x)\cdots\Psi(T^{i+1}x)\varphi(T^{i+1}x) - \sum_{i=0}^{k-1} \Psi(T^{k-1}x)\cdots\Psi(T^i x)\varphi(T^i x)\\
&= \varphi(T^k x) - \Psi(T^{k-1}x)\cdots\Psi(x)\varphi(x).
\end{aligned}$$

This implies that, as expected, the sequence of functions $S_k(\rho)$ is uniformly bounded, hence the orbit of every (x, v) is bounded. Note that if Ψ is constant, then the preceding relation becomes

$$S_k(\rho)(x) = \varphi(T^k x) - \Psi^{k-1}\varphi(x).$$

Let us concentrate on the particular case where $\mathcal{H} = \ell^2(\mathbb{Z})$ and Ψ is the (bilateral) shift on the canonical basis $\{v_n\}$ of $\mathcal{H}$ (see Example 17). Assuming that $\varphi\colon X \to \mathcal{H}$ solves (19), fix $x_0 \in X$ and set $x := T^{-k}x_0$ and $v := \varphi(T^{-k}x)$. Then

$$\varphi(x_0) = I(k, T^{-k}x_0)\varphi(T^{-k}x_0).$$

Taking the inner product against v_n, we get

$$\begin{aligned}\langle \varphi(x_0), v_n \rangle &= \langle \Psi^k \varphi(T^{-k}x_0), v_n \rangle + \Big\langle \sum_{j=0}^{k-1} \Psi^j \rho(T^{-(j+1)}x_0), v_n \Big\rangle \\ &= \langle \Psi^k \varphi(T^{-k}x_0), v_n \rangle + \sum_{j=0}^{k-1} \rho_{n-j}(T^{-(j+1)}x_0).\end{aligned}$$

Since φ is strongly continuous, the set $\Psi^k(\varphi(X))$ weakly converges to $\{0\}$ in the Hausdorff sense. Consequently, the first term above, namely $\langle \Psi^k \varphi(T^{-k}x_0), v_n\rangle$, converges to zero as n goes to infinity. Therefore, φ has the following form:

$$\varphi(x) = \sum_{n\in\mathbb{Z}} \left(\sum_{j=0}^{\infty} \rho_{n-j}\big(T^{-(j+1)}x\big) \right) v_n. \tag{20}$$

A closely related but slightly different case is that of a positive shift, that is, when $T\colon X \to X$ is a homeomorphism all of whose forward orbits are dense, and Ψ is constant and coincides with the shift on the canonical basis $\{v_n\}$ of $\mathcal{H} \sim \ell^2(\mathbb{N}_0)$. (Note that Ψ is not surjective). Indeed, among all (not necessarily continuous) sections $\varphi\colon X \to \mathcal{H}$, there is a unique solution to (19), and its expression is given by

$$\varphi(x) := \sum_{j=0}^{\infty} \left[\sum_{r=0}^{j} \rho_{j-r}\big(T^{-(r+1)}x\big) \right] v_j. \tag{21}$$

To see that $\varphi(x)$ belongs to $\mathcal{H}$, we first claim that for all $y \in X$ and all $n \in \mathbb{N}$, we have

$$\big\| I(n,y)0 \big\| \leq 2C. \tag{22}$$

Indeed, letting M be the closure of the (forward) orbit of $(x_0, 0)$ (where the topology on $\mathcal{H}$ is the weak one), we have $\|I(n,y)(v)\| \leq C$ for all $(y,v) \in M$ and all $n \in \mathbb{N}$. Since the forward orbits of T are dense, each fiber M_y is nonempty, hence we may take $v = v(y) \in M_y$. Using the triangle inequality and the fact that $I(n,y)$ is an isometry, we get

$$\|I(n,y)0\| \leq \|v\| + \|I(n,y)v\| \leq 2C.$$

Now, a simple computation yields

$$I(n,y)0 = \sum_{r=0}^{n-1} \Psi^r \rho\big(T^{-(r+1)}(T^n y)\big)$$

for all $n \in \mathbb{N}$ and all $y \in X$. Using (22), we obtain

$$\sum_{j=0}^{\infty} \left| \sum_{r=0}^{n-1} \rho_{j-r}\big(T^{-(r+1)}(T^n y)\big) \right|^2 \leq 4C^2, \tag{23}$$

where we let $\rho_k \equiv 0$ for $k < 0$. Given $N \in \mathbb{N}$, we choose $n > N$ and $y = T^{-n}x$ in (23), and we obtain

$$\sum_{j=0}^{N} \left| \sum_{r=0}^{j} \rho_{j-r}\left(T^{-(r+1)}x\right) \right|^2 \leq 4C^2.$$

Since this holds for all $N \in \mathbb{N}$, we finally get $\|\varphi(x)\| \leq 2C$. In particular, $\varphi(x)$ belongs to $\mathscr{H}$.

To see that φ is skew invariant, we just compute:

$$\begin{aligned}
I(1,x)\varphi(x) &= \Psi\varphi(x) + \rho(x) \\
&= \Psi\left(\sum_{j=0}^{\infty}\left[\sum_{r=0}^{j}\rho_{j-r}\left(T^{-(r+1)}x\right)\right]v_j\right) + \sum_{j=0}^{\infty}\rho_j(x)v_j \\
&= \sum_{j=0}^{\infty}\left[\sum_{r=0}^{j}\rho_{j-r}\left(T^{-(r+1)}x\right)\right]v_{j+1} + \sum_{j=0}^{\infty}\rho_j(x)v_j \\
&= \sum_{j=1}^{\infty}\left[\sum_{r=0}^{j-1}\rho_{j-1-r}\left(T^{-(r+1)}x\right)\right]v_j + \sum_{j=0}^{\infty}\rho_j(x)v_j \\
&= \sum_{j=1}^{\infty}\left[\sum_{r=0}^{j-1}\rho_{j-1-r}\left(T^{-(r+1)}x\right)\right]v_j + \sum_{j=0}^{\infty}\rho_j(x)v_j \\
&= \sum_{j=1}^{\infty}\left[\sum_{r=1}^{j}\rho_{j-r}\left(T^{-r}x\right)\right]v_j + \sum_{j=0}^{\infty}\rho_j(x)v_j \\
&= \sum_{j=0}^{\infty}\left[\sum_{r=0}^{j}\rho_{j-r}\left(T^{-(r+1)}(Tx)\right)\right]v_j \\
&= \varphi(Tx).
\end{aligned}$$

To see that φ is the unique skew-invariant function, we consider another such a function $\varphi^*\colon X \to \mathscr{H}$. For all $x \in X$ we have

$$\varphi^*(Tx) - \varphi(Tx) = I(1,x)\varphi^*(x) - I(1,x)\varphi(x) = \Psi\varphi^*(x) - \Psi\varphi(x),$$

that is, $(\varphi^* - \varphi)(Tx) = \Psi(\varphi^* - \varphi)(x)$. Defining $\phi_j\colon X \to \mathbb{R}$ by letting

$$\sum_{j=0}^{\infty}\phi_j(x)v_j := \varphi(x) - \varphi^*(x),$$

this yields $\phi_j(x) = \phi_{j-n}(T^{-n}x)$ for all $n \in \mathbb{N}$. For $n > j$, this implies that $\phi_j(x) = 0$, hence $\varphi^* = \varphi$.

Finally, since we know that there exists a continuous skew-invariant section, the expression (21) defines a continuous function.

REMARK 30. Since we know that the map $x \mapsto \mathrm{ctr}(M_x)$ is skew invariant for any skew-invariant bounded subset $M \subset X \times \mathscr{H}$ whose projection on the first coordinate is onto, the vector $\varphi(x)$ above must coincide with $\mathrm{ctr}(M_x)$ for all $x \in X$.

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