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A remarkable family of left-ordered groups: Central extensions of Hecke groups

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ABSTRACT

We provide an infinite family of left-ordered groups that have a positive cone that is generated by two elements as a semigroup. This family corresponds to that of certain central extensions of Hecke groups, and includes the Klein bottle group and the braid group B_3. Using the classical convex extension (flipping) procedure, on each of the groups of this family we define an ordering sharing many properties with Dehornoy's. Several related questions and problems are addressed.

Braid groups are relevant in many branches of Mathematics. In recent years, they have been studied as important examples of *left-orderable groups* (that is, groups admitting a total order which is invariant by left-multiplication). Historically, the first such order on B_n (for all $n \geqslant 3$) was defined by Dehornoy using pure algebraic (and quite deep) methods [5]. Some years later, an alternative geometric approach using Nielsen's theory was proposed by Thurston [19]. In this work we will, nevertheless, be more interested in other kinds of orders on braid groups, first introduced by Dubrovina and Dubrovin [9].

We will restrict the discussion to $B_3 = \langle \sigma_1, \sigma_2\colon\ \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \rangle$. (Potential generalizations to general B_n will be discussed in Section 5.) In [9], it is shown that there is a unique left-invariant total order $\preccurlyeq_{DD}$ on B_3 satisfying $\sigma_1\sigma_2 \succ_{DD} id$ and $\sigma_2^{-1} \succ_{DD} id$. This is a rather surprising fact (actually, it was conjectured as impossible in [6, Conjecture 10.3.1]) which gives a new insight on the combinatorial structure of the Cayley graph of B_3 (*cf.* Fig. 2).

The situation described above is reminiscent to that of the Klein bottle group $K_2 = \langle a, b\colon\ a^{-1}ba = b^{-1} \rangle$. Indeed, K_2 is left-orderable, and there exits a unique left-ordering $\preccurlyeq$ satisfying $a \succ id$ and $b \succ id$. However, K_2 is a less interesting example because it admits only four left-invariant total orders (each of which is completely determined by the "signs" of a and b), whereas B_3 admits uncountably many [16,19].

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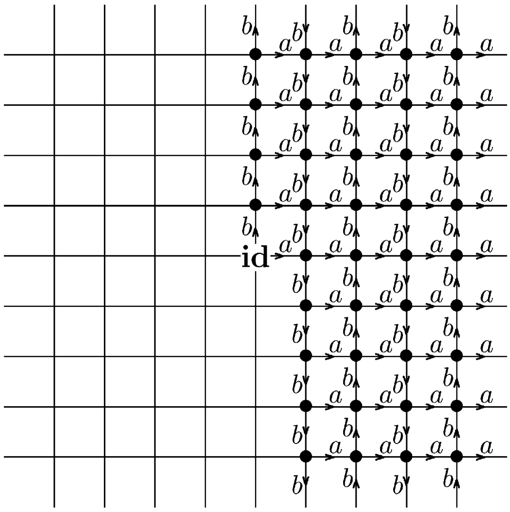

Fig. 1. The finitely generated positive cone $P^+ = \langle a, b\rangle^+$ on $K_2 = \langle a, b\colon\ a^{-1}ba = b^{-1}\rangle$.

The fact that certain left-orderings are determined by finitely many inequalities comes from the structure of their *positive cone*. This corresponds to the set of elements which are *positive* (that is, bigger than the identity), and it is easy to see that it is a semigroup. Actually, as is readily checked, the property of being left-orderable for a group Γ is equivalent to the existence of a (disjoint) decomposition

$$\Gamma = P^+ \sqcup P^- \sqcup \{id\},$$

where P^+ and P^- are semigroups, with $P^- = \{g^{-1}\colon\ g \in P^+\}$. Now the point is that such a decomposition exists, for both K_2 and B_3, with P^+ (and P^-) finitely generated. For instance, denoting by $\langle g_1, \ldots, g_k\rangle^+$ the semigroup generated by $\{g_1, \ldots, g_k\}$, we have

$$K_2 = \langle a, b\rangle^+ \sqcup \langle a^{-1}, b^{-1}\rangle^+ \sqcup \{id\}.$$

This decomposition can be visualized in Fig. 1, where the elements in $\langle a, b\rangle^+$ (that is, the positive elements of the induced ordering) are blackened.

A similar phenomenon occurs for B_3. Indeed, letting $a = \sigma_1\sigma_2$ and $b = \sigma_2^{-1}$, we also have the decomposition

$$B_3 = \langle a, b\rangle^+ \sqcup \langle a^{-1}, b^{-1}\rangle^+ \sqcup \{id\}.$$

The proof of this fact is given in [9]. It is very indirect and uses Dehornoy's theory. We will propose an alternative argument which applies to a larger family of groups. As a byproduct, we will retrieve (and generalize) the Dehornoy ordering and some of its properties by rather elementary methods (see Section 4).

As in the case of K_2, the decomposition of B_3 above may be easily illustrated: see Fig. 2. The Cayley graph of B_3 is, essentially, a product of $\mathbb{Z}^2$ by a dyadic rooted tree. The (quasi-isometric) copy of $\mathbb{Z}^2$ corresponds to the "upper level" of the graph, and the corresponding edges are slightly blackened. An arrow pointing to the right should be added to every horizontal edge of the graph. These edges represent multiplications by a, and all other (oriented) edges represent multiplications by b. Starting at id, every blackened element can be reached by a path that follows the direction of the arrows. Conversely, every element which is not blackened may be reached by a path starting at id following a direction opposite to that of the arrows. Finally, no (nontrivial) element can be reached both ways.

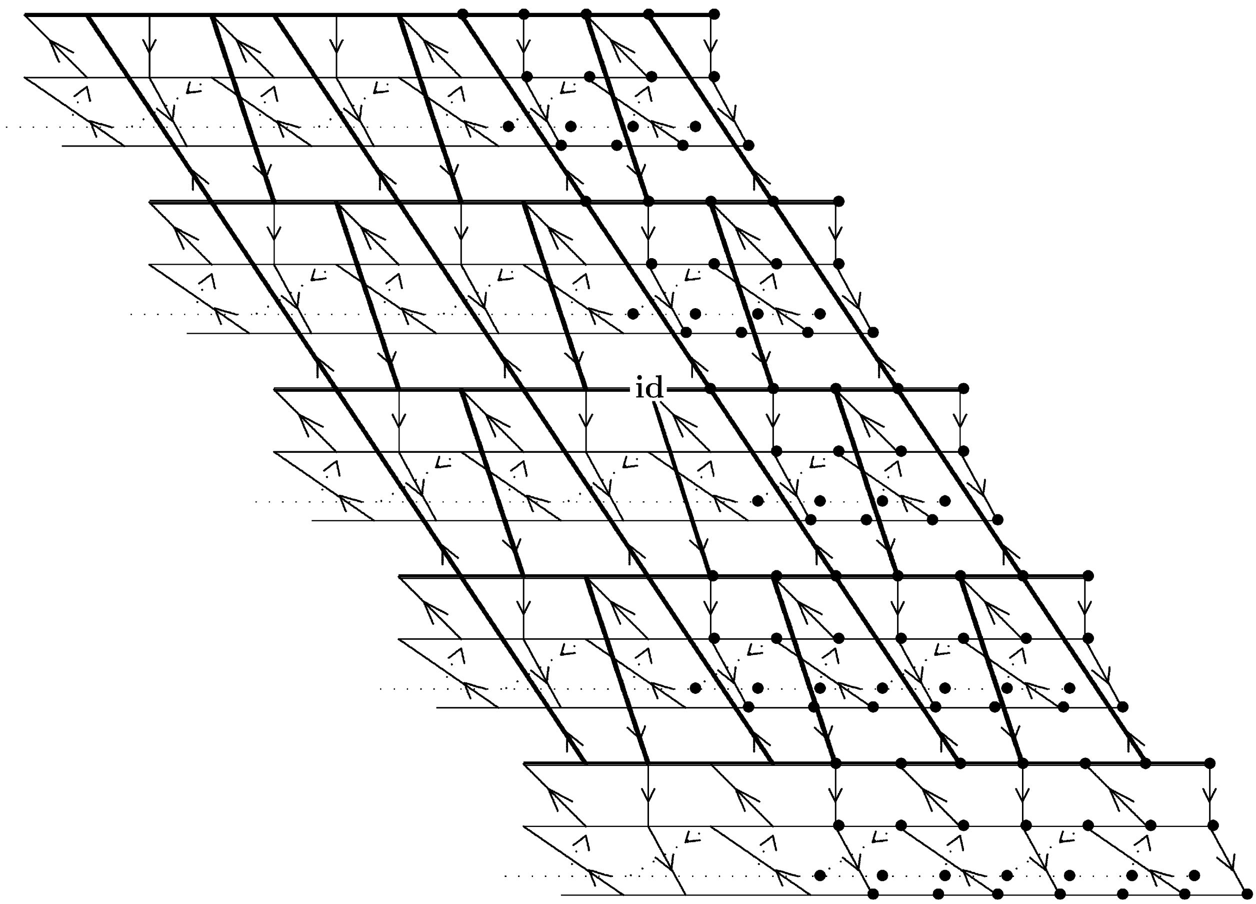

Fig. 2. The Cayley graph of $B_3 = \langle a, b \colon\ ba^2b = a\rangle$ and the DD-positive cone.

The Klein bottle group may be presented in the form

$$K_2 = \langle a, b \colon\ bab = a\rangle.$$

Moreover, with respect to the generators $a = \sigma_1\sigma_2$ and $b = \sigma_2^{-1}$, the standard presentation of B_3 becomes

$$B_3 = \langle a, b \colon\ ba^2b = a\rangle.$$

This makes natural the study of the groups

$$\Gamma_n = \langle a, b \colon\ ba^nb = a\rangle.$$

These groups have been already considered in [4,8] as examples lying on the border of the theory of Gaussian and Garside groups. We will show that, though they do not fit in these important categories, they share a remarkable combinatorial property with B_3.

Main Theorem. *For each $n \geqslant 1$, the group Γ_n admits the decomposition*

$$\Gamma_n = \langle a, b\rangle^+ \sqcup \langle a^{-1}, b^{-1}\rangle^+ \sqcup \{id\}.$$

The proof of this result involves two issues. First, we need to show that every nontrivial element $w \in \Gamma_n$ belongs to either $\langle a, b\rangle^+$ or $\langle a^{-1}, b^{-1}\rangle^+$. For this, we begin by appealing to the theory of

Garside groups and write w in the form $w = uv^{-1}$ for some u, v in $\langle a, b\rangle^+$ (see Section 1). This creates central *handles*, that is, expressions of the form $ab^k a^{-1}$. The main point here is that these handles belong to one of the semigroups above. Indeed, from the relation $ba^n b = a$ one easily deduces that $ab^k a^{-1} = (a^{-(n-1)} b^{-1})^k$. Due to this, the corresponding *reduction procedure* that we will perform in Section 2 for showing that $\Gamma_n \setminus \{id\} = \langle a, b\rangle^+ \cup \langle a^{-1}, b^{-1}\rangle^+$, though similar, is much simpler than the Dehornoy *handle reduction algorithm* [7]. In particular, its convergence follows from elementary combinatorial arguments.

The second issue consists in showing that $\langle a, b\rangle^+$ and $\langle a^{-1}, b^{-1}\rangle^+$ are disjoint, which is equivalent to showing that $\langle a, b\rangle^+$ does not contain the identity. This is done in Section 3 by means of a quite simple ping-pong type argument. As a motivation, recall the well-known representation of B_3 in $\mathrm{PSL}(2, \mathbb{R})$ given by

$$\sigma_1 \to \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \qquad \sigma_2 \to \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}.$$

This representation induces an action of B_3 on the circle (viewed as the projective line), and by looking at the dynamics of this action, this yields the desired property by a ping-pong type argument. The extension of this proof to Γ_n is straightforward, as one may easily produce an action of Γ_n on the circle satisfying similar dynamical properties. (Actually, Γ_n embeds into $\widetilde{\mathrm{PSL}}(2, \mathbb{R})$ for $n \geqslant 2$, and its image in $\mathrm{PSL}(2, \mathbb{R})$ is isomorphic to the so-called *Hecke groups* $\langle u, v\colon\ u^2 = v^{n+1}\rangle$ [11, Chapter II, Example 28].) Although this idea is new in the context, it is very natural. Indeed, the property to be shown implies that Γ_n is left-orderable, and left-orderability for countable groups is equivalent to the existence of faithful actions by (orientation-preserving) homeomorphisms of the real-line [10,14]. In the present case, the action on the circle appears by taking the quotient with respect to the central cyclic subgroup $\langle a^{n+1}\rangle$.

1. Γ_n as a group of fractions

Unless otherwise stated, in what follows we will only consider the case $n > 1$: the case $n = 1$ is elementary and we leave it to the reader.

We begin by noticing that $\Delta = a^{n+1}$ belongs to the center of Γ_n.[1] Indeed,

$$b\Delta = ba^{n+1} = \big(ba^n\big)a = \big(ab^{-1}\big)a = a\big(b^{-1}a\big) = a\big(a^n b\big) = a^{n+1}b = \Delta b, \qquad a\Delta = a^{n+2} = \Delta a.$$

A word in (positive powers of) a, b (resp. a^{-1}, b^{-1}) will be said to be *positive* (resp. *negative*). It is *non-positive* (resp. *non-negative*) if it is either trivial or negative (either trivial or positive).

Proposition 1.1. *Every element $w \in \Gamma_n$ may be written in the form $\bar{u}\Delta^\ell$ for some non-negative word $\bar{u}$ and $\ell \in \mathbb{Z}$.*

Proof. In any word representing w, we may rewrite the negative powers of a and b using the relations $a^{-1} = a^n\Delta^{-1}$ and $b^{-1} = \Delta^{-1}a^n ba^n$. Since Δ belongs to the center of Γ_n, this shows the proposition.[2] □

Let us take a more careful look at the positive words in a, b. Using the relation $ba^n b = a$, one easily concludes that every $v \in \langle a, b\rangle^+$ may be written in the form

[1] Although it will be not used in this work, it is worth mentioning that the center of Γ_n coincides with the cyclic group generated by a^{n+1}. This is a direct consequence of [17]. More elementary, this can be easily deduced by looking at the embedding of Γ_n in $\widetilde{\mathrm{PSL}}(2, \mathbb{R})$ to be discussed in Section 3.

[2] This argument is motivated by the fact that the presentation $\Gamma_n = \langle a, c\colon\ cac = a^n\rangle$ endows Γ_n with a structure of a *Garside group*: see [4, p. 268] and [8, Example 2]. Indeed, as is well known, Garside groups are groups of fractions of the corresponding monoids.

$$v = b^{n_0} a^{m_1} b^{n_1} \cdots b^{n_{k-1}} a^{m_k} b^{n_k} \Delta^{\ell},$$

where $n_i > 0$ for $i \in \{1, \ldots, k\}$, $n_0 \geqslant 0$, $n_k \geqslant 0$, $m_i \in \{1, \ldots, n\}$, and $\ell \geqslant 0$. Moreover, if $m_i = n$ and $0 < i < k$, then we may replace $b^{n_{i-1}} a^{m_i} b^{n_i} = b^{n_{i-1}-1}(ba^n b) b^{n_i - 1}$ by $b^{n_{i-1}-1} a b^{n_i - 1}$. This is also possible for $i = 0$ (resp. $i = k$) when $n_0 > 0$ (resp. $n_k > 0$). Performing these reductions as far as possible, we conclude that v may be written in the form

$$v = b^{n_0} a^{m_1} b^{n_1} \cdots b^{n_{k-1}} a^{m_k} b^{n_k} \Delta^{\ell},$$

so that the following properties are satisfied:

(i) $n_i > 0$ for $0 < i < k$, $n_0 \geqslant 0$, $n_k \geqslant 0$.
(ii) $m_i \in \{1, \ldots, n-1\}$ for $1 < i < k$.
(iii) m_1 lies in $\{1, \ldots, n-1\}$ (resp. $\{1, \ldots, n\}$) if $n_0 > 0$ (resp. $n_0 = 0$); similarly, m_k lies in $\{1, \ldots, n-1\}$ (resp. $\{1, \ldots, n\}$) if $n_k > 0$ (resp. $n_k = 0$).
(iv) $\ell \geqslant 0$.

Here, for $k = 0$, an expression as above should be understood as $b^{n_0} \Delta^{\ell}$, where $n_0 \geqslant 0$.

Therefore, by Proposition 1.1, every element $w \in \Gamma_n$ may be written in the form

$$w = b^{n_0} a^{m_1} b^{n_1} \cdots b^{n_{k-1}} a^{m_k} b^{n_k} \Delta^{\ell} = u \Delta^{\ell}, \tag{1}$$

where properties (i), (ii), and (iii) above are satisfied, and $\ell \in \mathbb{Z}$. Such an expression will be said to be a *normal form* for w.

2. Eliminating numerators or denominators

Let $w = u\Delta^{\ell}$ be a normal form of a nontrivial element $w \in \Gamma_n$. Our task consists in showing that w is either positive or negative in a, b. If u is trivial, then w is positive or negative according to the sign of ℓ. If u is nontrivial and $\ell \geqslant 0$, then w is positive. Assume throughout that u is nontrivial and $\ell < 0$. We will show that, in this situation, w is negative.

Case I. We have $u = b^r$ for some positive integer r.

In this case, the relation $ba^n b = a$ yields $aba^{-1} = a^{-(n-1)} b^{-1}$, hence

$$w = b^r a^{-1} a^{-n} \Delta^{\ell - 1} = a^{-1} \big(aba^{-1}\big)^r a^{-n} \Delta^{\ell-1} = a^{-1} \big(a^{-(n-1)} b^{-1}\big)^r a^{-n} \Delta^{\ell - 1}.$$

Case II. The element u does not belong to $\langle b \rangle$:

Let us consider the normal form (1). There are two possibilities:

(i) If $n_k = 0$, then using the relation $aba^{-1} = a^{-(n-1)} b^{-1}$ we obtain

$$\begin{aligned} w &= b^{n_0} a^{m_1} b^{n_1} \cdots a^{m_{k-1}} b^{n_{k-1}} a^{m_k} \Delta^{\ell} \\ &= b^{n_0} a^{m_1} b^{n_1} \cdots a^{m_{k-1} - 1} \underline{a b^{n_{k-1}} a}^{-1} a^{m_k - n} \Delta^{\ell - 1} \\ &= b^{n_0} a^{m_1} b^{n_1} \cdots a^{m_{k-1} - 1} \big(a^{-(n-1)} b^{-1}\big)^{n_{k-1}} a^{m_k - n} \Delta^{\ell - 1} \\ &= b^{n_0} a^{m_1} b^{n_1} \cdots a^{m_{k-2}} b^{n_{k-2}} a^{m_{k-1} - n} b^{-1} \big(a^{-(n-1)} b^{-1}\big)^{n_{k-1} - 1} a^{m_k - n} \Delta^{\ell - 1} \\ &= b^{n_0} a^{m_1} b^{n_1} \cdots a^{m_{k-2} - 1} \underline{a b^{n_{k-2}} a}^{-1} a^{m_{k-1} - n + 1} b^{-1} \big(a^{-(n-1)} b^{-1}\big)^{n_{k-1} - 1} a^{m_k - n} \Delta^{\ell - 1}. \end{aligned}$$

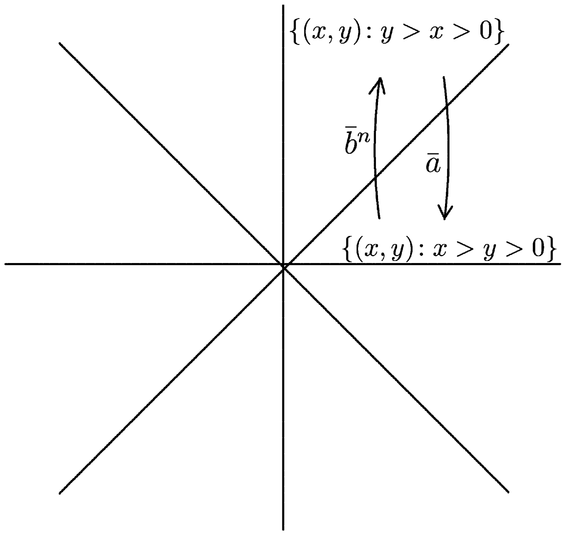

Fig. 3.

(ii) If $n_k > 0$, then

$$\begin{aligned} w &= b^{n_0} a^{m_1} b^{n_1} \cdots a^{m_k - 1} \underline{a b^{n_k} a}^{-1} a^{-n} \Delta^{\ell-1} \\ &= b^{n_0} a^{m_1} b^{n_1} \cdots a^{m_k - 1} \big(a^{-(n-1)} b^{-1}\big)^{n_k} a^{-n} \Delta^{\ell-1} \\ &= b^{n_0} a^{m_1} b^{n_1} \cdots a^{m_k - n} b^{-1} \big(a^{-(n-1)} b^{-1}\big)^{n_k - 1} a^{-n} \Delta^{\ell-1} \\ &= b^{n_0} a^{m_1} b^{n_1} \cdots a^{m_{k-1} - 1} \underline{a b^{n_{k-1}} a}^{-1} a^{m_k - n + 1} b^{-1} \big(a^{-(n-1)} b^{-1}\big)^{n_k - 1} a^{-n} \Delta^{\ell-1}. \end{aligned}$$

The main point here is that in (i) we have $m_k - n \leqslant 0$ and $m_{k-1} - n + 1 \leqslant 0$. Similarly, in (ii) we have $m_k - n + 1 \leqslant 0$. This allows repeating the argument. Proceeding in this way as far as possible, it is easy to see that the final output will be a negative word representing w.

3. No positive word represents the identity

We begin with a proof that applies to $B_3 = \langle a, b\colon\ ba^2b = a\rangle$. The case of Γ_n will be treated with a similar idea.

Proposition 3.1. *No element in $\langle a, b\rangle^+ \subset B_3$ represents the identity.*

Proof. Consider the representation of B_3 in $\mathrm{PSL}(2, \mathbb{R})$ given by

$$a \to \bar{a} = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}, \qquad b \to \bar{b} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

Denote by U (resp. V) the projection of $\{(x, y)\colon\ x > y > 0\}$ (resp. $\{(x, y)\colon\ 0 < x < y\}$) into $\mathbb{P}^1(\mathbb{R})$. A direct computation shows that

$$\bar{a}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ y - x \end{bmatrix}, \qquad \bar{b}^n \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ nx + y \end{bmatrix},$$

which easily yields $\bar{a}(V) \subset U$ and $\bar{b}^n(U \cup V) \subset V$, for all $n > 0$. (See Fig. 3.)

Now given an element $w \in \langle a, b\rangle^+$, let us write it in the form

$$w = b^{n_0} a^{m_1} b^{n_1} \cdots a^{m_k} b^{n_k} a^{3r},$$

where $k \geqslant 0$, n_0, n_k, r are non-negative, $n_i > 0$ for the other indexes i, and $m_i \in \{1, 2\}$, with $m_i = 1$ for $1 < i < k$, and $m_1 = 1$ (resp. $m_k = 1$) when $n_0 > 0$ (resp. $n_k > 0$). Notice that $\bar{a}^3 = id$. Assume that w is not a power of a^3. In this case, to show that $w \neq id$, it suffices to prove that

$$\bar{w} = \bar{b}^{n_0} \bar{a}^{m_1} \bar{b}^{n_1} \cdots \bar{a}^{m_k} \bar{b}^{n_k}$$

is nontrivial in $\mathrm{PSL}(2, \mathbb{R})$. Now using the relation $\bar{b}\bar{a}^2\bar{b} = \bar{a}$, one can easily check that, unless $\bar{w}$ is conjugate to a power of $ab = \sigma_1$, it is conjugate to a word $\bar{w}'$ in $\bar{a}$ (with no use of $\bar{a}^2$) and positive powers of $\bar{b}$ which either begins and finishes with a power of $\bar{b}$, or begins and finishes with $\bar{a}$. Since σ_1 is not a torsion element, $\bar{w} \neq id$ when $\bar{w}$ is conjugate to σ_1. Otherwise, a ping-pong type argument shows that either $\bar{w}'(U) \subset V$ or $\bar{w}'(V) \subset U$, hence $\bar{w}' \neq id$. $\square$

The representation considered above is obtained via the well-known identification of B_3 to $\widetilde{\mathrm{PSL}}(2, \mathbb{Z})$, followed by the quotient by the center $\langle a^3\rangle$. Indeed, with respect to the system of generators $\{f = a,\ h = b^{-1}a\}$, the presentation of B_3 becomes $\langle f, h\colon\ f^3 = h^2\rangle$.

It turns out that Γ_n also embeds into $\widetilde{\mathrm{PSL}}(2, \mathbb{R})$. To see this, we first rewrite the presentation of Γ_n in terms of $f = a$ and $h = b^{-1}a$:

$$\Gamma_n = \langle f, h\colon\ f^{n+1} = h^2\rangle.$$

This presentation shows that Γ_n corresponds to a central extension of the Hecke group

$$H(n+1) = \langle \bar{f}, \bar{h}\colon\ \bar{f}^{n+1} = \bar{h}^2 = id\rangle.$$

A concrete realization of $H(n+1)$ inside $\mathrm{PSL}(2, \mathbb{R})$ arises when identifying $\bar{f}$ to the circle rotation of angle $\frac{2\pi}{n+1}$, and $\bar{h}$ to the hyperbolic reflexion with respect to the geodesic joining $p_n = \bar{f}^n(p)$ and $p = p_0$ for some point $p \in \mathrm{S}^1$. This realization allows embedding Γ_n into $\widetilde{\mathrm{PSL}}(2, \mathbb{R})$ by identifying $f \in \Gamma_n$ to the lifting of $\bar{f}$ to the real line given by $x \mapsto x + \frac{2\pi}{n+1}$, and h to the (unique) lifting h of $\bar{h}$ satisfying $x \leqslant h(x) \leqslant x + 2\pi$ for all $x \in \mathbb{R}$.[3]

The dynamics of the action of $H(n+1)$ on the circle is illustrated in Fig. 4. Here, $\bar{g} = \bar{f}\bar{h}^{-1} = \bar{f}\bar{h}$ is a parabolic Möbius transformation fixing p_0 and sending p_n into p_1, where $p_i = \bar{f}^i(p)$ for $0 \leqslant i \leqslant n$. Using this action, we now proceed to show that no element w in $\langle a, b\rangle^+ \subset \Gamma_n$ represents the identity.

We begin by writing w in the form

$$w = b^{n_0} a^{m_1} b^{n_1} \cdots a^{m_k} b^{n_k} a^{(n+1)r},$$

where $k \geqslant 0$, n_0, n_k, r are non-negative, $n_i > 0$ for $0 < i < k$, and $m_i \in \{1, 2, \ldots, n\}$, with $m_i \neq n$ for $1 < i < k$, and $m_1 \neq n$ (resp. $m_k \neq n$) when $n_0 > 0$ (resp. $n_k > 0$). If w were equal to the identity, then

$$\bar{w} = \bar{g}^{n_0} \bar{f}^{m_1} \bar{g}^{n_1} \cdots \bar{f}^{m_k} \bar{g}^{n_k}$$

would act trivially on the circle. Assume that w is not a power of a. Then one easily checks that, unless $\bar{w}$ is a power of $\bar{f}\bar{g}$, it is conjugate either to some $\bar{w}' \in \langle \bar{f}, \bar{g}\rangle^+$ beginning and ending by $\bar{f}$

[3] Actually, the arguments given so far only show that the above identifications induce a group homomorphism from Γ_n into $\widetilde{\mathrm{PSL}}(2, \mathbb{R})$, and the injectivity follows from the arguments given below combined with the result of Section 2.

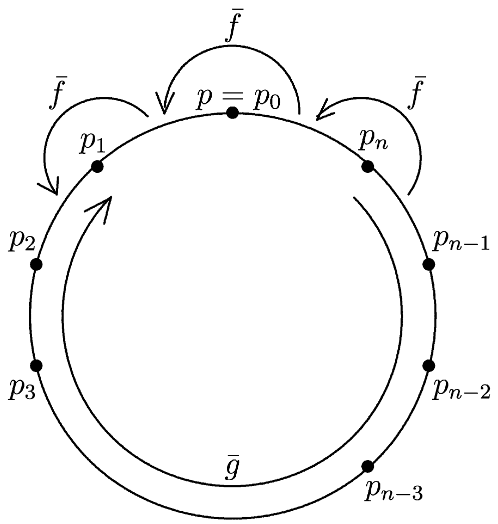

Fig. 4.

and so that all exponents of a lie in $\{1, \ldots, n-1\}$, or to some $\bar{w}'' \in \langle \bar{f}, \bar{g} \rangle^+$ beginning and ending with g with all exponents of a in $\{1, \ldots, n-1\}$. Now, an easy ping-pong type argument shows that $\bar{w}'(]p_0, p_1[) \subset]p_1, p_n[$ and $\bar{w}''(]p_1, p_n[) \subset]p_0, p_1[$, and hence $\bar{w}' \neq id$ and $\bar{w}'' \neq id$. Thus, to conclude the proof, we need to check that neither a nor $\bar{f}\bar{g}$ are torsion elements.

That $\bar{f}\bar{g}$ is not torsion follows from that it sends $[p_0, p_n]$ into the subinterval $[p_1, p_2]$, and hence no iterate of it can equal the identity. Finally, to see that a is not torsion, just notice that a maps to the translation by $\frac{2\pi}{n+1}$ in $\widetilde{\mathrm{PSL}}(2, \mathbb{R})$, and hence has infinite order.[4]

4. Dehornoy-like orderings

In what follows, we will denote by $\preccurlyeq_n$ the left-ordering on Γ_n whose positive cone is $\langle a, b \rangle^+$. Using $\preccurlyeq_n$, we will define an analog of the Dehornoy ordering.

We begin by recalling the Dehornoy ordering on B_3. Consider the Artin (standard) presentation

$$B_3 = \langle \sigma_1, \sigma_2\colon \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \rangle.$$

Following Dehornoy [5], an element of B_3 is said to be 1-positive if it may be written as a word of the form

$$\sigma_2^{n_0}\sigma_1\sigma_2^{n_1}\sigma_1 \cdots \sigma_2^{n_{k-1}}\sigma_1\sigma_2^{n_k},$$

where $n_i \in \mathbb{Z}$. It is said 2-positive if it is of the form σ_2^n for some $n > 0$. An element in B_3 is said to be D-positive if it is either 1-positive or 2-positive. The remarkable result of Dehornoy (in the case of B_3) asserts that the set of D-positive elements is the positive cone of a left-ordering $\preccurlyeq_D$ on B_3. The proof given by Dehornoy as well as many subsequent proofs are very intricate (see [7] for a detailed discussion on this). Nevertheless, a short proof using the ordering $\preccurlyeq_2$ may be given. What follows is inspired from [14] (see Examples 3.35 and 3.36 therein).

Before continuing our discussion, recall that a subgroup Γ_0 of a left-ordered group $(\Gamma, \preccurlyeq)$ is said to be *$\preccurlyeq$-convex* if g belongs to Γ_0 whenever $h_1 \prec g \prec h_2$ for some h_1, h_2 in Γ_0. Convex subgroups are very useful for defining new orders: If $(\Gamma, \preccurlyeq)$ and Γ_0 are as above and $\preccurlyeq'$ is any left-ordering on Γ_0, then the *extension* of $\preccurlyeq'$ by $\preccurlyeq$ is the left-ordering on Γ whose positive cone is

$$P^+ = P^+_{\preccurlyeq'} \cup \left(P^+_{\preccurlyeq} \setminus \Gamma_0\right).$$

[4] This also follows from the main result of [3].

Finally, we denote by $\bar{\preccurlyeq}$ the *reverse ordering* of $\preccurlyeq$, that is, the left-ordering defined by $g \bar{\succ} id$ if and only if $g \prec id$.

Lemma 4.1. *For each $n \in \mathbb{N}$, the subgroup $\langle b \rangle \subset \Gamma_n$ is $\preccurlyeq_n$-convex. Moreover, the only $\preccurlyeq_n$-convex subgroups of Γ_n are $\{id\}$, $\langle b \rangle$, and Γ_n itself.*

Proof. Let $c \in \Gamma_n$ be such that $b^r \prec_n c \prec_n b^s$. Assume that c is $\preccurlyeq_n$-positive (the other case is analogous). If c does not belong to $\langle b \rangle$, then it may be written in the form $w_1 a w_2$, where w_1 and w_2 are (perhaps empty) words on non-negative powers of a and b. The inequality $c \prec_n b^s$ yields $w = b^{-s} w_1 a w_2 \prec_n id$. Introducing the identity $a = ba^2b$ several times, one easily shows that w may be rewritten as $w = w_1' w_2$, where w_1' only uses positive powers of a and b. Thus, w is $\preccurlyeq_n$-positive, which is a contradiction.

To show that the only $\preccurlyeq_n$-convex subgroups of Γ_n are $\{id\}$, $\langle b \rangle$, and Γ_n itself, we proceed by contradiction. Clearly, $\langle b \rangle$ does not contain any nontrivial convex subgroup. Suppose that there exists a $\preccurlyeq_n$-convex subgroup N of Γ_n such that $\langle b \rangle \subsetneq N \subsetneq \Gamma_n$. Let $\preccurlyeq'$, $\preccurlyeq''$, and $\preccurlyeq'''$, be the left-orderings defined on $\langle b \rangle$, N, and Γ_n, respectively, by:

- $\preccurlyeq'$ is the restriction of $\preccurlyeq_n$ to $\langle b \rangle$,
- $\preccurlyeq''$ is the extension of $\preccurlyeq'$ by the restriction of $\bar{\preccurlyeq}_n$ to N,
- $\preccurlyeq'''$ is the extension of $\preccurlyeq''$ by $\preccurlyeq_n$.

The order $\preccurlyeq'''$ is different from $\preccurlyeq_n$ (the $\preccurlyeq_n$-negative elements in $N \setminus \langle b \rangle$ are $\preccurlyeq'''$-positive), but its positive cone still contains the elements a, b. Nevertheless, this is impossible, since these elements generate the positive cone of $\preccurlyeq_n$. □

Now let $\bar{\preccurlyeq}_n$ be the reverse ordering of $\preccurlyeq_n$, and let $\preccurlyeq_n'$ be the ordering of Γ_n obtained as the extension of $\bar{\preccurlyeq}_n$ (restricted to $\langle b \rangle$) by $\preccurlyeq_n$. We claim that, for $n = 2$ (*i.e.* for B_3), $\preccurlyeq_n'$ coincides with the Dehornoy ordering $\preccurlyeq_D$. Indeed, if $c \in \langle b \rangle$ is $\preccurlyeq_2'$-positive, then it is a negative power of $b = \sigma_2^{-1}$, hence $\preccurlyeq_D$-positive. If $c \in B_3 \setminus \langle b \rangle$ is $\preccurlyeq_2'$-positive, then it may be written as a word using only positive powers of a and b. Replacing $a = \sigma_1 \sigma_2$ and $b = \sigma_2^{-1}$, this allows writing c as a word where only positive powers of σ_1 are used. In particular, c is $\preccurlyeq_D$-positive. We thus conclude that the positive cone of $\preccurlyeq_D$ contains that of $\preccurlyeq_2'$. Conversely, if a nontrivial element c is not $\preccurlyeq_2'$-positive, then c^{-1} is $\preccurlyeq_2'$-positive, hence $\preccurlyeq_D$-positive; thus, c is neither $\preccurlyeq_D$-positive. This shows that the positive cone of $\preccurlyeq_2'$ contains that of $\preccurlyeq_D$.

The equivalence between $\preccurlyeq_D$ and $\preccurlyeq_2'$ gives a new proof of Dehornoy's theorem (for B_3). It also motivates the following definition.

Definition 4.2. For each $n \geqslant 2$ the left-ordering $\preccurlyeq_n'$ on Γ_n will be called the *Dehornoy-like ordering* of Γ_n.

As in the case of B_3, an element $c \in \Gamma_n$ is $\preccurlyeq_n'$-positive if either $c = b^{-k}$ for some $k \geqslant 1$, or it may be written in the form

$$c = b^{n_0} a b^{n_1} a \cdots b^{n_{k-1}} a b^{n_k}$$

for some $n_i \in \mathbb{Z}$ (with $k \geqslant 1$). Notice that the smallest positive element of $\preccurlyeq_n'$ is b^{-1}. Moreover, the family of $\preccurlyeq_n'$-convex subgroups of Γ_n coincides with that of $\preccurlyeq_n$-convex ones, that is, $\{id\}, \langle b \rangle, \Gamma_n$ (see [14, Remark 3.34]). The following proposition (and its proof) extends [14, Theorem D].

Proposition 4.3. *The positive cone of the Dehornoy-like ordering $\preccurlyeq_n'$ of Γ_n is not finitely generated as a semigroup.*

Proof. Following [16, Example 8.2], we will show that the sequence of conjugates $b^k a(\preccurlyeq_n')$ converges to $\preccurlyeq_n'$ in a nontrivial way. Here, $b^k a(\preccurlyeq_n')$ is the left-ordering whose positive cone is the conjugate $b^k a P_{\preccurlyeq_n'} (b^k a)^{-1}$ of $P_{\preccurlyeq_n'}$. Saying that $b^k a(\preccurlyeq_n')$ converges to $\preccurlyeq_n'$ in a nontrivial way means that, though $b^k a(\preccurlyeq_n')$ does not coincide with $\preccurlyeq_n'$ for k large enough, given finitely many $\preccurlyeq_n'$-positive elements $c_1, \ldots, c_r$, these elements are also positive with respect to $b^k a(\preccurlyeq_n')$ for k large enough. Such a convergence implies that the positive cone of $\preccurlyeq_n'$ cannot be finitely generated. Indeed, if it were generated by $c_1, \ldots, c_r$, then these elements would be positive for $b^k a(\preccurlyeq_n')$ for k large enough. This would imply that $b^k a(\preccurlyeq_n')$ coincides with $\preccurlyeq_n'$ for large k, which is a contradiction.

If c_i does not belong to $\langle b \rangle$, then c_i may be written in the form $c_i = b^{n_0} a \bar{w}$ for some $n_0 \in \mathbb{Z}$ and a certain $\bar{w}$ containing no negative power of a. We then have

$$\big(b^k a\big)^{-1} c_i b^k a = a^{-1} b^{-k+n_0} a \bar{w} b^k a.$$

For $k > n_0$, the relation $a^{-1} b^{-1} a = a^{n-1} b$ yields

$$\big(b^k a\big)^{-1} c_i b^k a = \big(a^{n-1} b\big)^{k-n_0} \bar{w} b^k a.$$

The right-side expression above contains only positive powers of a, thus showing that c_i is positive with respect to $b^k a(\preccurlyeq_n')$ provided that $k > n_0$.

If c_i belongs to $\langle b \rangle$, then $c_i = b^{-r}$ for some $r \in \mathbb{N}$. This yields

$$\big(b^k a\big)^{-1} c_i b^k a = a^{-1} b^{-k} b^{-r} b^k a = a^{-1} b^{-r} a = \big(a^{n-1} b\big)^r.$$

The right-side expression contains only positive powers of a, hence it is $\preccurlyeq_n'$-positive.

Finally, to show that $b^k a(\preccurlyeq_n')$ and $\preccurlyeq_n'$ do not coincide, it suffices to notice that the smallest positive element of the former ordering, namely $(b^k a)^{-1} b^{-1} b^k a = a^{-1} b^{-1} a$, is different from b^{-1}, which is the smallest positive element of $\preccurlyeq_n'$. □

Remark 4.4. The very same argument of the proof above shows that $b^k a(\preccurlyeq_n)$ also converges to $\preccurlyeq_n'$ as k goes to infinity.

Another relevant property of the Dehornoy ordering on B_3 is the so-called *Property S*: All conjugates of σ_1 and σ_2 are $\preccurlyeq_D$-positive. We were not able to reprove this property with our methods. More importantly, we do not know whether an analog of this property holds for all Dehornoy-like orderings.

5. Some questions and comments

One may address plenty of questions on the structure of the groups Γ_n. However, we would like to focus on certain aspects related to group orderability.

***C*-orderability and local indicability.** Let us recall that a group is said to be *C-orderable* if it admits a left-ordering $\preccurlyeq$ satisfying $fg^k \succ g$ for all f, g positive and all $k \geqslant 2$ (see [14]). Such an ordering is said to be *Conradian*. A remarkable theorem of Brodskii [2] asserts that torsion-free, 1-relator groups are C-orderable. Indeed, such a group is necessarily *locally indicable* [2,12] (that is, each of its finitely generated subgroups surjects into $\mathbb{Z}$), and local indicability is equivalent to C-orderability (see [14, §3] for a discussion on this point).

Now notice that, since the groups Γ_n are left-orderable, they are torsion-free. (This also follows from [3].) By the discussion above, they are C-orderable.[5] For example, the local indicability of $B_3 \sim \Gamma_2$ comes from the well-known exact sequence

$$0 \to [B_3, B_3] \sim F_2 \to B_3 \to B_3/[B_3, B_3] \sim \mathbb{Z} \to 0$$

and the fact that free groups are locally indicable (this last result goes back to Magnus [13]). We point out, however, that the orderings $\preccurlyeq_n$ and $\preccurlyeq'_n$ are not Conradian (for $n > 1$):

- For $\preccurlyeq_n$, notice that $a \succ_n id$ and $b \succ_n id$, though $a^{-1}ba^n = b^{-1} \prec_n id$, thus $ba^n \prec_n a$.
- For $\preccurlyeq'_n$, we have $ab^2 \succ'_n id$ and $ab \succ'_n id$. Now from $a^{-1}ba = b^{-1}a^{-(n-1)}$ we obtain

$$\begin{aligned}(ab)^{-2}\big(ab^2\big)(ab)^4 &= b^{-1}\underline{a^{-1}ba}b(ab)^3 = b^{-1}b^{-1}a^{-(n-1)}b(ab)^3\\ &= b^{-2}a^{-(n-2)}\underline{a^{-1}ba}b(ab)^2\\ &\ \ \vdots\\ &= b^{-2}a^{-(n-2)}b^{-1}a^{-(n-2)}b^{-1}a^{-(n-2)}b^{-1}a^{-(n-1)}b \prec'_n id,\end{aligned}$$

hence $ab^2((ab)^2)^2 \prec'_n (ab)^2$.

As a more sophisticated argument, let us mention Conradian orderings with finitely many convex groups (*cf.* Lemma 4.1) may only exist on solvable groups (see for instance [15, §1.3]), and the groups Γ_n (with $n > 1$) are non-amenable (to see this, just notice that the actions on the circle constructed in Section 3 have no invariant probability measure).

Other positive cones generated by two elements. It is interesting to compare the groups Γ_n with the Baumslag–Solitar groups $BS_{1,n} = \langle a, b\colon\ b^{-1}a^n b = a\rangle$. Indeed, $BS_{1,n}$ is locally indicable (hence C-orderable), admits uncountably many left-orderings, but only four C-orderings (all of which are bi-invariant). Actually, this is nearly a characterization of these groups (see [18]). This gives some "evidence" for a positive answer to the following

Main Question. Let Γ be a group admitting a left-ordering whose positive cone is generated by (no more than) two elements. Is Γ isomorphic to either $\mathbb{Z}$ or Γ_n for some $n \geqslant 1$?

Notice that the group $\Gamma_{m,n} = \langle a, b\colon\ ba^n b = a^m\rangle$ is isomorphic to Γ_{m+n-1} for all positive m, n, thus it belongs to the family above. Indeed, the relator of $\Gamma_{m,n}$ may be written as $(ba^{n-1})^{-1}a^{m+n-1} \times (ba^{n-1})^{-1} = a$.

Positive cones generated by $k > 2$ elements. According to [9], for each $n \geqslant 1$, the braid group B_n admits a left-ordering whose positive cone is generated by $n-1$ elements, namely

$$\sigma_1\sigma_2\cdots\sigma_{n-1}, (\sigma_2\sigma_3\cdots\sigma_{n-1})^{-1}, \sigma_3\sigma_4\cdots\sigma_{n-1}, \ldots, (\sigma_{n-1})^{(-1)^n}.$$

Once again, the proof of this fact given in [9] uses Dehornoy's theory. We were not able to extend our approach to simplify and/or generalize this phenomenon. One of the difficulties lies in that, with the generators above, the natural presentations of B_n are not Garside. We expect, however, that some alternative approach should yield an answer for the following

[5] Notice that the C-orderability of Γ_n does not follows from that it embeds into $\widetilde{\mathrm{PSL}}(2,\mathbb{R})$. Indeed, $\widetilde{\mathrm{PSL}}(2,\mathbb{R})$ contains finitely generated groups with trivial first cohomology, as for example the lifting of the $(2,3,7)$-triangle group [1,20].

Main Problem. For each $k > 3$, find an infinite family of groups (including both B_{k-1} and the Tararin groups T_k from [18, §4.2]) all of which admit left-orderings with a positive cone generated (as a semigroup) by k elements.

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