

## SYMMETRIC RANDOM WALKS ON $\text{Homeo}^+(\mathbb{R})$

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*Dedicated to John Milnor on his 80th anniversary*

We study symmetric random walks on finitely generated groups of orientation-preserving homeomorphisms of the real line. We establish an oscillation property for the induced Markov chain on the line that implies a weak form of recurrence. Except for a few special cases, which can be treated separately, we prove a property of “global stability at a finite distance”: roughly speaking, there exists a compact interval such that any two trajectories get closer and closer whenever one of them returns to the compact interval. The probabilistic techniques employed here lead to interesting results for the study of group actions on the line. For instance, we show that under a suitable change of the coordinates, the drift of every point becomes zero provided that the action is minimal. As a byproduct, we recover the fact that every finitely generated group of homeomorphisms of the real line is topologically conjugate to a group of (globally) Lipschitz homeomorphisms. Moreover, we show that such a conjugacy may be chosen in such a way that the displacement of each element is uniformly bounded.

**1. Introduction.** In this article, we study symmetric random walks on finitely generated groups of (orientation-preserving) homeomorphisms of the real line. The results presented here fit into the general framework of systems of iterated random functions [8]. However, besides the lack of compactness of the phase space, there is a crucial point that separates our approach from the classical ones—the complete absence of any hypothesis of contraction. So to carry out our study, we need to use some extra structure, and this is provided by the natural ordering of the real line. In this direction, the results herein are also closely related to [14], where general Markov processes on ordered spaces are examined. However, since we only consider symmetric measures, there is no zero drift condition required for our

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processes, unlike [14] where this is a crucial assumption. In fact, one of our main results is that when the action is minimal, an appropriate change of coordinates on the real line makes the drift of *every point* equal to zero. This follows from a reparametrization that utilizes the stationary measure and a straightforward argument that employs the one-dimensional structure of the phase space in a decisive manner.

It is quite remarkable that, in presence of the linear order structure, we recover several phenomena that in more complex phase spaces are very specific to particular classes of groups. For instance, in [1], Babillot, Bougerol and Elie consider random walks on the group of affine homeomorphisms of  $\mathbf{R}^n$  in the difficult case where the logarithm of the expansion rate vanishes in mean. In this situation, they show the existence of an infinite Radon measure that is invariant by the transition operator, and for the case where the Lebesgue measure is not totally invariant, they establish a property of “global stability at a finite distance”: any two trajectories get closer and closer whenever one of them returns to a fixed compact set. Using this property, they obtain the uniqueness of the stationary measure (up to a constant multiple). It turns out that three of our main results here are analogues of these facts for groups of homeomorphisms of the real line. Furthermore, these results are also analogous to—though much more elaborate than—the previously established results for groups of circle homeomorphisms; see, for instance, [5], Section 5.1. As in the case of [5], the proofs here involve a prior study of the general structure of the associated dynamics, which is the core of this paper.

The motivation for studying groups of homeomorphisms of the real line comes from many sources. Algebraically, these groups are characterized by the existence of a left-invariant total order [6], which fits into well developed and quite formal theories [13]. More recently, many results about groups acting on the real line or the circle have focused on the relation with “rigidity theory,” a kind of non-linear version of representation theory where one seeks to understand the nature of the obstructions to the existence of (faithful) group actions on specific phase spaces (see [10] for a survey of these ideas). In this direction, it is conjectured that some particular groups, like groups with Kazhdan’s property (T) or lattices in higher-rank simple Lie groups, do not act on the real line (or equivalently, are not left-orderable). We strongly believe that our probabilistic approach opens new and promising avenues of study that bear the potential to yield important results in the investigation of these and many other open questions concerning left-orderable groups.

**2. A description of the results.** Given a symmetric probability measure  $\mu$ , we consider the group  $G$  generated by its support. Although some of our results apply to the case where this support is countably infinite, we restrict our discussion to the case where it is finite. Assume throughout that the action of  $G$  is *irreducible*; that is, there are no global fixed points. Otherwise, one may consider the action the connected components of the complement of the set of these global fixed points

(on each of these components, the action is irreducible). We then consider the random walk induced by  $\mu$  as a Markov process on the real line. In Section 4, the recurrence properties of this process are studied. We first prove that almost every trajectory oscillates between  $-\infty$  and  $+\infty$  (Proposition 4.2). Moreover, there exists a compact interval  $K$  such that these trajectories pass through  $K$  infinitely many times (Theorem 4.3). Using standard arguments à la Chacon–Orstein, this allows us to show the existence of a stationary Radon measure on the line (Theorem 5.1).

In Section 6, general properties of the stationary measures are examined. If the atomic part of the stationary measure is nontrivial, then it is supported on the union of discrete orbits and is totally invariant (Lemmas 6.2 and 6.3). If there are no discrete orbits, there exists a unique minimal nonempty closed invariant set  $\mathcal{M}$  that is the support of any stationary measure (Proposition 6.1 and Lemma 6.4). Furthermore, the stationary measure is unique up to a constant factor (Theorem 6.5). This result follows from an argument due to S. Brofferio in [2] and a nondivergence property for the trajectories of the Markov process established in Lemma 6.6.

In Section 7, we obtain the property of global stability at a finite distance provided that no invariant Radon measure exists and  $G$  is not centralized by any homeomorphism without fixed points. Roughly speaking, this last condition means that the action does not appear as the lift of an action on the circle. If this is not the case, a weak form of the contraction property is established (all of this is summarized in Theorem 7.2).

In Section 8, we provide a connection to the beautiful work [7], where Derriennic studies Markov processes on the real line satisfying  $\mathbb{E}(X_1^x) = x$  for large values of  $|x|$ . For every finitely generated group of homeomorphisms of the real line acting minimally, we produce a coordinate change for which the Derriennic property  $[\mathbb{E}(X_1^x) = x]$  holds for every  $x \in \mathbf{R}$  (Theorem 8.1). This is done by appropriately integrating the associated stationary measure. A careful analysis of the invariant Radon measure is carried out before establishing this result. In particular, we prove that the measure is infinite on every unbounded interval (Lemma 4.1). As a consequence of the existence of these Derriennic coordinates, we recover a rather surprising fact: every finitely generated group of homeomorphisms of the real line is topologically conjugate to a group of Lipschitz homeomorphisms (Theorem 8.5). (This result also follows from the (probabilistic) techniques introduced in [5].) Moreover, we show that such a conjugacy may be taken so that the displacement function  $x \mapsto g(x) - x$  becomes bounded uniformly in  $x$  for all  $g \in G$ .

**3. Notation.** Let  $\{g_n\}$  be a sequence of i.i.d.  $\text{Homeo}^+(\mathbf{R})$ -valued random variables, whose distribution is a *symmetric* measure  $\mu$ . The left random walk on  $\text{Homeo}^+(\mathbf{R})$  is defined by the random variables

$$f_n := g_n \circ \cdots \circ g_1.$$

More precisely, let  $G$  be the group generated by the support of  $\mu$  and consider the probability space  $\Omega := (G^{\mathbf{N}}, \mu^{\otimes \mathbf{N}})$ . Then  $g_n$  is defined to be the  $n$ th coordinate of  $\omega \in \Omega$ . The group  $G$  is assumed to be countable, and in fact,  $G$  will be finitely generated in most cases.

We introduce the Markov chain  $X$  on the real line, that is, for any  $x \in \mathbf{R}$  and any nonnegative integer  $n$ ,

$$X_n^x := f_n(x).$$

Let  $C_b(\mathbf{R})$  and  $C_c(\mathbf{R})$  denote the spaces of continuous bounded functions and compactly supported continuous functions, respectively. Let  $P : C_b(\mathbf{R}) \rightarrow C_b(\mathbf{R})$  be the transition operator, defined as usual by

$$P(\varphi)(x) := \mathbb{E}(\varphi(X_1^x)),$$

where  $\varphi \in C_b(\mathbf{R})$  and  $x \in \mathbf{R}$ . The operator  $P$  acts by duality on the set of finite measures on the real line, and if  $\mu$  is finitely supported,  $P$  preserves  $C_c(\mathbf{R})$  and hence acts by the duality on the set of Radon measures.

It should come as no surprise that in the investigation of a group action on the real line, one is led to study the action on the components of  $\mathbf{R} \setminus \text{Fix}(G)$ . On any such component, no global fixed point exists, and so, we may (and we always will) assume that the action is *irreducible*, that is, for every  $x \in \mathbf{R}$  there exists a  $g \in G$  such that  $g(x) \neq x$ . If  $\mu$  is a symmetric probability measure on  $\text{Homeo}^+(\mathbf{R})$ , then we will say that  $\mu$  is irreducible if the group generated by its support satisfies this property.

**4. Recurrence.** In this section, we establish the recurrence of the Markov chain  $X$  when the measure  $\mu$  is irreducible, symmetric, and has finite support. We begin with a lemma that extends [5], Proposition 5.7 (see also [9]) and that will be crucial in Sections 6.1 and 8. The proof is based on the second proof proposed in [5]; for a proof based on the first proof therein, see [6].

LEMMA 4.1. *Let  $\mu$  be an irreducible, symmetric probability measure on the group  $\text{Homeo}^+(\mathbf{R})$ . Then any nonvanishing  $P$ -invariant Radon measure  $\nu$  on the real line is bi-infinite [i.e.,  $\nu(x, \infty) = \infty$  and  $\nu(-\infty, x) = \infty$ , for all  $x \in \mathbf{R}$ ].*

PROOF. Suppose that there exists an  $x \in \mathbf{R}$  such that  $\nu(x, \infty) < \infty$ . Since the action is irreducible, for every  $y \in \mathbf{R}$ , there is an element  $g \in G$  such that  $g(x) < y$ . Select  $n > 0$  such that  $\mu^{*n}(g^{-1}) > 0$  and then observe that

$$\nu(y, \infty) \leq \nu(g(x), \infty) \leq \frac{1}{\mu^{*n}(g^{-1})} \nu(x, \infty) < \infty.$$

This argument implies that  $\nu(y, \infty) < \infty$  for all  $y \in \mathbf{R}$ .

Now let  $f : \mathbf{R} \rightarrow (0, \infty)$  be the function defined by  $f(x) := \nu(x, \infty)$ . Since  $\mu$  is symmetric, this function is harmonic on the orbits. Fix a real number  $A$  satisfying  $0 < A < \nu(-\infty, \infty)$ , and then let  $h := \max(0, A - f)$ . The function  $h$  is subharmonic, that is,  $h \leq P(h)$ . Moreover, it vanishes on a neighborhood of  $-\infty$  and is bounded on a neighborhood of  $\infty$ . This implies that  $h$  is  $\nu$ -integrable, and since  $\int Ph d\nu = \int h d\nu$ , the function  $h$  must be  $P$ -invariant  $\nu$ -a.e. Now a classical lemma in [11] asserts that a measurable function which is in  $L^1(\mathbf{R}, \nu)$  and  $P$ -invariant must be  $G$ -invariant almost everywhere. Thus,  $h$  is constant on almost every orbit. However, this is impossible since every orbit intersects every neighborhood of  $-\infty$  (where  $h$  vanishes) and of  $\infty$  (where  $h$  is positive). This contradiction establishes the desired result.  $\square$

PROPOSITION 4.2 (Oscillation). *Let  $\mu$  be an irreducible, symmetric probability measure on  $\text{Homeo}^+(\mathbf{R})$ . Then for every  $x \in \mathbf{R}$ , almost surely we have*

$$\limsup_{n \rightarrow \infty} X_n^x = +\infty \quad \text{and} \quad \liminf_{n \rightarrow \infty} X_n^x = -\infty.$$

PROOF. Given points  $A$  and  $x$  on the real line, let

$$p_A(x) := \mathbb{P}\left\{\limsup_{n \rightarrow \infty} X_n^x > A\right\}.$$

Since  $G$  acts by orientation-preserving homeomorphisms, for each  $x \leq y$  we have

$$\left\{(g_n)_n \in G^{\mathbf{N}} \mid \limsup_{n \rightarrow \infty} X_n^x > A\right\} \subset \left\{(g_n)_n \in G^{\mathbf{N}} \mid \limsup_{n \rightarrow \infty} X_n^y > A\right\}.$$

In particular,  $p_A(x) \leq p_A(y)$ , that is,  $p_A$  is nondecreasing. Moreover, since  $p_A$  is the probability of the tail event

$$\left\{\limsup_{n \rightarrow \infty} X_n^x > A\right\},$$

and  $X$  is a Markov chain,  $p_A$  is harmonic: for every  $x \in \mathbf{R}$  and every integer  $n \geq 0$ ,

$$p_A(x) = \mathbb{E}(p_A(X_n^x)).$$

We would like to think of  $p_A$  as the distribution function of a finite measure on  $\mathbf{R}$ . Since this is possible only if  $p_A$  is continuous on the right, we are led to consider the right-continuous function

$$\bar{p}_A(x) := \lim_{y \rightarrow x, y > x} p_A(y).$$

This function is still nondecreasing; hence there exists a finite measure  $\nu$  on  $\mathbf{R}$  such that for all  $x < y$ ,

$$\nu(x, y] = \bar{p}_A(y) - \bar{p}_A(x).$$

Since  $p_A$  is harmonic and  $G$  acts by homeomorphisms, the function  $\bar{p}_A$  must be harmonic. Thus  $\nu$  is also harmonic, and furthermore, the measure  $\nu$  is  $P$ -invariant

since  $\mu$  is symmetric. Now recall that Lemma 4.1 implies that any  $P$ -invariant finite measure vanishes identically (see also [5], Proposition 5.7), and therefore,  $\nu = 0$  and  $\bar{p}_A$  is constant. The 0–1 law can be applied here to conclude that (for any fixed  $A$ ) either  $p_A(x) \equiv 0$  or  $p_A(x) \equiv 1$ .

Let us now show that  $p_A$  equals to 1 for any  $A$ . Indeed, fix any  $x_0 > A$ . Then, for any  $g \in \text{Homeo}^+(\mathbf{R})$ , we have either  $g(x_0) \geq x_0$ , or  $g^{-1}(x_0) \geq x_0$ , and hence, due to the symmetry of measure  $\mu$ , for every  $n$  the inequality  $X_n^{x_0} \geq x_0$  holds with probability at least  $1/2$ . It is easy then to see that

$$p_A = p_A(x_0) \geq \limsup_{n \rightarrow \infty} \mathbb{P}\{X_n^{x_0} \geq x_0\} \geq 1/2.$$

As we have already shown that  $p_A$  is equal to 0 or to 1, this implies that  $p_A$  is identically equal to 1.

The latter means that for every  $x \in \mathbf{R}$ ,

$$\limsup_{n \rightarrow \infty} X_n^x = +\infty$$

holds almost surely. Analogously, for every  $x \in \mathbf{R}$ , almost surely we have

$$\liminf_{n \rightarrow \infty} X_n^x = -\infty.$$

This completes the proof of the proposition.  $\square$

We are now ready to prove the main result of this section.

**THEOREM 4.3 (Recurrence).** *Let  $\mu$  be an irreducible, finitely supported, symmetric probability measure on  $\text{Homeo}^+(\mathbf{R})$ . Then there exists a compact interval  $K$  such that, for every  $x$ , almost surely the sequence  $(X_n^x)$  intersects  $K$  infinitely often.*

**PROOF.** Consider an interval  $K = [A, B]$ , where  $A < B$  are points in the real line such that for every element  $g$  of the support of  $\mu$ , we have  $g(A) < B$ . By Proposition 4.2, for every  $x \in \mathbf{R}$ , almost surely the sequence  $(X_n^x)$  will pass from  $(-\infty, A]$  to  $[B, +\infty)$  infinitely often. Now the desired conclusion follows from the observation that the choices of  $A$  and  $B$  imply that every time this happens,  $\{X_n^x\}$  must traverse the interval  $K$ .  $\square$

**5. Existence of a stationary measure.** An important consequence of the previous result is the existence of a  $P$ -invariant Radon measure on the real line.

**THEOREM 5.1 (Existence of a  $P$ -invariant measure).** *Let  $\mu$  be an irreducible, finitely supported, symmetric probability measure on  $\text{Homeo}^+(\mathbf{R})$ , and let  $P$  be the associated transition operator. Then there exists a  $P$ -invariant Radon measure on the real line.*

PROOF. Fix a continuous, compactly-supported function  $\xi : \mathbf{R} \rightarrow [0, 1]$  such that  $\xi \equiv 1$  on  $K$ . For any initial point  $x$ , let us stop the process  $X_n^x$  at a random stopping time  $T = T(w)$  chosen in a Markovian way so that, for all  $n \in \mathbb{N}$ ,

$$\mathbb{P}(T = n + 1 \mid T \geq n) = \xi(X_{n+1}^x).$$

In other words, after each iteration of the initial random walk, when we arrive at some point  $y = X_{n+1}^x$  we stop with the probability  $\xi(y)$ , and we continue the iterations with probability  $1 - \xi(y)$ .

Denote by  $Y^x$  the random stopping point  $X_T^x$ , and consider its distribution  $p^x$  (notice that  $T$  is almost surely finite since the process  $X_n^x$  almost surely visits  $K$  and  $\xi \equiv 1$  on  $K$ ). Due to the continuity of  $\xi$ , the measure  $p^x$  on  $\mathbf{R}$  depends continuously (in the weak topology) on  $x$ . Therefore, the corresponding diffusion operator  $P_\xi$  defined by

$$P_\xi(\varphi)(x) = \mathbb{E}(\varphi(Y^x)) = \int_{\mathbf{R}} \varphi(y) dp^x(y)$$

acts on the space of continuous bounded functions on  $\mathbf{R}$ , and hence it acts by duality on the space of probability measures on  $\mathbf{R}$ . Notice that for any such probability measure, its image under  $P_\xi$  is supported on  $\tilde{K} := \text{supp}(\xi)$ . Thus, applying the Krylov–Bogolubov procedure of time averaging (and extracting a convergent subsequence), we see that there exists a  $P_\xi$ -invariant probability measure  $\nu_0$ .

To construct a Radon measure that is stationary for the initial process, we proceed as follows. For each point  $x \in \mathbb{R}$ , let us take the sum of the Dirac measures supported in its random trajectory before the stopping moment  $T$ . In other words, we consider the “random measure”  $m_x(\omega) := \sum_{j=0}^{T(w)-1} \delta_{X_j^x}$ . We then consider its expectation

$$m_x = \mathbb{E} \left( \sum_{j=0}^{T(w)-1} \delta_{X_j^x} \right)$$

as a measure on  $\mathbf{R}$ . Finally, we integrate  $m_x$  with respect to the measure  $\nu_0$  on  $x$ , thus yielding a Radon measure  $\nu := \int m_x d\nu_0(x)$  on  $\mathbf{R}$ . Formally speaking, for any compactly supported function  $f$ , we have

$$(1) \quad \int_{\mathbf{R}} f d\nu = \int_{\mathbf{R}} \mathbb{E} \left( \sum_{j=0}^{T(w)-1} f(X_j^x) \right) d\nu_0(x).$$

Notice that the right-hand side expression of (1) is well defined and finite. Indeed, there exist  $N \in \mathbb{N}$  and  $p_0 > 0$  such that with probability at least  $p_0$  a trajectory starting at any point of  $\text{supp}(f)$  hits  $K$  in at most  $N$  steps. Thus, the distribution of the measure  $m_x(w)$  on  $\text{supp}(f)$  [the number of steps that are spent in  $\text{supp}(f)$  until the stopping moment] has an exponentially decreasing tail. Thus, its expectation is

finite and bounded uniformly on  $x \in \text{supp}(f)$ , which implies the finiteness of the integral.

Now, let us check that the measure  $\nu$  is  $P$ -invariant. Let us first rewrite the measure  $\nu$ . To do this, notice that using the full probability formula, one can check that the measure  $m_x$  equals

$$\sum_{n \geq 0} \sum_{g_1, \dots, g_n \in G} \prod_{j=1}^n \mu(g_j) \cdot \prod_{j=1}^n [1 - \xi(g_j \circ \dots \circ g_1(x))] \cdot \delta_{g_n \circ \dots \circ g_1(x)}.$$

Thus

$$\begin{aligned} P(m_x) &= \sum_{g \in G} \mu(g) \cdot g_\star m_x \\ &= \sum_{g \in G} \mu(g) \cdot g_\star \left( \sum_{n \geq 0} \sum_{g_1, \dots, g_n \in G} \prod_{j=1}^n \mu(g_j) \cdot \prod_{j=1}^n [1 - \xi(g_j \circ \dots \circ g_1(x))] \right. \\ &\quad \left. \times \delta_{g_n \circ \dots \circ g_1(x)} \right) \\ &= \sum_{n \geq 0} \sum_{g_1, \dots, g_n, g \in G} \left( \mu(g) \cdot \prod_{j=1}^n \mu(g_j) \right) \cdot \prod_{j=1}^n [1 - \xi(g_j \circ \dots \circ g_1(x))] \\ &\quad \times g_\star \delta_{g_n \circ \dots \circ g_1(x)} \\ &= \sum_{n \geq 0} \sum_{g_1, \dots, g_n, g_{n+1} \in G} \left( \prod_{j=1}^{n+1} \mu(g_j) \right) \cdot \prod_{j=1}^{(n+1)-1} [1 - \xi(g_j \circ \dots \circ g_1(x))] \\ &\quad \times \delta_{g_{n+1} \circ g_n \circ \dots \circ g_1(x)}. \end{aligned}$$

In the same way as before, one can check that the last expression equals the expectation of the random measure  $\sum_{j=1}^{T(\omega)} \delta_{X_j^x}$ . In this sum, we are counting the stopping time, but not the initial one, and therefore

$$Pm_x = m_x - \delta_x + \mathbb{E}(\delta_{Y^x}).$$

Integration with respect to  $\nu_0$  yields

$$\begin{aligned} P\nu &= P\left(\int_{\mathbf{R}} m_x d\nu_0(x)\right) = \int_{\mathbf{R}} P(m_x) d\nu_0(x) \\ &= \int_{\mathbf{R}} m_x d\nu_0(x) - \int_{\mathbf{R}} \delta_x d\nu_0(x) + \int_{\mathbf{R}} \mathbb{E}(\delta_{Y^x}) d\nu_0(x) = \nu - \nu_0 + P_\xi \nu_0. \end{aligned}$$

Since  $\nu_0$  is  $P_\xi$ -invariant, we finally obtain  $P\nu = \nu$ , as we wanted to show.  $\square$



**6. Properties of  $P$ -invariant measures.** This section is devoted to the study of the properties of the  $P$ -invariant Radon measures constructed in Section 4. The following topological fact is well known; we recall its proof for completeness.

**PROPOSITION 6.1.** *Let  $G$  be a finitely generated, irreducible group of homeomorphisms of the real line. Then either  $G$  carries a discrete orbit or there is a unique minimal nonempty closed  $G$ -invariant set  $\mathcal{M}$ . In the latter case, the closure of every orbit contains  $\mathcal{M}$ .*

**PROOF.** Let  $K$  be a compact interval that intersects every orbit; the existence of such an interval follows from the proof of Theorem 4.3. Let  $\mathcal{E}$  be the family of nonempty compact subsets  $\mathcal{K}$  of  $K$  such that  $\mathcal{K} = (G\mathcal{K}) \cap K$ . If  $\{\mathcal{K}_\lambda, \lambda \in \Lambda\}$  is a chain (with respect to inclusion) in  $\mathcal{E}$ , then  $\mathcal{K}_\Lambda := \bigcap_\lambda \mathcal{K}_\lambda$  also belongs to  $\mathcal{E}$ . By Zorn's lemma,  $\mathcal{E}$  has a maximal element  $\mathcal{K}_0$ . Notice that  $\mathcal{M} := G\mathcal{K}_0$  is a nonempty minimal  $G$ -invariant closed subset of the real line. When  $\mathcal{M}$  is not a discrete set, every point of  $\mathcal{M}$  is an accumulation point, and  $\mathcal{M}$  is locally compact. Hence, there are only two possibilities when  $G$  has no discrete orbits—either  $\mathcal{M} = \mathbf{R}$  or  $\mathcal{M}$  is locally homeomorphic to a Cantor set. In the first case, the proposition is proved. In the second case, the orbit of every point of  $\mathcal{M}$  is dense in  $\mathcal{M}$ . We will now prove that the closure of orbits of points in  $\mathbf{R} \setminus \mathcal{M}$  contains  $\mathcal{M}$ ; this will establish the uniqueness of the set  $\mathcal{M}$ . Let  $C$  be an arbitrary connected component of  $\mathbf{R} \setminus \mathcal{M}$ . Then  $C$  is bounded and its right endpoint  $r$  belongs to  $\mathcal{M}$ . Therefore, there is a sequence of elements  $g_n \in G$  such that  $g_n(r)$  tends to  $r$  as  $n$  tends to infinity and  $g_n(r) \neq r$  for every  $n$  (otherwise, the set of accumulation points of the orbit of  $r$  would be a closed  $G$ -invariant set strictly contained in  $\mathcal{M}$ ). Now  $g_n(C)$  tends uniformly to  $r$  as  $n$  tends to infinity. Since the closure of the orbit of  $r$  equals  $\mathcal{M}$ , this shows that the closure of the orbit of any point in  $C$  contains  $\mathcal{M}$ .  $\square$

If there is a discrete orbit, then the counting measure on it is a Radon measure that is  $G$ -invariant, and in particular, it is also  $P$ -invariant for any probability measure  $\mu$  on  $G$ . The next two lemmas provide converses to this fact.

**LEMMA 6.2.** *Let  $\mu$  be a symmetric probability measure on  $\text{Homeo}^+(\mathbf{R})$  whose support is finite and generates an irreducible group  $G$ . Let  $\nu$  be a  $P$ -invariant Radon measure on the real line. If there is a discrete orbit, then  $\nu$  is supported on the union of discrete orbits and is totally invariant.*

**PROOF.** If there is a discrete orbit  $O$ , it can be parametrized by  $\mathbf{Z}$  and then the action of  $G$  on  $O$  is by integer translations. In this situation, the normal subgroup  $G^1$  formed by the elements acting trivially on  $O$  is recurrent by Polya's theorem [18]. Let  $\mu^1$  be the (symmetric) measure on  $G^1$  obtained by balayage of  $\mu$  to  $G^1$ . Observe that the restriction of  $\nu$  to each component  $C$  of  $\mathbf{R} \setminus O$  is a finite

measure that is invariant for the Markov chain induced by  $\mu^1$  on  $C$ . It now follows from Lemma 4.1 (or from [5], Proposition 5.7) that this measure is supported on  $\text{Fix}(G^1) \cap \bar{C}$ , the set of global fixed points for the group  $G^1$  contained in the closure of  $C$ . As a consequence,  $G$  acts by “integer translations” on the support of  $\nu$ , which consists of discrete orbits. To see that  $\nu$  is invariant, notice that for each atom  $x \in \mathbf{R}$ , the function  $g \mapsto \nu(g(x))$  viewed as a function defined on  $G/G^1 \sim \mathbf{Z}$  is harmonic and positive and hence constant.  $\square$

LEMMA 6.3. *Let  $\mu$  be a symmetric probability measure on  $\text{Homeo}^+(\mathbf{R})$  whose support is finite and generates an irreducible group  $G$ . Let  $\nu$  be a  $P$ -invariant Radon measure on the real line. If the atomic part  $\nu_a$  of  $\nu$  is nontrivial, then it is supported on a union of discrete orbits.*

PROOF. Let  $x \in \mathbf{R}$  be a point such that  $\nu(x) > 0$ . Let  $O = G(x)$  be the orbit of  $x$  endowed with the discrete topology, and let  $\bar{\nu}$  be the measure on  $O$  defined by  $\bar{\nu}(y) := \nu(y)$ . Then  $\bar{\nu}$  is an invariant measure for the Markov process induced by  $\mu$  on  $O$ .

Let  $L$  be an arbitrary compact interval containing the compact interval  $K$  constructed in the proof of Theorem 4.3 and let  $R := L \cap O$ . We want to show that  $R$  is finite. To do this, first observe that  $R$  is a recurrent subset of  $O$ , by Theorem 4.3. Let  $Y$  be the Markov chain on  $R$  defined by the first return of  $X$  to  $R$ . This Markov chain is symmetric because  $X$  is symmetric. Moreover, the restriction of  $\bar{\nu}$  to  $R$  is invariant. Now since  $\sum_{y \in R} \bar{\nu}(y) < \infty$ , there must be an atom  $y \in R$  such that  $\bar{\nu}(y)$  is maximal. The  $P_Y$ -invariance of  $\bar{\nu}|_R$  and the symmetry of the transition probabilities  $p_Y(\cdot, \cdot)$  yield

$$\sum_{z \in R} p_Y(y, z) \bar{\nu}(z) = \sum_{z \in R} p_Y(z, y) \bar{\nu}(z) = \bar{\nu}(y).$$

The maximum principle now implies that  $\bar{\nu}(z) = \bar{\nu}(y)$  for all  $z \in O$ . Thus, all the atoms of  $\bar{\nu}$  contained in  $R$  have the same mass and hence there is only a finite number of them. In particular, this argument shows that  $O$  is discrete.  $\square$

Next we consider the case where  $G$  has no discrete orbits. As in Proposition 6.1, let  $\mathcal{M}$  be the unique nonempty minimal  $G$ -invariant closed subset of the real line.

LEMMA 6.4. *Let  $\mu$  be an irreducible, symmetric measure on  $\text{Homeo}^+(\mathbf{R})$  whose support is finite and generates a group  $G$  without discrete orbits on the real line. Then any  $P$ -invariant Radon measure is supported on  $\mathcal{M}$ .*

PROOF. Let  $\nu$  be a  $P$ -invariant Radon measure on the real line. The measure  $\nu$  is quasi-invariant by  $G$ , because for all  $h$  in the support of  $\mu$  we have

$$h_\star \nu \leq \frac{1}{\mu(h)} \sum_{g \in G} \mu(g) g_\star \nu = \frac{1}{\mu(h)} \nu.$$

So the support of  $\nu$  is a closed  $G$ -invariant subset of the real line, and hence, it contains  $\mathcal{M}$ . Therefore, it suffices to verify that  $\nu$  does not charge any component of  $\mathcal{M}^c$ . If  $\mathcal{M}^c$  is nonempty, we may collapse each connected component of  $\mathcal{M}^c$  to a point, thus obtaining a topological real line carrying a  $G$ -action for which every orbit is dense. The  $P$ -invariant measure  $\nu$  can be pushed to a  $P$ -invariant Radon measure  $\bar{\nu}$  for this new action. If a component of  $\mathcal{M}^c$  has a positive charge, then  $\bar{\nu}$  has atoms. By Lemma 6.3, this implies that the  $G$ -action cannot be minimal after the collapsing, which is a contradiction. We thus conclude that the original  $P$ -invariant measure  $\nu$  does not charge the components of  $\mathcal{M}^c$ , and so,  $\nu$  must be supported on  $\mathcal{M}$ .  $\square$

6.1. *Uniqueness of the  $P$ -invariant Radon measure.* When the action of  $G$  possesses discrete orbits, we know that every stationary Radon measure must be  $G$ -invariant; however, two such measures may be supported on different orbits. We now establish the uniqueness (up to a scalar factor) of the stationary measure in the case where  $G$  is a finitely generated, irreducible subgroup of  $\text{Homeo}^+(\mathbf{R})$  without discrete orbits. Recall that, in this case, there exists a unique minimal closed  $G$ -invariant set  $\mathcal{M}$ , and the orbit of every point in  $\mathcal{M}$  is dense in  $\mathcal{M}$ ; see Proposition 6.1.

**THEOREM 6.5.** *Let  $\mu$  be a symmetric measure on  $\text{Homeo}^+(\mathbf{R})$  whose support is finite and generates an irreducible group  $G$  without discrete orbits. Then the  $P$ -invariant Radon measure  $\nu$  is unique up to a scalar factor, and its support is  $\mathcal{M}$ . Moreover, for all continuous functions  $\varphi, \psi$  with compact support, with  $\varphi \geq 0$  and  $\int \varphi d\nu > 0$ , and for every  $x \in \mathbf{R}$ , we have a.s. the convergence*

$$(2) \quad \frac{\psi(X_1^x) + \dots + \psi(X_N^x)}{\varphi(X_1^x) + \dots + \varphi(X_N^x)} \rightarrow \frac{\int \psi d\nu}{\int \varphi d\nu}$$

as  $N$  tends to infinity.

For the proof of this theorem, we first consider the case when every  $G$ -orbit is dense. Let  $\nu$  be a  $P$ -invariant measure. We know that  $\nu$  is fully supported and has no atoms. By Lemma 4.1, we may consider the distance  $d$  on the real line defined by

$$d(x, y) := \nu[x, y], \quad x \leq y.$$

**LEMMA 6.6.** *For any fixed number  $0 < p < 1$  and all  $x, y$ , with probability at least  $p$  we have*

$$\lim_{n \rightarrow \infty} d(X_n^x, X_n^y) \leq \frac{d(x, y)}{1 - p}.$$

PROOF. Since  $\nu$  is  $P$ -invariant, the sequence of random variables  $\omega \mapsto d(X_n^x, X_n^y)$  is a martingale. In particular, for every integer  $n \geq 1$  we have

$$\mathbb{E}(d(X_n^x, X_n^y)) = d(x, y).$$

By the martingale convergence theorem, the sequence  $d(X_n^x, X_n^y)$  converges a.s. to a nonnegative random variable  $v(x, y)$ . By Fatou's inequality, for every  $x < y$  we have

$$\mathbb{E}(v(x, y)) \leq \liminf_{n \rightarrow \infty} \mathbb{E}(d(X_n^x, X_n^y)) = d(x, y).$$

The lemma then follows from Chebyshev's inequality.  $\square$

We will combine the preceding lemma with an argument from [2]. For this, recall that a  $P$ -stationary measure  $\nu$  is said to be *ergodic* if every  $G$ -invariant measurable subset either has measure 0 or its complement has measure 0.

LEMMA 6.7. *Assume that the hypotheses of Theorem 6.5 are satisfied and that every  $G$ -orbit is dense. If  $\nu$  is an ergodic  $P$ -invariant Radon measure, then the convergence (2) holds a.s. for every  $x \in \mathbf{R}$ .*

PROOF. The diffusion operator acting on  $L^1(\mathbf{R}, \nu)$  is a positive contraction. Moreover, because of the recurrence of the Markov process, this operator is conservative. We may hence apply the Chacon–Ornstein theorem [3], which together with the ergodicity of  $\nu$  shows that for  $\nu$ -almost every point  $x \in \mathbf{R}$  and all functions  $\varphi, \psi$  in  $C_c(\mathbf{R})$  such that  $\varphi \geq 0$  and  $\varphi = 1$  on the interval of recurrence  $K$  constructed in the proof of Theorem 4.3, we have almost surely

$$(3) \quad \lim_{n \rightarrow \infty} \frac{S_n \psi(x, \omega)}{S_n \varphi(x, \omega)} = \frac{\int \psi d\nu}{\int \varphi d\nu},$$

where  $S_n \psi(x, \omega) := \psi(X_1^x) + \dots + \psi(X_n^x)$  [and similarly for  $S_n \varphi(x, \omega)$ ]. Let  $y \in \mathbf{R}$  and the functions  $\varphi, \psi$  be fixed. We claim that, for any  $k \geq 1$ , with probability at least  $1 - 1/k$  we have

$$(4) \quad \limsup_{n \rightarrow \infty} \left| \frac{S_n \psi(y, \omega)}{S_n \varphi(y, \omega)} - \frac{\int \psi d\nu}{\int \varphi d\nu} \right| \leq \frac{1}{k}.$$

This obviously implies that (3) holds almost surely.

Since  $\nu$  has total support, one can find a point  $x$  generic in the sense of (4) and sufficiently close to  $y$  so that  $d(x, y) \leq \varepsilon$ . From Lemma 6.6, with probability at least  $1/2$  we have for all  $n$  sufficiently large, say  $n \geq n_0(\omega)$ ,

$$(5) \quad d(X_n^y, X_n^x) \leq (k + 1)\varepsilon.$$

Now, as we already know that (with probability 1)

$$\lim_{n \rightarrow \infty} \frac{S_n \psi(x, \omega)}{S_n \varphi(x, \omega)} = \frac{\int \psi d\nu}{\int \varphi d\nu},$$

instead of estimating the difference in (4), it suffices to obtain estimates of the “relative errors”

$$(6) \quad \limsup_{n \rightarrow \infty} \left| \frac{S_n \psi(y, \omega) - S_n \psi(x, \omega)}{S_n \varphi(x, \omega)} \right| \leq \delta_1(\varepsilon)$$

and

$$(7) \quad \limsup_{n \rightarrow \infty} \left| \frac{S_n \varphi(y, \omega) - S_n \varphi(x, \omega)}{S_n \varphi(x, \omega)} \right| \leq \delta_2(\varepsilon)$$

in such a way that  $\delta_1(\varepsilon) \rightarrow 0$  and  $\delta_2(\varepsilon) \rightarrow 0$  as  $\varepsilon \rightarrow 0$ .

Since the estimate (7) for  $\varphi$  is a particular case of the estimate (6), we will only check (6). Now, (5) implies that

$$\begin{aligned} & |S_n \psi(y, \omega) - S_n \psi(x, \omega)| \\ & \leq \text{mod}((k + 1)\varepsilon, \psi) \text{card}\{n_0(\omega) \leq j \leq n \mid \text{either } X_j^x \text{ or } X_j^y \text{ is in } \text{supp } \psi\} \\ & \quad + 2n_0(\omega) \max|\psi| \\ & \leq \text{mod}((k + 1)\varepsilon, \psi) \text{card}\{j \leq n \mid X_j^x \in U_{(k+1)\varepsilon}(\text{supp } \psi)\} + \text{const}(\omega). \end{aligned}$$

Here,  $\text{mod}(\cdot, \psi)$  stands for the modulus of continuity of  $\psi$  with respect to the distance  $d$  on the variable, and  $U_{(k+1)\varepsilon}(\text{supp } \psi)$  denotes the  $(k + 1)\varepsilon$ -neighborhood of the support of  $\psi$ , again with respect to  $d$ .

Let  $\chi$  be a continuous function satisfying  $0 \leq \chi \leq 1$  and that is equal to 1 on  $U_{(k+1)\varepsilon}(\text{supp } \psi)$  and to 0 outside  $U_{(k+2)\varepsilon}(\text{supp } \psi)$ . We have

$$\text{card}\{j \leq n \mid X_j^x \in U_{(k+1)\varepsilon}(\text{supp } \psi)\} \leq S_n \chi(x, \omega).$$

Thus

$$\begin{aligned} & \left| \frac{S_n \psi(y, \omega) - S_n \psi(x, \omega)}{S_n \varphi(x, \omega)} \right| \\ & \leq \frac{\text{const}(\omega) + \text{mod}((k + 1)\varepsilon, \psi) \cdot S_n \chi(x, \omega)}{S_n \varphi(x, \omega)} \\ & \xrightarrow{n \rightarrow \infty} \text{mod}((k + 1)\varepsilon, \psi) \cdot \frac{\int \chi \, d\nu}{\int \varphi \, d\nu} =: \delta_1(\varepsilon). \end{aligned}$$

[Notice here that we have applied the fact that, by our choice of  $x$ , the equality (3) holds for the functions  $\chi$  and  $\varphi$ .] Since  $\text{mod}((k + 1)\varepsilon, \psi)$  tends to 0 as  $\varepsilon \rightarrow 0$  and the quotient

$$\frac{\int \chi \, d\nu}{\int \varphi \, d\nu} \leq \frac{\nu(U_{(k+2)\varepsilon}(\text{supp } \varphi))}{\int \varphi \, d\nu}$$

remains bounded, this yields  $\delta_1(\varepsilon) \rightarrow 0$  as  $\varepsilon \rightarrow 0$ .  $\square$

It is now easy to finish the proof of Theorem 6.5 in the case where all the  $G$ -orbits are dense. Indeed, given any two ergodic  $P$ -invariant Radon measures  $\nu_1, \nu_2$ ,

for all  $x \in \mathbf{R}$  and all compactly supported, real-valued function  $\psi$ , we have almost surely

$$\frac{S_N \psi(x, \omega)}{S_N \varphi(x, \omega)} \rightarrow \frac{\int \psi \, d\nu_i}{\int \varphi \, d\nu_i},$$

where  $i = 1, 2$ . Thus,  $\int \psi \, d\nu_1 = \lambda \int \psi \, d\nu_2$ , with  $\lambda := \int \varphi \, d\nu_1 / \int \varphi \, d\nu_2$ . This proves that  $\nu_1 = \lambda \nu_2$ . The case of nonnecessarily ergodic  $\nu_1, \nu_2$  follows from standard ergodic decomposition type arguments.

The proof of Theorem 6.5 in the nonminimal case is more technical, because the argument of collapsing the connected components of the complement of the unique minimal invariant closed set  $\mathcal{M}$  is delicate. Indeed, although this procedure induces a minimal action from which the uniqueness of the stationary measure (up to a scalar factor) may be easily deduced, establishing (2) is much more complicated, mainly due to the fact that, after collapsing, the functions  $\psi, \varphi$  are no longer continuous. Below we propose two different solutions to this problem.

**FIRST PROOF OF THEOREM 6.5 IN THE NONMINIMAL CASE.** As before, the main point consists in obtaining a good estimate of the form (6). To do this, we fix  $\varepsilon_0 > 0$ , and we consider all the connected components of the complement of  $\mathcal{M}$  over which the oscillation of  $\psi$  is at least  $\varepsilon_0$ . Since  $\psi$  has compact support, there are only finitely many such components, say  $C_1, \dots, C_k$ . Given  $\varepsilon_1 > 0$ , let us consider a continuous function  $\chi_1$  satisfying  $0 \leq \chi_1 \leq 1$  and that is equal to 1 on each  $U_{\varepsilon_1}(C_i)$  and to 0 outside  $\bigcup_i U_{2\varepsilon_1}(C_i)$ . Now, take  $\varepsilon_2 > 0$  such that, if  $d(x, y) \leq 3\varepsilon_2$ , then either  $|\psi(x) - \psi(y)| \leq 2\varepsilon_0$  or  $x$  belongs to  $\bigcup_i U_{\varepsilon_1}(C_i)$ . (The existence of such an  $\varepsilon_2$  is easy to establish.) Finally, let  $\chi$  be a continuous function satisfying  $0 \leq \chi \leq 1$  and that is equal to 1 on the set

$$S_1 := \{x \mid \text{there is } y \in \text{supp } \psi \text{ such that } d(x, y) \leq 3\varepsilon_2\}$$

and to 0 outside  $\{x \mid d(x, y) \geq 4\varepsilon_2 \text{ for all } y \in \text{supp } \psi\}$ .

Notice that, although  $d$  is not a metric on the line, we still have that, if  $d(x, y) \leq \varepsilon$ , then with probability at least  $1 - 1/k$  there is  $n_0(\omega)$  such that, for all  $n \geq n_0(\omega)$ ,

$$d(X_n^x, X_n^y) \leq (k + 1)\varepsilon.$$

Fix  $\varepsilon \leq 3\varepsilon_2/(k + 1)$ . Given  $y \in \mathbf{R}$ , take a point  $x$  that is generic in the sense of (3) and such that  $d(x, y) \leq \varepsilon$ . With probability at least  $1 - 1/k$  we have

$$\begin{aligned} |S_n \psi(x, \omega) - S_n \psi(y, \omega)| &\leq \sum_{j=1}^n |\psi(X_j^x) - \psi(X_j^y)| \\ &\leq \text{const}(\omega) + 2 \max |\psi| \text{card} \left\{ j \leq n \mid X_j^x \in \bigcup_i U_{\varepsilon_1}(C_i) \right\} \\ &\quad + 2\varepsilon_0 \text{card} \{j \leq n \mid X_j^x \in S_1\} \\ &\leq \text{const}(\omega) + 2 \max |\psi| S_n \chi_1(x, \omega) + 2\varepsilon_0 S_n \chi(x, \omega). \end{aligned}$$

Dividing by  $S_n\varphi(x, \omega)$  and passing to the limit we obtain

$$\begin{aligned} \limsup_{n \rightarrow \infty} \frac{|S_n\psi(x, \omega) - S_n\psi(y, \omega)|}{S_n\varphi(x, \omega)} &\leq 2 \max |\psi| \frac{\int \chi_1 dv}{\int \varphi dv} + 2\varepsilon_0 \frac{\int \chi dv}{\int \varphi dv} \\ &\leq \frac{2 \max |\psi|}{\int \varphi dv} \sum_i \nu(U_{2\varepsilon_1}(C_i)) + 2\varepsilon_0 \frac{\int \chi dv}{\int \varphi dv}. \end{aligned}$$

To conclude, notice that the first term can be made arbitrarily small by taking  $\varepsilon_1$  very small, since the  $\nu$ -measure of the set  $\cup_i C_i$  is zero.  $\square$

SECOND PROOF OF THEOREM 6.5 IN THE NONMINIMAL CASE. To obtain an estimate of the form (6), we collapse the connected components of  $\mathcal{M}^c$ , thus obtaining a topological real line carrying a minimal  $G$ -action. However, after collapsing, the functions  $\psi$  and  $\varphi$  are no longer continuous. To solve this problem, we consider a nonnegative function  $\varphi_1 \in C_c(\mathbf{R})$  that is positive on the recurrence interval  $K$  and is constant on each connected component of  $\mathcal{M}^c$ . If we are able to estimate (6) but for  $\varphi_1$  instead of  $\varphi$  and for any function  $\psi$ , then we will have

$$\frac{S_n\psi(y, \omega)}{S_n\varphi(y, \omega)} = \frac{S_n\psi(y, \omega)/S_n\varphi_1(y, \omega)}{S_n\varphi(y, \omega)/S_n\varphi_1(y, \omega)} \rightarrow \frac{\int \psi dv / \int \varphi_1 dv}{\int \varphi dv / \int \varphi_1 dv} = \frac{\int \psi dv}{\int \varphi dv}$$

as we want to show.

Fix  $\varepsilon > 0$ . We leave to the reader the task of showing the existence of  $\psi_1, \chi_1$  in  $C_c(\mathbf{R})$  that are constant on each connected component of  $\mathcal{M}^c$  and satisfy:

- (i)  $|\psi - \psi_1| \leq \chi_1$ ;
- (ii)  $\int \chi_1 dv \leq \varepsilon$ .

Then using

$$\frac{S_n\psi(y, \omega)}{S_n\varphi_1(y, \omega)} = \frac{S_n\psi_1(y, \omega)}{S_n\varphi_1(y, \omega)} + \frac{S_n(\psi - \psi_1)(y, \omega)}{S_n\varphi_1(y, \omega)}$$

we obtain

$$\begin{aligned} &\left| \frac{S_n\psi(y, \omega)}{S_n\varphi_1(y, \omega)} - \frac{\int \psi dv}{\int \varphi_1 dv} \right| \\ &\leq \left| \frac{S_n\psi_1(y, \omega)}{S_n\varphi_1(y, \omega)} - \frac{\int \psi_1 dv}{\int \varphi_1 dv} \right| + \left| \frac{S_n(\psi - \psi_1)(y, \omega)}{S_n\varphi_1(y, \omega)} \right| + \left| \frac{\int (\psi_1 - \psi) dv}{\int \varphi_1 dv} \right| \\ &\leq \left| \frac{S_n\psi_1(y, \omega)}{S_n\varphi_1(y, \omega)} - \frac{\int \psi_1 dv}{\int \varphi_1 dv} \right| + \left| \frac{S_n\chi_1(y, \omega)}{S_n\varphi_1(y, \omega)} - \frac{\int \chi_1 dv}{\int \varphi_1 dv} \right| + 2 \left| \frac{\int \chi_1 dv}{\int \varphi_1 dv} \right|. \end{aligned}$$

As in the minimal case, we have

$$\left| \frac{S_n\psi_1(y, \omega)}{S_n\varphi_1(y, \omega)} - \frac{\int \psi_1 dv}{\int \varphi_1 dv} \right| \rightarrow 0, \quad \left| \frac{S_n\chi_1(y, \omega)}{S_n\varphi_1(y, \omega)} - \frac{\int \chi_1 dv}{\int \varphi_1 dv} \right| \rightarrow 0.$$

Since

$$\left| \frac{\int \chi_1 d\nu}{\int \varphi_1 d\nu} \right| \leq \frac{\varepsilon}{\int \varphi_1 d\nu},$$

this shows the desired convergence

$$\frac{S_n \psi(y, \omega)}{S_n \varphi_1(y, \omega)} \rightarrow \frac{\int \psi d\nu}{\int \varphi_1 d\nu},$$

thus finishing the proof.  $\square$

**7. Global stability at a finite distance.** We say that an irreducible subgroup  $G$  of  $\text{Homeo}^+(\mathbf{R})$  has the *strong contraction property* if there exists a compact interval  $L$  such that, for every compact interval  $I$ , there is a sequence of elements  $h_n$  of  $G$  such that  $h_n(I) \subset L$  for all  $n$ , and the diameter of  $h_n(I)$  tends to zero as  $n$  tends to infinity. The group  $G$  has the *weak contraction property* if the property above holds for all compact intervals  $I$  of length less than 1.

For example, every non-Abelian subgroup of the affine group has the strong contraction property. In the opposite direction, the group  $\text{Homeo}^+(\mathbf{S}^1)$  of homeomorphisms of the real line commuting with the translation  $x \mapsto x + 1$  does not have the strong contraction property, since no interval of length greater than 1 can be contracted to an interval of length less than 1. However, this group has the weak contraction property.

Recall that the action of a subgroup  $G \subset \text{Homeo}^+(\mathbf{R})$  is *semi-conjugate* to that of an homomorphic image  $\bar{G} \subset \text{Homeo}^+(\mathbf{R})$  if there exists a surjective, nondecreasing, continuous map  $D: \mathbf{R} \rightarrow \mathbf{R}$  such that  $D(g(x)) = \bar{g}(D(x))$  for all  $x \in \mathbf{R}$  and all  $g \in G$ , where  $\bar{g}$  denotes the image of  $g$  under the homomorphism. (We have already met this situation in the proof of Lemma 6.4.) The following result was obtained by Mal'jutin [15], although an analogous statement due to McCleary (see, e.g., [13], Theorem 7.E) was already known in the context of orderable groups. We include a proof for completeness.

**THEOREM 7.1.** *Let  $G$  be a finitely generated, irreducible subgroup of  $\text{Homeo}^+(\mathbf{R})$ . Then one of the following possibilities occur:*

- $G$  has a discrete orbit;
- $G$  is semi-conjugate to a minimal group of translations;
- $G$  is semi-conjugate to a subgroup of  $\text{Homeo}^+(\mathbf{S}^1)$  having the weak contraction property;
- $G$  has the strong contraction property.

**PROOF.** Assume that there are no discrete orbits. By Proposition 6.1, there is a unique minimal nonempty closed  $G$ -invariant subset  $\mathcal{M}$ . Now collapse each connected component of  $\mathcal{M}^c$  to a point to semi-conjugate  $G$  to a group  $\bar{G}$  whose



action is minimal. If  $G$  preserves a Radon measure, then after semi-conjugacy this measure becomes a  $\bar{G}$ -invariant Radon measure of total support and no atoms. Therefore,  $\bar{G}$  (resp.,  $G$ ) is conjugate (resp., semi-conjugate) to a group of translations.

Now suppose that  $G$  has no invariant Radon measure. We claim that the action of  $\bar{G}$  cannot be free. If the action was free,  $\bar{G}$  would be conjugate to a group of translations by Hölder’s theorem; see either [12] or [17]. Pulling back the Lebesgue measure by the semi-conjugacy would provide a  $G$ -invariant Radon measure, which is contrary to our assumption. So the action of  $\bar{G}$  is not free.

Let  $\bar{g} \in \bar{G}$  be a nontrivial element having fixed points, and let  $\bar{x}_0$  be a point in the boundary of  $\text{Fix}(\bar{g})$ . Then there is a left or right neighborhood  $I$  of  $\bar{x}_0$  that is contracted to  $\bar{x}_0$  under iterates of either  $\bar{g}$  or its inverse. By minimality, every  $\bar{x}$  has a neighborhood that can be contracted to a point by elements in  $\bar{G}$ . Coming back to the original action, we conclude that every  $x \in \mathbf{R}$  has a neighborhood that can be contracted to a point by elements in  $G$ . Since  $G$  is finitely generated, such a point can be chosen to belong to a compact interval  $L$  that intersects every orbit (compare with Theorem 4.3).

For each  $x \in \mathbf{R}$  define  $T(x) \in \mathbf{R} \cup \{+\infty\}$  as the supremum of the  $y > x$  such that the interval  $(x, y)$  can be contracted to a point in  $L$  by elements of  $G$ . Then either  $T \equiv +\infty$ , in which case the group  $G$  has the strong contraction property, or  $T(x)$  is finite for every  $x \in \mathbf{R}$ . In the last case,  $T$  induces a nondecreasing map  $\bar{T} : \mathbf{R} \rightarrow \mathbf{R}$  commuting with all the elements in  $\bar{G}$ . Since the union of the intervals on which  $\bar{T}$  is constant is invariant by  $\bar{G}$ , the minimality of the action implies that there is no such interval, that is,  $\bar{T}$  is strictly increasing. Moreover, the interior of  $\mathbf{R} \setminus \bar{T}(\mathbf{R})$  is also invariant, hence empty because the action is minimal. In other words,  $\bar{T}$  is continuous. All of this shows that  $\bar{T}$  induces a homeomorphism of  $\mathbf{R}$  into its image. Since the image of  $\bar{T}$  is  $\bar{G}$ -invariant, it must be the whole line. Therefore,  $\bar{T}$  is a homeomorphism from the real line to itself. Observe that  $\bar{T}(x) > x$  for any point  $x$ , which implies that  $\bar{T}$  is conjugate to the translation  $x \mapsto x + 1$ . After this conjugacy,  $\bar{G}$  becomes a subgroup of  $\widetilde{\text{Homeo}}^+(\mathbf{S}^1)$ . This completes the proof.  $\square$

We now establish a probabilistic version of Theorem 7.1. Notice that in the first two cases given by this theorem, the Markov chain  $X$  induces a random walk on a (finitely generated) subgroup of the group of translations. In the other two cases, we establish the global stability at a finite distance. More precisely, we obtain the following result.

**THEOREM 7.2.** *Let  $\mu$  be an irreducible, finitely supported, symmetric probability measure on  $\text{Homeo}^+(\mathbf{R})$  such that the group  $G$  generated by the support of  $\mu$  acts minimally on  $\mathbf{R}$ . If  $G$  has the strong contraction property, then for any  $x < y$  and any compact interval  $J$ , almost surely we have*

$$(8) \quad \mathbf{1}_J(X_n^x) |X_n^y - X_n^x| \longrightarrow 0 \quad \text{as } n \rightarrow \infty.$$

If  $G$  satisfies only the weak contraction property, then viewed (after conjugacy) as a subgroup of  $\widetilde{\text{Homeo}}^+(\mathbf{S}^1)$ , convergence (8) holds with positive probability for any  $x < y < x + 1$ .

We will assume below that  $G$  has the strong contraction property, since the case of the weak contraction property is analogous and may be left to the reader. Moreover, the result in the latter context is not new. Indeed, by conjugacy into a subgroup of  $\widetilde{\text{Homeo}}^+(\mathbf{S}^1)$ , the  $P$ -invariant Radon measures become invariant by the translation  $x \mapsto x + 1$ . Therefore, these measures are proportional to the “pull-back” of the unique  $P$ -stationary probability measure of the associated action of  $G$  on the circle  $\mathbf{R}/\mathbf{Z}$ . Furthermore, for this associated action, a natural property of strong contraction for random compositions holds. See [5], Section 5.1, for more details.

The main technical ingredient of the proof of Theorem 7.2 is the next lemma, which has an obvious extension to more general Markov processes.

LEMMA 7.3. *In the context of Theorem 7.2, assume that  $G$  has the strong contraction property, and let  $K$  be any compact interval of recurrence. Fix  $k \in \mathbb{N}$  and  $h_1, \dots, h_k$  in the support of  $\mu$ . Then almost surely the following happens for infinitely many  $n \geq 0$ : the point  $X_n^x$  belongs to  $K$  and  $g_{n+1}, \dots, g_{n+k}$  coincide with  $h_1, \dots, h_k$ , respectively.*

PROOF. Due to the Markov property, it suffices to show that this situation almost surely happens at least once. Let  $\xi : \mathbf{R} \rightarrow [0, 1]$  be the function defined by letting  $\xi(z)$  be the probability that there exists  $n \geq 0$  such that  $X_n^z \in K$  and  $g_{n+i} = h_i$  for  $i = 1, \dots, k$ . We need to show that  $\xi(x) = 1$ , and we will actually show that  $\xi(z) = 1$  holds for all  $z \in \mathbf{R}$ . To do this, let  $p := \mu(h_1) \cdots \mu(h_k) > 0$ . For each  $\omega \in \Omega$  and  $z \in \mathbf{R}$ , let  $n(z) \geq 0$  be the first-entry time of  $z$  into  $K$ . A moment reflexion shows that

$$\xi(z) = p + (1 - p)\mathbb{E}(\xi(X_{n(z)+k}^z) | (g_{n(z)+1}, \dots, g_{n(z)+k}) \neq (h_1, \dots, h_k)).$$

Letting  $\Phi := \inf_{z \in \mathbf{R}} \xi(z)$ , this yields

$$\Phi \geq p + (1 - p)\Phi.$$

This easily implies that  $\Phi = 1$ , as we wanted to show.  $\square$

The proof of Theorem 7.2 when  $G$  has the strong contraction property is now easy. Indeed, let  $L$  be the interval of contraction, and let  $K := [a, b]$  be a compact interval of recurrence containing  $J$  and  $x, y$ . As in Lemma 6.6, the value of  $\nu(f_n(K))$  converges to a limit  $l(\omega)$  almost surely. Given  $\omega \in \Omega$  for which it converges, we fix  $M > 0$  such that  $\nu(f_n(K)) \leq M$  holds for all  $n$ . Choose an interval  $I := [\bar{a}, \bar{b}]$  containing  $K$  so that  $\nu[\bar{a}, a] > M$  and  $\nu[b, \bar{b}] > M$ . Given  $\varepsilon > 0$ , let

$h \in \Gamma$  be such that  $h(I) \subset J$  and  $|h(I)| \leq \varepsilon$ . Write  $h$  in the form  $h_k \cdots h_1$ , where each  $h_i$  is in the support of  $\mu$ . Lemma 7.3 shows that if  $\omega$  is generic there exist infinitely many  $n \in \mathbb{N}$  such that:

- $f_n(x)$  belongs to  $K$ ;
- $\nu(f_n(K)) \leq M$ ;
- $(g_{n+1}, \dots, g_{n+k}) = (h_1, \dots, h_k)$ .

Since  $f_n(K)$  intersects  $K$  and its  $\nu$ -measure is bounded from above by  $M$ , it must be contained in  $I$ . Therefore,

$$f_{n+k}(K) = g_{n+k} \cdots g_{n+1} f_n(K) = h_k \cdots h_1 f_n(K) \subset h_k \cdots h_1(I) = h(I) \subset L,$$

hence

$$|f_{n+k}(K)| \leq |h(I)| < \varepsilon.$$

Since  $n$  can be taken as large as required and  $\nu$  has no atoms, we must necessarily have  $l(\omega) \leq \varepsilon$ . Since this is true for all  $\varepsilon > 0$ , we conclude that  $l(\omega) = 0$ . This implies the desired result.

It should be emphasized that the distance  $d$  induced by  $\nu$  and the usual distance on  $\mathbf{R}$  may be very different in neighborhoods of  $\pm\infty$ . As an example, consider the case of a non-Abelian subgroup  $G$  of the affine group generated by an expansion and a translation. Writing  $g(x) = a_g x + b_g$  for each  $g \in G$ , the homomorphism  $g \mapsto \log(a_g)$  induces a (symmetric) random walk on  $\mathbf{Z}$ , which is therefore recurrent. As a consequence, the length of the interval  $[X_n^x, X_n^y]$  oscillates between 0 and  $\infty$  even though its  $\nu$ -measure converges to zero.

**8. Derriennic’s property and Lipschitz actions.** Let  $\mu$  be a symmetric probability measure on  $\text{Homeo}^+(\mathbf{R})$  with finite support generating an irreducible group  $G$ . We will say that the pair  $(G, \mu)$  has the *Derriennic property* if, for every  $x \in \mathbf{R}$ ,

$$x = \int_G g(x) d\mu(g).$$

This terminology is inspired by [7], where Derriennic studies Markov processes on the real line satisfying  $\mathbb{E}(X_1^x) = x$  for large values of  $|x|$ . As we demonstrate below, under very general conditions, this property is always guaranteed after a suitable semi-conjugacy.

**PROPOSITION 8.1.** *Let  $\mu$  be a finitely supported, symmetric measure on  $\text{Homeo}^+(\mathbf{R})$  whose support generates an irreducible group  $G$  without discrete orbits. Then  $G$  is semi-conjugate to a group  $\tilde{G}$  so that the pair  $(\tilde{G}, \mu)$  has the Derriennic property.*

PROOF. Since  $G$  has no discrete orbits, there is a unique nonempty  $G$ -invariant closed subset  $\mathcal{M}$  in which every orbit is dense. By Lemma 6.4, the support of the  $P$ -invariant measure  $\nu$  coincides with  $\mathcal{M}$ . Moreover, Lemma 6.3 shows that no  $P$ -invariant measure  $\nu$  has atoms. Now fix a point  $x_0$  in the real line and consider the map

$$x \in \mathbf{R} \mapsto D(x) := \begin{cases} \nu[x_0, x], & \text{if } x \geq x_0, \\ -\nu[x, x_0], & \text{if } x \leq x_0. \end{cases}$$

This map is continuous and nondecreasing. Furthermore, Lemma 4.1 implies that this map is also surjective. Consequently, since the support of  $\nu$  is  $G$ -invariant,  $D$  induces a semi-conjugacy from  $G$  to a group  $\bar{G}$  whose action is minimal. We claim that the pair  $(\bar{G}, \mu)$  has the Derriennic's property. Let  $\bar{P}_\mu$  be the transition operator associated to the Markov process. Notice that  $D$  maps the measure  $\nu$  to the Lebesgue measure, which is then  $\bar{P}_\mu$ -invariant. Now, for any  $x < y$ , we have

$$y - x = \int_{\bar{G}} (\bar{g}(y) - \bar{g}(x)) d\mu(\bar{g}),$$

which implies that the value of the drift,

$$\text{Dr}(\bar{G}, \mu) := \int_{\bar{G}} (\bar{g}(x) - x) d\mu(\bar{g})$$

is independent of  $x$ . To complete the proof, we need to show that the drift vanishes. To do this, we closely follow the argument of the first proof of [5], Proposition 5.7.

Fix any  $a < b$ , and let us integrate (9) over  $[a, b]$ , then doubling the integral in order to couple  $g$  and  $g^{-1}$ ,

$$\begin{aligned} 2 \int_a^b \text{Dr}(\bar{G}, \mu) dx &= \int_a^b \left( \int_G (\bar{g}(x) - x) d\mu(\bar{g}) + \int_G (\bar{g}^{-1}(x) - x) d\mu(\bar{g}) \right) dx \\ &= \int_G \left( \int_a^b [(\bar{g}(x) - x) + (\bar{g}^{-1}(x) - x)] dx \right) d\mu(\bar{g}). \end{aligned}$$

Now, we will transform the value under the integral by means of the following notion.

DEFINITION 8.2. For any  $c \in \mathbb{R}$  and  $\bar{g} \in \text{Homeo}^+(\mathbf{R})$ , let

$$\begin{aligned} \Phi_{\bar{g}}(c) &= \Phi_{\bar{g}^{-1}}(c) \\ &= \text{mes}\{(x, y) \mid \text{either } x < c < y < \bar{g}(x) \text{ or } x < c < y < \bar{g}^{-1}(x)\}, \end{aligned}$$

where  $\text{mes}$  is the two-dimensional Lebesgue measure; see Figure 1. Equivalently,

$$\Phi_{\bar{g}}(c) = \Phi_{\bar{g}^{-1}}(c) = \begin{cases} \int_{\bar{g}^{-1}(x)}^x [\bar{g}(s) - s] ds, & \text{if } \bar{g}(x) \geq x, \\ \int_{\bar{g}(x)}^x [\bar{g}^{-1}(s) - s] ds, & \text{if } \bar{g}(x) \leq x. \end{cases}$$

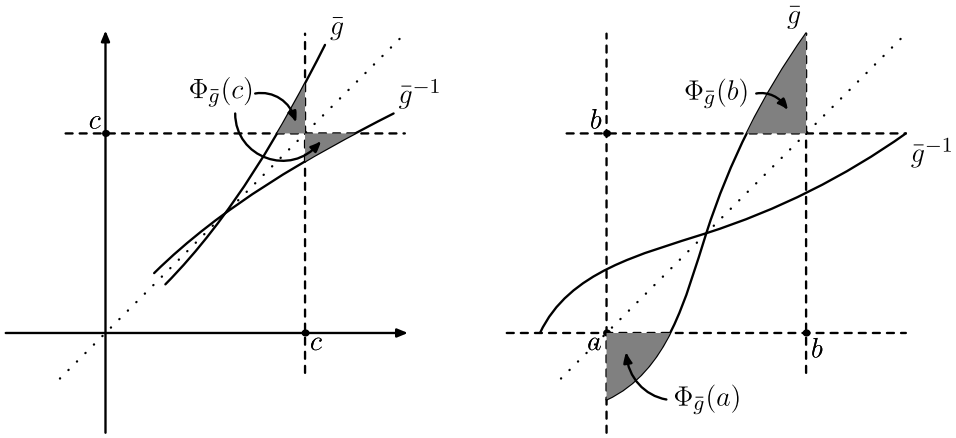


FIG. 1. Definition of  $\Phi_{\bar{g}}$  and illustration for the proof of Lemma 8.3.

A geometric argument based on symmetry yields the following lemma.

LEMMA 8.3. For any  $\bar{g} \in \text{Homeo}^+(\mathbf{R})$  and for any interval  $[a, b]$  we have

$$(9) \quad \int_a^b [(\bar{g}(x) - x) + (\bar{g}^{-1}(x) - x)] dx = \Phi_{\bar{g}}(b) - \Phi_{\bar{g}}(a).$$

PROOF. Notice that  $\int_a^b (\bar{g}(x) - x) dx$  equals

$\text{mes}\{(x, y) \mid a < x < b, x < y < \bar{g}(x)\} - \text{mes}\{(x, y) \mid a < x < b, \bar{g}(x) < y < x\}$ ,  
 which may be rewritten as

$$(10) \quad \begin{aligned} & \text{mes}\{(x, y) \mid a < x < b, b < y < \bar{g}(x)\} \\ & + \text{mes}\{(x, y) \mid a < x < b, a < y < b, x < y < \bar{g}(x)\} \\ & - \text{mes}\{(x, y) \mid a < x < b, \bar{g}(x) < y < a\} \\ & - \text{mes}\{(x, y) \mid a < x < b, a < y < b, \bar{g}(x) < y < x\}. \end{aligned}$$

A similar equality holds when changing  $\bar{g}$  by  $\bar{g}^{-1}$ . Now, when taking the sum of  $\int_a^b (\bar{g}(x) - x) dx$  and  $\int_a^b (\bar{g}^{-1}(x) - x) dx$ , we see that the corresponding second and fourth terms from (10) cancel each other. Indeed, these terms correspond to the couples  $(x, y) \in [a, b]^2$ , and we have  $x < y < \bar{g}(x)$  if and only if  $\bar{g}^{-1}(y) < x < y$ . The symmetry argument then shows that the second term for  $\bar{g}$  is exactly the negative of the fourth term for  $\bar{g}^{-1}$ , and vice versa.

Therefore, the value of

$$\int_a^b [(\bar{g}(x) - x) + (\bar{g}^{-1}(x) - x)] dx$$

equals

$$\begin{aligned} & [\text{mes}\{(x, y) \mid a < x < b, b < y < \bar{g}(x)\} \\ & + \text{mes}\{(x, y) \mid a < x < b, b < y < \bar{g}^{-1}(x)\}] \\ & - [\text{mes}\{(x, y) \mid a < x < b, \bar{g}(x) < y < a\} \\ & + \text{mes}\{(x, y) \mid a < x < b, \bar{g}^{-1}(x) < y < a\}], \end{aligned}$$

and one can easily see that the expressions inside the brackets are equal to  $\Phi_{\bar{g}}(b)$  and  $\Phi_{\bar{g}}(a)$ , respectively; see Figure 1. This proves the desired equality.  $\square$

Now, we can complete the proof of Proposition 8.1: by integrating (9) over  $G$  we obtain, for any  $a < b$ ,

$$2(b - a) \text{Dr}(\bar{G}, \mu) = \int_G (\Phi_{\bar{g}}(b) - \Phi_{\bar{g}}(a)) d\mu(\bar{g}).$$

Denoting now  $\Phi_\mu(c) := \int_G \Phi_{\bar{g}}(c) d\mu(\bar{g})$ , this yields

$$2(b - a) \text{Dr}(\bar{G}, \mu) = \Phi_\mu(b) - \Phi_\mu(a).$$

The last equality shows that  $\Phi_\mu$  is an affine function. On the other hand,  $\Phi_\mu$  is an average of nonnegative functions, and thus it is nonnegative. Therefore,  $\Phi_\mu$  must be constant, which implies that  $\text{Dr}(\bar{G}, \mu) = 0$ .  $\square$

The next proposition demonstrates the relevance of the Derriennic property in the study of the smoothness of a group action.

**PROPOSITION 8.4.** *If a pair  $(G, \mu)$  has the Derriennic property, then every element of  $G$  is a Lipschitz map. Moreover, the displacement function  $x \mapsto g(x) - x$  is uniformly bounded in  $x$  for every  $g \in G$ .*

**PROOF.** It suffices to prove the lemma for the elements of the support of  $\mu$ .

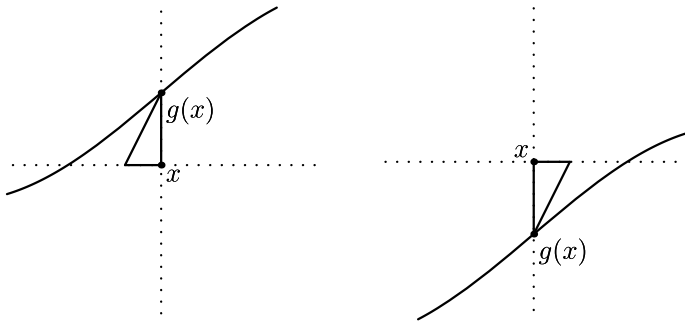
To check the Lipschitz property, notice that for any  $g_0 \in \text{supp } \mu$  and any  $x < y$  we have

$$\mu(g_0) \cdot (g_0(y) - g_0(x)) \leq \int_G \mu(g) \cdot (g(y) - g(x)) d\mu(g) = y - x$$

and thus

$$g_0(y) - g_0(x) \leq \frac{1}{\mu(g_0)} \cdot (y - x).$$

To obtain the bounded displacement property, notice that for any  $g \in \text{supp } \mu$  and for any  $x \in \mathbb{R}$ , the domain that we have used to define  $\Phi_g(x)$  contains [as  $g$  is  $1/\mu(g)$ -Lipschitz] a rectangular triangle with sides  $|x - g(x)|$  and  $\mu(g_0) \cdot |x - g(x)|$ ; see Figure 2.

FIG. 2. *Triangles.*

Hence,  $\Phi_g(x) \geq \frac{\mu(g)}{2}|x - g(x)|^2$ , which implies that  $\Phi_\mu \geq \frac{\mu(g)^2}{2}|x - g(x)|^2$ . Since  $\Phi_\mu$  does not depend on  $x$ , we obtain the desired uniform upper bound for the displacement  $|g(x) - x|$ .  $\square$

As a consequence, we have the following result.

**THEOREM 8.5.** *If  $G$  is an irreducible, finitely generated subgroup of  $\text{Homeo}^+(\mathbf{R})$ , then there exists a homeomorphism  $D: \mathbf{R} \rightarrow \mathbf{R}$  such that, for every  $g \in G$ , the map  $D \circ g \circ D^{-1}$  is Lipschitz and has uniformly bounded displacement.*

**PROOF.** Without loss of generality, assume that the  $G$ -action is minimal—otherwise, consider the subgroup of  $\text{Homeo}^+(\mathbf{R})$  generated by  $G$  and two rationally independent translations. By Theorem 8.1,  $G$  is semi-conjugate to a group satisfying the Derriennic property. Since the orbits of  $G$  are dense, the semiconjugacy is in fact a conjugacy. Now the desired conclusion follows as an application of Proposition 8.4.  $\square$

It should be pointed out that the result above also follows from [5], Theorem D (see also [4]) by means of rather tricky—and less conceptual—arguments. The reader is referred to [6] for a detailed discussion on this. Finally, a conjugacy into a group of  $C^1$  diffeomorphisms of the line is not always possible; see [16] and references therein.

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