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ON THE DYNAMICS OF (LEFT) ORDERABLE GROUPS

by Andrés NAVAS

ABSTRACT. — We develop dynamical methods for studying left-orderable groups as well as the spaces of orderings associated to them. We give new and elementary proofs of theorems by Linnell (if a left-orderable group has infinitely many orderings, then it has uncountably many) and McCleary (the space of orderings of the free group is a Cantor set). We show that this last result also holds for countable torsion-free nilpotent groups which are not rank-one Abelian. Finally, we apply our methods to the case of braid groups. In particular, we show that the positive cone of the Dehornoy ordering is not finitely generated as a semigroup. To do this, we define the Conradian soul of an ordering as the maximal convex subgroup restricted to which the ordering is Conradian, and we elaborate on this notion.

RÉSUMÉ. — Nous développons des méthodes dynamiques pour étudier les groupes ordonnables ainsi que leurs espaces d'ordres associés. Nous donnons des preuves nouvelles et élémentaires de théorèmes dus à Linnell (si un groupe ordonnable possède une infinité d'ordres, alors il possède une infinité non dénombrable) et McCleary (l'espace des ordres du groupe libre est un Cantor). Nous montrons que ce dernier résultat est valable aussi pour les groupes nilpotents dénombrables et sans torsion qui ne sont pas abéliens de rang un. Finalement, nous appliquons nos méthodes au cas des groupes de tresses. En particulier, nous démontrons que le cone positif de l'ordre de Dehornoy n'est pas de type fini en tant que semi-groupe. Pour ce faire, nous définissons le noyau conradien d'un ordre comme étant le plus grand sous-groupe convexe sur lequel la relation est conradienne, et nous travaillons avec cette notion.

Introduction

The theory of orderable groups (that is, groups admitting a left-invariant total order relation) is a well developed subject in group theory whose starting points correspond to seminal works by Dedekind and Hölder at the end of the nineteenth century and the beginning of the twentieth century, respectively. Starting from the fifties, this theory was strongly pursued by

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several mathematical schools. Widely known modern references for all of this are the books [3] and [33]. (We should point out that, in general, this theory is presented as a particular subject of the much bigger one of lattice-orderable groups [16, 24, 32].) In the recent years, the possibility of ordering many interesting groups (Thompson's group F [50], braid groups [18], mapping class groups of punctured surfaces with boundary [58], fundamental groups of some hyperbolic 3-dimensional manifolds [4, 9, 15, 57], *etc.*), and the question of knowing whether some particular classes of groups can be ordered (higher rank lattices [35, 34, 42], groups with Kazhdan's property (T) [10, 44], *etc.*), have attracted the interest to this area of people coming from different fields in mathematics as low dimensional geometry and topology, combinatorial and geometric group theory, rigidity theory, mathematical logic, and model theory.

Orderable groups have mostly been studied using pure algebraic methods. Nevertheless, the whole theory should have a natural dynamical counterpart. Indeed, an easy and well-known argument shows that every countable orderable group admits a faithful action by orientation-preserving homeomorphisms of the real line; moreover, the converse is true even without the countability hypothesis (see Proposition 2.1). Quite surprisingly, this very simple remark has not been exploited as it should have been, as the following examples show:

— The first example of an orderable group which is non locally indicable is generally attributed to Bergman [2] (see also [62]). This group is contained in $\widetilde{\mathrm{PSL}}(2,\mathbb{R})$, and it corresponds to the universal cover of the $(2,3,7)$-triangle group. Nevertheless, the fact that this group acts on the line and its first cohomology is trivial had been already remarked (almost twenty years before) by Thurston in relation to his famous stability theorem for codimension-one foliations [63].

— A celebrated result by Dehornoy establishes that braid groups B_n are orderable (see for instance [17]). However, readily soon after Dehornoy's work, Thurston pointed out to the mathematical community that the fact that these groups act faithfully on the line had been already noted by Nielsen in 1927 (see for instance the remark at the end of [31]). Indeed, the geometric techniques by Nielsen allow to produce many (left-invariant and total) orders on B_n, and it turns out that one of them coincides with Dehornoy's ordering [58]. We refer the reader to [18] for a nice exposition of all of these ideas.

— In the opposite direction, many results about the existence of invariant Radon measures for actions on the line are closely related to the prior algebraic theory of Conradian orders: See §3.3 for more explanation on this.

This work represents a systematic study of some of the aspects of the theory of orderable groups. This study is done preferably, though not only, from a dynamical viewpoint. In §1, we begin by revisiting some classical orderability criteria, as for instance the decomposition into positive and negative cones. We also recall the construction of the *space of orderings* associated to an orderable group, which corresponds to a (Hausdorff) topological space on which the underlying group acts naturally by conjugacy (or equivalently, by right multiplication). Roughly, two orderings are close if they coincide over large finite subsets. Although the author learned this idea from Ghys almost ten years ago, the first reference on this is Sikora's seminal work [59] (see also [14]). The main issue here is to establish a relationship with a classical criterion of orderability due to Conrad, Fuchs, Loś, and Ohnishi. This approach allows us, in particular, to give a short and simple proof of the known fact that every locally indicable group admits a left-invariant total order satisfying the so called *Conrad property* (*cf.* Proposition 3.11).

In §2, we recall the classical dynamical criterion for orderability of countable groups. After elaborating a deep further on this, we use elementary perturbation type arguments for giving a new proof of the following result first established (in a different context) by McCleary [41].(1)

Theorem A. — *For every integer $n \geqslant 2$, the space of orderings of the free group F_n is homeomorphic to the Cantor set.*

Using a short argument due to Linnell [36], this allows us to answer by the affirmative a question from [12].

Corollary. — *If $\preceq$ is a left-invariant total order relation on F_n (where $n \geqslant 2$), then the semigroup formed by the elements $g \in F_n$ satisfying $g \succ \mathrm{id}$ is not finitely generated.*

In the general case, if the space of orderings of an orderable group is infinite, then it may have a very complicated structure. A quite interesting example illustrating this fact is given by braid groups which, according to a nice construction by Dubrovina and Dubrovin [21], do admit orders that

(1) Added in Proof: Notice that Theorem A was presented as a conjecture in [59]. Although it was already known, we have decided to include our proof here in order to illustrate our methods. Let us point out that Clay has recently shown that the space of orderings of F_n contains points which are recurrent for the dynamics of the conjugacy action and whose orbits are dense, thus straightening Theorem A (see [11]). A dynamical proof of this result (inspired on our dynamical ideas) appears in [56].

are isolated (in the corresponding space of orders). The rest of this work is a tentative approach for studying this type of phenomenon. For this, in §3 we revisit some classical properties for orders on groups. We begin by recalling Hölder's theorem concerning Archimedean orders (*cf.* Proposition 3.3) and free actions on the line (*cf.* Proposition 3.2). In the same spirit, Proposition 3.4 shows (for countable groups) the equivalence of being bi-orderable and admitting *almost free* actions on the line. Very important for our approach is the dynamical counterpart of the Conrad property for left-invariant orders, namely the nonexistence of *crossed elements* (or *resilient orbits*) for the corresponding actions (*cf.* Propositions 3.14 and 3.18). We then define the notion of *Conradian soul* of an order as the maximal convex subgroup such that the restriction of the original order to it satisfies the Conrad property. The pertinence of this concept is showed by providing an equivalent dynamical definition for countable orderable groups (*cf.* Proposition 3.30). Section 3 finishes with a little discussion on the notion of right-recurrence for orders, which has been introduced by Morris-Witte in his beautiful work on amenable orderable groups [43].

In §4, we study of the structure of spaces of orderings for general orderable groups. In §4.1, we begin by using pure algebraic arguments to show that, if $\preceq$ is a Conradian ordering on a group Γ, then $\preceq$ cannot be isolated when Γ has infinitely many orders (*cf.* Proposition 4.1). As a consequence we obtain the following result, which extends [59, Proposition 1.7]. For the statement, recall that the *rank* of a torsion-free Abelian group is the minimal dimension of a vector space over $\mathbb{Q}$ in which the group embeds.

Theorem B. — *The space of orderings of every (non-trivial) countable torsion-free nilpotent group which is not rank-one Abelian is homeomorphic to the Cantor set. Consequently, for each left-invariant total order $\preceq$ on such a group Γ, the semigroup formed by the elements $g \in \Gamma$ satisfying $g \succ \mathrm{id}$ is not finitely generated.*

Continuing in this direction, in §4.2 we use the results of §3.3 to give a very short proof of the fact that, if a left-invariant total order $\preceq$ on a countable group Γ has trivial Conradian soul, then $\preceq$ is not isolated in the space of orderings of Γ (*cf.* Proposition 4.7). Finally, by elaborating on the arguments of §4.1 and §4.2, in §4.3 we give a slightly different (though equivalent) version of a recent result of Linnell.[2]

[2] Added in Proof: This corresponds essentially to [36, Proposition 1.7], and is included in [37]. Let us point out that a different proof covering the case of uncountable groups was subsequently given in [49].

THEOREM C. — *The space of orderings of a countable (orderable) group is either finite or contains a homeomorphic copy of the Cantor set.*

Perhaps more interesting than the statement above are the techniques involved in the proof, which are completely different from those of Linnell. These techniques allow us to identify (and partially understand) a very precise bifurcation phenomenon in some spaces of orderings. Indeed, if an ordering is isolated inside an infinite space of orderings, then its Conradian soul is non-trivial but admits only finitely many orderings. Thus, one can consider the finitely many associated orderings on the group obtained by changing the original one on the Conradian soul and keeping it outside (this procedure of *convex extension* is classical: See §3.3.5). It appears that at least one of these new orderings is an accumulation point of its orbit under the action of the group (*cf.* Proposition 4.9). For instance, for the case of Dubrovina-Dubrovin's ordering on B_3, the Conradian soul is isomorphic to $\mathbb{Z}$, which admits only two different orderings. It turns out that the associated ordering on B_3 is Dehornoy's one. Since the former is isolated in the space of orderings of B_3, this yields to the following result.(3)

THEOREM D. — *Dehornoy's ordering is an accumulation point of its orbit under the right action of B_n. (In other words, this ordering may be approximated by its conjugates.) Consequently, its positive cone is not finitely generated as a semigroup. Moreover, there exists a sequence of conjugates of Dubrovina-Dubrovin's ordering that converges to Dehornoy's ordering as well.*

The rough idea of the proofs of Theorems A, C, and D is that, starting from a left-invariant total order on a countable group, one can induce an action on the line, and from this action one may produce very many new order relations, except for some specific and well understood cases where the group structure is quite particular, and only finitely many orderings exist. Orderable groups appear in this way as a very flexible category despite the fact that, at first glance, it could seem very rigid because the underlying phase space is ordered and 1-dimensional. According to a general principle by Gromov [25], this mixture between flexibility and rigidity should contain some of the essence of the richness of the theory.(4)

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(3) Added in Proof: Subsequent simpler and/or shorter proofs appear in [19] and [51] (see also [47]).

(4) It is important to point out that this remark applies only to left-orderable groups, and not to the very interesting bi-orderable case: This theory remains completely out of reach of our methods. We point out, however, that Theorem C has no analogue in this context, since there exist bi-orderable groups admitting infinite but countably many

We have made an effort to make this article mostly self-contained, with the mild cost of having to reproduce some classical material. Several natural questions are left open. We hope that some of them are of genuine mathematical value and will serve as a guide for future research on the topic.

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1. The space of orderings of an orderable group

An order relation $\preceq$ on a group Γ is *left-invariant* (resp. *right-invariant*) if for all g, h in Γ such that $g \preceq h$ one has $fg \preceq fh$ (resp. $gf \preceq hf$) for all $f \in \Gamma$. The relation is *bi-invariant* if it is simultaneously invariant by the left and by the right. To simplify, we will use the term *ordering* for referring to a left-invariant total order on a group, and we will say that a group Γ is *orderable* (resp. *bi-orderable*) if it admits a total order which is invariant by the left (resp. by the right and by the left simultaneously).(5)

If $\preceq$ is an order relation on a group Γ, we will say that $f \in \Gamma$ is *positive* (resp. *negative*) if $f \succ \mathrm{id}$ (resp. if $f \prec \mathrm{id}$). Note that if $\preceq$ is a total order relation then every non-trivial element is either positive or negative. Moreover, if $\preceq$ is left-invariant and $P^+ = P^+_{\preceq}$ (resp. $P^-_{\preceq} = P^-$) denotes the set of positive (resp. negative) elements in Γ (sometimes called the positive

bi-orderings [7]. Whether there is an analogue of Theorem A for bi-orderings remains as an open question.

(5) Some authors use the term *orderable* for groups admitting a total bi-invariant order, and call *left orderable* the groups that we just call *orderable*.

(resp. negative) cone), then P^+ and P^- are semigroups and Γ is the disjoint union of P^+, P^-, and $\{\mathrm{id}\}$. In fact, one can characterize the orderability in this way: A group Γ is orderable if and only if it contains semigroups P^+ and P^- such that Γ is the disjoint union of them and $\{\mathrm{id}\}$. (It suffices to define $\prec$ by declaring $f \prec g$ when $f^{-1}g$ belongs to P^+.) Moreover, Γ is bi-orderable exactly when these semigroups may be taken invariant by conjugacy (that is, when they are normal subsemigroups).

Example 1.1. — The category of orderable groups include torsion-free nilpotent groups, free groups, surface groups, *etc.* Another relevant example is given by braid groups B_n. Recall that the group B_n has a presentation of the form

$$B_n = \Big\langle \sigma_1, \ldots, \sigma_{n-1} \colon \quad \begin{array}{ll} \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} & \text{for } 1 \leqslant i \leqslant n-2, \\ \sigma_i\sigma_j = \sigma_j\sigma_i & \text{for } |i-j| \geqslant 2 \end{array} \Big\rangle.$$

Following Dehornoy [17], for $i \in \{1, \ldots, n-1\}$ an element of B_n is said to be σ_i-positive if it may be written as a word of the form

$$w_1\sigma_i^{n_1}w_2\sigma_i^{n_2}\cdots w_k\sigma_i^{n_k}w_{k+1},$$

where the w_i are words on $\sigma_{i+1}^{\pm 1}, \ldots, \sigma_{n-1}^{\pm 1}$, and all the exponents n_i are positive. An element in B_n is said to be σ-positive if it is σ_i-positive for some $i \in \{1, \ldots, n-1\}$. The remarkable result by Dehornoy establishes that the set of σ-positive elements form the positive cone of a left-invariant total order $\preceq_D$ on B_n. We will refer to this order as the Dehornoy's ordering.

We remark that, for each $j \in \{2, \ldots, n\}$, the subgroup of B_n generated by $\sigma_j, \sigma_{j+1}, \ldots, \sigma_{n-1}$ is naturally isomorphic to B_{n-j+1} by an isomorphism which respects the corresponding Dehornoy's orderings.

Remark 1.2. — The characterization of orderings in terms of positive and negative cones shows immediately the following: If $\preceq$ is an ordering on a group Γ, then the order $\overline{\preceq}$ defined by $g \overline{\succ} \mathrm{id}$ if and only if $g \prec \mathrm{id}$ is also left-invariant and total.

Given an orderable group Γ we denote by $\mathcal{O}(\Gamma)$ the set of all the orderings on Γ. As it was pointed out to the author by Ghys, the group Γ acts on $\mathcal{O}(\Gamma)$ by conjugacy (or equivalently, by right multiplication): Given an order $\preceq$ with positive cone P^+ and an element $f \in \Gamma$, the image of $\preceq$ under f is the order $\preceq_f$ whose positive cone is fP^+f^{-1}. In other words, one has $g \preceq_f h$ if and only if $fgf^{-1} \preceq fhf^{-1}$, which is equivalent to $gf^{-1} \preceq hf^{-1}$.

Remark 1.3. — If Γ is an orderable group, then the whole group of automorphisms of Γ (and not only the conjugacies) acts on $\mathcal{O}(\Gamma)$. This may be useful for studying bi-orderable groups. Indeed, since the fixed points for the right action of Γ on $\mathcal{O}(\Gamma)$ correspond to the bi-invariant orderings, the group of outer automorphisms of Γ acts on the corresponding *space of bi-orderings*.

The *space of orderings* $\mathcal{O}(\Gamma)$ has a natural (Hausdorff) topology first introduced (and exploited) by Sikora in [59]. A sub-basis of this topology is the family of the sets of the form $U_{f,g} = \{\preceq : f \prec g\}$. Note that the right action of Γ on $\mathcal{O}(\Gamma)$ becomes in this way an action by homeomorphisms. Similarly, the map sending $\preceq$ to $\overline{\preceq}$ from Example 1.2 is a continuous involution of $\mathcal{O}(\Gamma)$. To understand the topology on $\mathcal{O}(\Gamma)$ better, associated to the symbols $-$ and $+$ let us consider the space $\{-,+\}^{\Gamma\setminus\{\mathrm{id}\}}$. We claim that there exists a one-to-one correspondence between the set $\mathcal{O}(\Gamma)$ and the subset $\mathcal{X}(\Gamma)$ of $\{-,+\}^{\Gamma\setminus\{\mathrm{id}\}}$ formed by the functions $\mathrm{sign}\colon \Gamma\setminus\{\mathrm{id}\} \to \{-,+\}$ satisfying:

- For every $g \in \Gamma\setminus\{\mathrm{id}\}$ one has $\mathrm{sign}(g) \neq \mathrm{sign}(g^{-1})$,
- If f, g in $\Gamma\setminus\{\mathrm{id}\}$ are such that $\mathrm{sign}(f) = \mathrm{sign}(g)$, then $\mathrm{sign}(fg) = \mathrm{sign}(f) = \mathrm{sign}(g)$.

Indeed, to each $\preceq$ in $\mathcal{O}(\Gamma)$ we may associate the function $\mathrm{sign}_{\preceq}\colon \Gamma\setminus\{\mathrm{id}\} \to \{-,+\}$ defined by $\mathrm{sign}_{\preceq}(g) = +$ if and only if $g \succ \mathrm{id}$. Conversely, given a function sign with the properties above, we may associate to it the unique order $\preceq_{\mathrm{sign}}$ in $\mathcal{O}(\Gamma)$ which satisfies $f \succ_{\mathrm{sign}} g$ if and only if $\mathrm{sign}(g^{-1}f)$ equals $+$. Now if we endow $\{-,+\}^{\Gamma\setminus\{\mathrm{id}\}}$ with the product topology and $\mathcal{X}(\Gamma)$ with the subspace one, then the induced topology on $\mathcal{O}(\Gamma)$ via the preceding identification coincides with the topology previously defined by prescribing the sub-basis elements. As a consequence, since $\{-,+\}^{\Gamma\setminus\{\mathrm{id}\}}$ is compact and $\mathcal{X}(\Gamma)$ is closed therein, this shows that the topological space $\mathcal{O}(\Gamma)$ is always compact.

The compactness of $\mathcal{O}(\Gamma)$ is by no means a new result. It was first established for countable groups by Sikora [59]. Subsequent proofs covering the case of uncountable groups appear in [14] and [43]. Although our approach is not the simplest possible one, it allows us revisiting some classical orderability criteria essentially due to Conrad, Fuchs, Loś, and Ohnishi (see for instance [3, 24, 33]). This is summarized in Proposition 1.4 below. For the statement, let us consider the following two conditions:

(i) For every finite family of elements $g_1, \ldots, g_k$ which are different from the identity, there exists a family of exponents $\eta_i \in \{-1, 1\}$ such that id does not belong to the semigroup generated by the elements of the form $g_i^{\eta_i}$,

(ii) For every finite family of elements $g_1, \ldots, g_k$ which are different from the identity, there exists a family of exponents $\eta_i \in \{-1, 1\}$

such that id does not belong to the smallest semigroup which simultaneously satisfies the following two properties:

- It contains all the elements $g_i^{\eta_i}$;
- For all f, g in the semigroup, the elements fgf^{-1} and $f^{-1}gf$ also belong to it.

In each case such a choice of the exponents η_i will be said to be *compatible.*

PROPOSITION 1.4. — *A group* Γ *is orderable (resp. bi-orderable) if and only if it satisfies condition* (i) *(resp. condition* (ii)*) above.*

Proof. — The necessity of the conditions (i) or (ii) is clear: It suffices to chose each exponent η_i so that $g_i^{\eta_i}$ becomes a positive element.

To prove the converse claim in case (i), for each finite family $g_1, \ldots, g_k$ of elements in Γ which are different from the identity, and for each compatible choice of exponents $\eta_i \in \{-1, 1\}$, let us consider the (closed) subset $\mathcal{X}(g_1, \ldots, g_k; \eta_1, \ldots, \eta_k)$ of $\{-,+\}^{\Gamma \setminus \{\mathrm{id}\}}$ formed by all of the sign functions which satisfy the following property: One has $\mathrm{sign}(g) = +$ and $\mathrm{sign}(g^{-1}) = -$ for every g belonging to the semigroup generated by the elements $g_i^{\eta_i}$. (It easily follows from the hypothesis that this subset is non-empty.) Now for fixed $g_1, \ldots, g_k$ let $\mathcal{X}(g_1, \ldots, g_k)$ be the union of all the sets of the form $\mathcal{X}(g_1, \ldots, g_k; \eta_1, \ldots, \eta_k)$, where the choice of the exponents η_i is compatible. Note that, if $\{\mathcal{X}_i = \mathcal{X}(g_{i,1}, \ldots, g_{i,k_i}), i \in \{1, \ldots, n\}\}$ is a finite family of subsets of this form, then the intersection $\mathcal{X}_1 \cap \cdots \cap \mathcal{X}_n$ contains the (non-empty) set $\mathcal{X}(g_{1,1}, \ldots, g_{1,k_1}, \ldots, g_{n,1}, \ldots, g_{n,k_n})$, and it is therefore non-empty. Since $\{-,+\}^{\Gamma \setminus \{\mathrm{id}\}}$ is compact, a direct application of the Finite Intersection Property shows that the intersection $\mathcal{X}$ of all the sets of the form $\mathcal{X}(g_1, \ldots, g_k)$ is (closed and) non-empty. It is quite clear that $\mathcal{X}$ is actually contained in $\mathcal{X}(\Gamma)$, and this shows that Γ is orderable.

The case of condition (ii) is similar. We just need to replace the sets $\mathcal{X}(g_1, \ldots, g_k; \eta_1, \ldots, \eta_k)$ by the sets $B\mathcal{X}(g_1, \ldots, g_k; \eta_1, \ldots, \eta_k)$ formed by all of the sign functions satisfying $\mathrm{sign}(g) = +$ and $\mathrm{sign}(g^{-1}) = -$ for every g belonging to the smallest semigroup satisfying simultaneously the following properties:

- It contains all of the elements $g_i^{\eta_i}$;
- For every f, g in the semigroup, the elements fgf^{-1} and $f^{-1}gf$ also belong to it.

□

What is relevant with the previous conditions (i) and (ii) is that they involve only finitely many elements. This shows in particular that the properties of being orderable or bi-orderable are "local", that is, if they are satisfied by every finitely generated subgroup of a group Γ, then they are satisfied by Γ itself. As we have already mentioned, all these facts are well-known. The classical proofs use the Axiom of Choice, and our approach just uses its topological equivalent, namely Tychonov's theorem. This point of view is more appropriate in relation to spaces of orderings. It will be used once again when dealing with Conradian orders, and it will serve to justify the pertinence of Question 3.42.

If Γ is a countable orderable group, then the topology on $\mathcal{O}(\Gamma)$ is metrizable. Indeed, if $\mathcal{G}_0 \subset \mathcal{G}_1 \subset \cdots$ is a complete exhaustion of Γ by finite sets, then we can define the distance between two different orderings $\leqslant$ and $\preceq$ by letting $\mathrm{dist}(\leqslant, \preceq) = e^{-n}$, where n is the maximum non negative integer number such that $\leqslant$ and $\preceq$ coincide on $\mathcal{G}_n$. An equivalent metric dist' is obtained by letting $\mathrm{dist}'(\leqslant, \preceq) = e^{-n'}$, where n' is the maximum non negative integer such that the positive cones of $\leqslant$ and $\preceq$ coincide on $\mathcal{G}_{n'}$, that is, $P_{\leqslant} \cap \mathcal{G}_{n'} = P_{\preceq} \cap \mathcal{G}_{n'}$. One easily checks that these metrics are ultrametric. Moreover, the fact that $\mathcal{O}(\Gamma)$ is compact becomes more transparent in this case.

When Γ is finitely generated, one may choose $\mathcal{G}_n$ as being the ball of radius n with respect to some finite and symmetric system of generators $\mathcal{G}$ of Γ, that is, the set of elements g which can be written in the form $g = g_{i_1} g_{i_2} \cdots g_{i_m}$, where $g_{i_j} \in \mathcal{G}$ and $0 \leqslant m \leqslant n$. (In this case the action of Γ on $\mathcal{O}(\Gamma)$ is by bi-Lipschitz homeomorphisms.) One easily checks that the metrics on $\mathcal{O}(\Gamma)$ resulting from two different finite systems of generators are not only topologically equivalent but also Hölder equivalent. Therefore, according to Theorem A, the following question (suggested to the author by L. Flaminio) makes sense.

Question 1.5. — What can be said about the metric structure (up to Lipschitz equivalence) of the Cantor set viewed as the space of orderings of the free groups F_n? For instance, are the corresponding Hausdorff dimensions positive and finite? If so, what can be said about the supremum or the infimum value of the Hausdorff dimensions when ranging over all finite systems of generators? (Note that using the arguments of [59], one can easily show that the Hausdorff dimension of $\mathcal{O}(\mathbb{Z}^n)$ is equal to zero.)

In general, the study of the dynamics of the action of Γ on $\mathcal{O}(\Gamma)$ should reveal useful information. This is indeed the main idea behind the proof of

Morris-Witte's theorem [43]: See §3.4. Let us formulate two simple questions on this (see also Question 2.7).

Question 1.6. — For which countable orderable groups the action of Γ on $\mathcal{O}(\Gamma)$ is uniformly equicontinuous? The same question makes sense for topological transitivity, or for having a dense orbit.

Question 1.7. — What can be said in general about the space $\mathcal{O}(\Gamma)/\Gamma$? For instance, is the set of isolated orderings modulo the right action of Γ always finite? (Compare [58, Theorem 3.5].)

To close this Section, we recall a short argument due to Linnell [36] showing that if an ordering $\preceq$ on a group Γ is non isolated in $\mathcal{O}(\Gamma)$, then its positive cone is not finitely generated as a semigroup. This shows why the Corollary in the Introduction of this work follows directly from Theorem A.

Proposition 1.8. — *If $\preceq$ is a left-invariant total order on a group Γ and $\preceq$ is non isolated in $\mathcal{O}(\Gamma)$, then the corresponding positive cone is not finitely generated as a semigroup.*

Proof. — If $g_1, \ldots, g_k$ generate $P^+_{\preceq}$, then the only ordering on Γ which coincides with $\preceq$ on any set containing these generators and the identity element is $\preceq$ itself... □

2. The dynamical realization of countable orderable groups

2.1. A dynamical criterion for orderability

The following dynamical criterion for group orderability is classical. We refer to [23] for more details (see also [28] for an extension to the case of partially ordered groups).

Proposition 2.1. — *For every countable group Γ, the following properties are equivalent:*

(i) Γ *acts faithfully on the real line by orientation-preserving homeomorphisms,*

(ii) Γ *is an orderable group.*

Proof. — Assume that Γ acts faithfully by orientation-preserving homeomorphisms of the line. Let us consider a dense sequence (x_n) in $\mathbb{R}$, and let us define $g \prec h$ if for the smallest index n such that $g(x_n) \neq h(x_n)$ one has $g(x_n) < h(x_n)$. One easily checks that $\preceq$ is a total left-invariant order relation. (Note that this direction does not use the countability hypothesis.)

Suppose now that Γ admits a left-invariant total order $\preceq$. Choose a numbering $(g_i)_{i\geqslant 0}$ for the elements of Γ, put $t(g_0) = 0$, and define $t(g_k)$ by induction in the following way: Assuming that $t(g_0), \ldots, t(g_i)$ have been already defined, if g_{i+1} is bigger (resp. smaller) than $g_0, \ldots, g_i$ then put $t(g_{i+1}) = \max\{t(g_0), \ldots, t(g_i)\} + 1$ (resp. $\min\{t(g_0), \ldots, t(g_i)\} - 1$), and if $g_m \prec g_{i+1} \prec g_n$ for some m, n in $\{0, \ldots, i\}$ and g_j is not between g_m and g_n for any $0 \leqslant j \leqslant i$ then let $t(g_{i+1})$ be equal to $(t(g_m) + t(g_n))/2$.

Note that Γ acts naturally on $t(\Gamma)$ by $g(t(g_i)) = t(gg_i)$. It is not difficult to see that this action extends continuously to the closure of the set $t(\Gamma)$. (Compare Lemma 2.8.) Finally, one can extend the action to the whole line by extending the maps g affinely to each interval of the complementary set of $t(\Gamma)$. $\square$

It is worth analyzing the preceding proof carefully. If $\preceq$ is an ordering on a countable group Γ and $(g_i)_{i\geqslant 0}$ is a numbering of the elements of Γ, then we will call the (associated) *dynamical realization* the action of Γ on $\mathbb{R}$ constructed in this proof. It is easy to see that this realization has no global fixed point unless Γ is trivial. Moreover, if f is an element of Γ whose dynamical realization has two fixed points $a < b$ (which may be equal to $\pm\infty$) and has no fixed point in $]a, b[$, then there must exist some point of the form $t(g)$ inside $]a, b[$. Finally, it is not difficult to show that the dynamical realizations associated to different numberings of the elements of Γ are all topologically conjugate. (Compare Lemma 2.8.) Therefore, we can speak of any dynamical property for the dynamical realization without referring to a particular numbering.

More interesting is to analyze the order obtained from an action on the line. First, note that if the dense sequence (x_n) is such that the orbit of the first point x_0 is *free* (that is, one has $g(x_0) \neq x_0$ for all $g \neq \mathrm{id}$), then the tail $(x_n)_{n\geqslant 1}$ of the sequence is irrelevant for the definition of the associated order. This remark is non innocuous since many group actions on the line have free orbits, as the following examples show.

Example 2.2. — Let Γ be the affine group over the rationals (that is, the group of maps of the form $x \mapsto bx + a$, where a, b belong to $\mathbb{Q}$). Clearly, the orbit of every irrational number ε by the natural action of Γ on the line is free. Therefore, we may define an ordering $\preceq_\varepsilon$ on Γ by declaring that $g \succ_\varepsilon \mathrm{id}$ if and only if $g(1/\varepsilon) > 1/\varepsilon$. Note that for $g(x) = bx + a$, this is equivalent to $b + \varepsilon a > 1$. The orderings $\preceq_\varepsilon$ were introduced by Smirnov in [60].

Example 2.3. — As it is well explained in [58], the actions of braid groups on the line constructed using Nielsen's geometrical arguments have (plenty of) free orbits.

Perhaps the most important (and somehow "universal") case of actions with free orbits corresponds to dynamical realizations of left-invariant total orders $\preceq$ on countable groups: The orbit of the point $t(\mathrm{id})$ — and therefore the orbit of each point of the form $t(h)$ — is free, since $g(t(\mathrm{id})) = t(g) \neq t(\mathrm{id})$ for every $g \neq \mathrm{id}$.

The existence of free orbits allows showing that not all actions without global fixed points of (countable) orderable groups appear as dynamical realizations. For instance, this is the case of non-Abelian groups of piecewise-linear homeomorphisms of the line which coincide with translations outside a compact subset, as for example Thompson's group F (see [5]). Indeed, non-trivial commutators in such a group have intervals of fixed points; by suitable conjugacies, the intervals so obtained cover the line, hence no point has free orbit.

Question 2.4. — What are the (countable) orderable groups all of whose actions by orientation-preserving homeomorphisms of the line without global fixed points are semiconjugate to dynamical realizations? (For example, this is the case of the group $(\mathbb{Z}, +)$.)

Question 2.5. — For countable orderable groups, what can be said on the structure of the space of faithful actions on the line up to topological semiconjugacy? (Compare Question 1.7.)

Remark that, for each $g \in \Gamma$, the order relation for which an element $h \in \Gamma$ is positive if and only if $g(t(h)) > t(h)$ is no other thing than the conjugate of $\preceq$ by h^{-1}. Indeed, by construction, the condition $g(t(h)) > t(h)$ is equivalent to $t(gh) > t(h)$, and therefore to $gh \succ h$, that is, to $h^{-1}gh \succ \mathrm{id}$. Letting $h = \mathrm{id}$, this allows to recover the original ordering $\preceq$ from its dynamical realization.

Remark 2.6. — The involution $\preceq \mapsto \overline{\preceq}$ of $\mathcal{O}(\Gamma)$ introduced in Remark 1.2 has also a dynamical interpretation. Indeed, let Γ be a group of orientation-preserving homeomorphisms of the line, and let (x_n) be a dense sequence of points in $\mathbb{R}$. If $\preceq$ is the order on Γ induced from this sequence and $\varphi\colon \mathbb{R} \to \mathbb{R}$ is an orientation-reversing homeomorphism, then the order on Γ induced by the dense sequence $(\varphi(x_n))$ and the action $g \mapsto \varphi \circ g \circ \varphi^{-1}$ corresponds to $\overline{\preceq}$.

In general, the homeomorphisms appearing in dynamical realizations are not smooth. However, according to [20, Théorème D], the dynamical realization of every countable orderable group is topologically conjugate to a group of locally Lipschitz homeomorphisms of the line.

Although faithful actions on the line contain all the algebraic information of the corresponding orderable group, these actions are not always easy to deal with. For instance, according to [20, Proposition 5.7], for a countable orderable group Γ, none of its actions on the line provides relevant probabilistic information when the initial distribution is symmetric (see however [30] for some interesting examples in the non symmetric case; see also [52]). Nevertheless, a probabilistic approach may be useful for the study of the action of Γ on $\mathcal{O}(\Gamma)$. A basic question on this is the following.

Question 2.7. — If Γ is a countable group having infinitely many left-invariant total orders, under what conditions is the space $\mathcal{O}(\Gamma)$ a Γ-boundary (in the sense of [22])?

2.2. On the space of orderings of free groups

A natural strategy for proving Theorem A is the following. Starting with an ordering on the free group F_n, one considers the corresponding dynamical realization. By slightly perturbing the homeomorphisms corresponding to a system of free generators of F_n, one obtains an action on the line of a group which "in most cases" will still be free [23, Proposition 4.5]. From the perturbed action one may induce a new ordering on F_n, which will be near the original one if the perturbation is very small (with respect to the compact-open topology). Finally, in general this new ordering should be different, because if not then the original action would be "structurally stable", and this cannot be the case for free group actions on the line.

To put all these ideas in practice there are some technical difficulties. Although the strategy that we will actually follow uses a similar idea, it does not rely on any genericity type argument. This will allow us to provide an elementary and self-contained proof for Theorem A.

Recall that given two faithful actions $\phi_i\colon \Gamma \to \mathrm{Homeo}_+(\mathbb{R})$, $i \in \{1,2\}$, the action ϕ_2 is said to be *topologically semiconjugate* to ϕ_1 if there exists a continuous non-decreasing surjective map $\varphi\colon \mathbb{R} \to \mathbb{R}$ such that $\phi_1(g) \circ \varphi = \varphi \circ \phi_2(g)$ for every $g \in \Gamma$. The following criterion will allow us to distinguish two orderings obtained from actions on the line.

Lemma 2.8. — *Let $\preceq$ be an ordering on a non-trivial countable group Γ, and let ϕ_1 be the action corresponding to a dynamical realization of $\preceq$. Let*

ϕ_2 be an action of Γ by orientation-preserving homeomorphisms of the line for which there is no global fixed point and such that the orbit of the origin is free. If $\preceq'$ denotes the ordering on Γ induced from the ϕ_2-orbit of the origin, then $\preceq$ and $\preceq'$ coincide if and only if ϕ_2 is topologically semiconjugate to ϕ_1.

Proof. — If ϕ_2 is topologically semiconjugate to ϕ_1, then the relative positions of the points in $\{\phi_i(g),\ g \in \Gamma\}$ are the same for $i = 1$ and $i = 2$. From this one easily concludes that the induced orderings $\preceq$ and $\preceq'$ coincide.

Conversely, if $\preceq$ and $\preceq'$ coincide, then we may define a map φ from the ϕ_2-orbit of the origin to the set $t(\Gamma)$ by sending $\phi_2(g)(0)$ to $t(g) = \phi_1(g)(0)$. This map φ is strictly increasing because both conditions $\phi_2(g)(0) > \phi_2(h)(0)$ and $t(g) > t(h)$ are equivalent to $g \succ h$. Moreover, φ satisfies $\phi_1(g) \circ \varphi = \varphi \circ \phi_2(g)$ for every $g \in \Gamma$.

Claim. — *The map φ extends continuously to a non-decreasing map defined on the closure of the ϕ_2-orbit of the origin.*

Indeed, to show that φ has a continuous extension to the closure, it suffices to show that, if two sequences (g_n), (h_n) of elements of Γ, the first of which being strictly increasing and the second strictly decreasing, are such that $\lim_n \phi_2(g_n)(0) = p = \lim_n \phi_2(h_n)(0)$, then the points $a = \lim_n t(g_n)$ and $b = \lim_n t(h_n)$ coincide. Suppose not, and let $\varepsilon = b - a$. Let $n \in \mathbb{N}$ be such that $t(h_n) - b < \varepsilon/3$ and $a - t(g_n) < \varepsilon/3$. Since for each n there exist elements between g_n and h_n, the method of construction of the dynamical realization implies that the midpoint between $t(g_n)$ and $t(h_n)$ must belong to $t(\Gamma)$. By the definition of ε, this midpoint $t(f_1)$ belongs to $]a, b[$. Similarly, the midpoint of between $t(f_1)$ and $t(h_n)$ belongs to $]a, b[\cap t(\Gamma)$, thus it is of the form $t(f_2)$ for some $f_2 \in \Gamma$.

Now, let $f \in \Gamma$ be any element such that $t(f) \in]a, b[$. We have, $t(g_n) < t(f) < t(h_n)$, hence $g_n \prec f \prec h_n$, for all $n \in \mathbb{N}$. As a consequence, $\phi_2(g_n)(0) < \phi(f)(0) < \phi_2(h_n)(0)$. Passing to the limit this yields $\phi(f) = p$. Applying this to the elements $f_1 \neq f_2$, we obtain $\phi_2(f_1)(0) = \phi_2(f_2)(0) = p$. However, this contradicts the fact that the ϕ_2-orbit of the origin is free. Thus, φ extends continuously, and since it is strictly increasing when defined on $\phi_2(\Gamma)(0)$, its extension to the closure of this set is non-decreasing.

Now notice that, if $t(\Gamma)$ is dense in the line, then there is only one way to extending φ into a non-decreasing continuous and surjective map realizing the semiconjugacy. If not, let $]a, b[$ be a connected component of the complementary set of the closure of $t(\Gamma)$. Choosing an arbitrary orientation-preserving homeomorphism between the intervals $[\varphi^{-1}(c), \varphi^{-1}(d)]$ and

$[c,d]$, and extending it to the orbits by Γ of these intervals in an equivariant way, we may enlarge the domain of definition of φ still preserving the semiconjugacy relation $\phi_1(g) \circ \varphi = \varphi \circ \phi_2(g)$. Doing this with all the connected components of the complementary set of the closure of $t(\Gamma)$, we can extend φ to a semiconjugacy from ϕ_2 to ϕ_1 defined on the whole real line. □

During the proof of Theorem A, we will need to approximate a given homeomorphism of the interval by a real-analytic one. Although there exist many results of this type for general compact manifolds with boundary, the one-dimensional version of this fact is elementary.

Lemma 2.9. — *Every orientation-preserving homeomorphism of the interval* $[0,1]$ *can be approximated (in the* sup*-norm) by a sequence of real-analytic orientation-preserving diffeomorphisms.*

Proof. — Let f be an orientation-preserving homeomorphism of $[0,1]$. For each $n \in \mathbb{N}$ let f_n be a C^1 diffeomorphism sending the point i/n into $f(i/n)$, for all $i \in \{0,1,\ldots,n\}$. Such an f_n can be easily constructed by using an interpolation method. Alternatively, one may use piecewise-linear homeomorphisms, and then smoothing the derivative at the break-points by conjugating with (a translate of) the map $x \mapsto \exp(-1/x)$ (see [64]).

Now, for each $n \in \mathbb{N}$, let us consider the derivative $f_n' \colon [0,1] \to \mathbb{R}$ of f_n. This is a continuous function satisfying $f_n'(x) \geqslant \lambda_n$ for some $\lambda_n > 0$ and all $x \in [0,1]$. By the Stone-Weierstrass Theorem, each f_n' can be approximated by a sequence of real-analytic functions (even polynomials) $h_{n,k}$. For k large enough we have $\left|g_{n,k}(x) - f_n'(x)\right| \leqslant \min\{1/n, \lambda_n/2\}$ for all $x \in [0,1]$. We choose such a $k = k_n$, and we let $g_n = g_{n,k_n}$.

By integrating g_n, we obtain a diffeomorphism F_n from $[0,1]$ to a certain interval $[0,y_n]$. Since g_n and f_n' are close and y_n is the total integral of g_n, the sequence (y_n) converges to 1. Thus, by rescaling the image of each F_n, we get the desired sequence of real-analytic diffeomorphisms approximating f. □

We can now proceed to the proof of Theorem A. Let $\preceq$ be an ordering on the free group F_n. Given an arbitrary finite family of positive elements $h_j \in F_n$, where $j \in \{1,\ldots,m\}$, we need to show the existence of a distinct ordering $\preceq'$ on F_n for which all of these elements are still positive. To do this, let us fix a free system of generators $\{g_1,\ldots,g_n\}$ of F_n. Let us also consider the corresponding generators $g_{1,0},\ldots,g_{n,0}$ of a dynamical realization of $\preceq$ associated to a numbering of the elements of F_n starting with id. We first claim that, given $i \in \{1,\ldots,n\}$, there exists a sequence of real-analytic diffeomorphisms $g_{i,k} \in \mathrm{Homeo}_+(\mathbb{R})$ that converges to $g_{i,0}$ in the compact-open topology and such that, for each fixed k, the group Γ_k generated by $g_{1,k}, \ldots, g_{n,k}$ has no global fixed point. Indeed, let us fix a real-analytic diffeomorphism $\varphi \colon \mathbb{R} \to \,]0,1[$. By Lemma 2.9, the conjugate homeomorphisms $\bar{g}_{i,0} = \varphi \circ g_{i,0} \circ \varphi^{-1}$, $i \in \{1, \ldots, n\}$, may be approximated in the strong topology on $[0,1]$ by sequences of real-analytical diffeomorphisms $\bar{g}_{i,k}$ of $[0,1]$. This easily implies that each $g_{i,0}$ may be approximated in the compact-open topology by the sequence of real-analytic diffeomorphisms $g_{i,k} = \varphi^{-1} \circ \bar{g}_{i,k} \circ \varphi$. Finally, by conjugating each of these maps by a very small translation $T_{i,k}$, we may assume that for each fixed $k \in \mathbb{N}$ the maps $g_{i,k}$ have no common fixed point, and therefore the group Γ_k generated by them has no global fixed point in the line.

Case 1. — Passing to a subsequence if necessary, for every k the elements $g_{1,k}, \ldots, g_{n,k}$ satisfy some non-trivial relation.

In this case $\Gamma_k \sim F_n/N_k$ for some non-trivial normal subgroup N_k in F_n. Let us write one of the elements h_j above as a product of the generators of F_n, say $h_j = g_{i_1}^{\eta_1} \cdots g_{i_\ell}^{\eta_\ell}$. If we identify F_n to its dynamical realization (and therefore h_j to $g_{i_1,0}^{\eta_1} \cdots g_{i_\ell,0}^{\eta_\ell}$), then from the fact that $h_j(0) > 0$ and that $(g_{i,k})_k$ converges to g_i in the compact-open topology, one easily deduces that, if k is large enough, then $g_{i_1,k}^{\eta_1} \cdots g_{i_\ell,k}^{\eta_\ell}$ sends the origin into a positive real number. This means that the element in Γ_k corresponding to h_j is positive with respect to any ordering obtained from the action of Γ_k on the line using any dense sequence of points (x_n) starting at the origin. Since this is true for each index $j \in \{1, \ldots, m\}$, for k large enough all of the elements in Γ_k corresponding to the h_j's are simultaneously positive for all of such orderings. Let us fix one of these orderings $\preceq_k'$ on Γ_k, as well as an ordering $\preceq_{N_k}$ on N_k. Denoting by $[h]$ the class modulo N_k of an element $h \in F_n$, let us consider the ordering $\preceq_k^1$ (resp. $\preceq_k^2$) on F_n defined by $h \succ \mathrm{id}$ if and only if $[h] \succ_k' \mathrm{id}$, or if $h \in N_k$ and $h \succ_{N_k} \mathrm{id}$ (resp. $h \prec_{N_k} \mathrm{id}$). The elements h_j are still positive with respect to $\preceq_k^1$ and $\preceq_k^2$ for k large enough. On the other hand, $\preceq_k^1$ and $\preceq_k^2$ are different, because they do not coincide on N_k. Therefore, at least one of them is distinct from $\preceq$, which concludes the proof in this case.

Case 2. — Passing to a subsequence if necessary, for every k the elements $g_{1,k}, \ldots, g_{n,k}$ do not satisfy any non-trivial relation.

We first claim that it is possible to change the $g_{i,k}$'s into homeomorphisms of the real line so that the dynamical realization of F_n is not topologically semiconjugate to the action of Γ_k but the latter group still satisfies the properties above (namely, it has no global fixed point, and for each

$i \in \{1, \dots, n\}$ the maps $g_{i,k}$ converge to $g_{i,0}$ in the compact-open topology). To show this let us first note that, since the $g_{i,k}$'s are topologically conjugate to maps which extend to real analytic diffeomorphism of the closed interval $[0,1]$, they have only finitely many fixed points. Since topological semiconjugacies send fixed points into fixed points for corresponding elements, if one of the generators $g_{1,0}, \dots, g_{n,0}$ of the dynamical realization of $\preceq$ has fixed points outside every compact interval of the line, then this realization cannot be topologically semiconjugate to the action of Γ_k. If the sets of fixed points of the $g_{i,0}$'s are contained in some compact interval, then for each k let us consider an increasing sequence of points $y_l \geqslant 2^l$ which are not fixed by the generators $g_{1,k}, \dots, g_{n,k}$. Let us change $g_{1,k}$ into a homeomorphisms of the real line which coincides with the original one on the interval $[-2^k, 2^k]$ and whose set of fixed points outside $[-2^k, 2^k]$ coincides with the set $\{y_l \colon l \geqslant k\}$. The new maps $g_{1,k}$ still converge to $g_{1,0}$ in the compact-open topology. Moreover, by the choice of the sequence (y_l), there is no global fixed point for the group generated by (the new homeomorphism) $g_{1,k}$ and $g_{2,k}, \dots, g_{n,k}$. Finally, by looking at the sets of fixed points of $g_{1,k}$ and $g_{1,0}$, one easily concludes the nonexistence of a topological semiconjugacy between the action of the (new group) Γ_k and the dynamical realization of $\preceq$.

Now for each k the new homeomorphisms $g_{1,k}, \dots, g_{n,k}$ may satisfy some non-trivial relation. If this is the case for infinitely many $k \in \mathbb{N}$, then one proceeds as in Case 1. If not, then (passing to subsequences if necessary) we just need to consider the following two subcases.

Subcase i. — The orbit of the origin by each Γ_k is free.

For each k we may consider the order relation $\preceq_k$ on $F_n \sim \Gamma_k$ obtained from the corresponding action on the line using the orbit of the origin. A simple continuity argument as before shows that, for k large enough, the elements h_j are $\preceq_k$-positive. On the other hand, since the action of Γ_k is not topologically semiconjugate to the dynamical realization of $\preceq$, Proposition 2.8 implies that $\preceq_k$ and $\preceq$ do not coincide, thus finishing the proof for this case.

Subcase ii. — The orbit of the origin by each Γ_k is non free.

For a fixed k let us consider a positive element $h = g_{i_1}^{\eta_1} \cdots g_{i_\ell}^{\eta_\ell} \in F_n$ of minimal length $\ell = \ell_k$ for which the map $g_{i_1,k}^{\eta_1} \cdots g_{i_\ell,k}^{\eta_\ell}$ fixes the origin (here the exponents η_i belong to $\{-1, 1\}$). By the choice of h, the points $0, g_{i_\ell,k}^{\eta_\ell}(0), g_{i_{\ell-1},k}^{\eta_{\ell-1}} g_{i_\ell,k}^{\eta_\ell}(0), \dots, g_{i_2,k}^{\eta_{i_2}} \cdots g_{i_\ell,k}^{\eta_\ell}(0)$ are two-by-two distinct. By perturbing slightly the generator g_{i_1} near the latter point, we obtain a new group Γ_k' such that the new map $g_{i_1,k}^{\eta_1} \cdots g_{i_\ell,k}^{\eta_\ell}$ corresponding to h

sends the origin into a negative real number, but all of the elements in Γ'_k corresponding to the h_j's still send the origin into positive real numbers. If the generators of Γ'_k satisfy no non-trivial relation, then using any dense sequence of points on the line starting with the origin we may induce a new ordering $\preceq'$ on $F_n \sim \Gamma'_k$ which still satisfies $h_j \succ'$ id, but which is different from $\preceq$ since $h \succ$ id and $h \prec'$ id. If there is some non-trivial relation between the generators of Γ'_k, then one may proceed as in Case 1. This finishes the proof of Theorem A.

Example 2.10. — In contrast to Theorem A, we will see in Examples 3.34 and 3.35 that braid groups admit orderings which are isolated in the corresponding space of orderings (although these spaces contain homeomorphic copies of the Cantor set !).

3. A dynamical approach to some properties of left-invariant orders

3.1. Archimedean orders and Hölder's theorem

The main results of this Section are essentially due to Hölder. Roughly, they state that free actions on the line can exist only for groups admitting an order relation satisfying an *Archimedean* type property. Moreover, these groups are necessarily isomorphic to subgroups of $(\mathbb{R},+)$, and the corresponding actions are semiconjugate to actions by translations.

Definition 3.1. — *A left-invariant total order relation $\preceq$ on a group Γ is said to be Archimedean if for all g,h in Γ such that $g \neq \mathrm{id}$ there exists $n \in \mathbb{Z}$ such that $g^n \succ h$.*

Proposition 3.2. — *If Γ is a group acting freely by homeomorphisms of the real line, then Γ admits a total bi-invariant order which is Archimedean.*

Proof. — Let us consider the left-invariant order relation $\preceq$ in Γ such that $g \prec h$ if $g(x) < h(x)$ for some (equivalently, for all) $x \in \mathbb{R}$. This order relation is total, and since the action is free, one easily checks that it is also right-invariant and Archimedean. □

The converse to the proposition above is a direct consequence to the following one. As we will see in the next Section, the hypothesis of bi-invariance for the order is superfluous: It suffices for the order to be left-invariant (*cf.* Proposition 3.6).

Proposition 3.3. — *Every group admitting a bi-invariant Archimedean order is isomorphic to a subgroup of* $(\mathbb{R},+)$.

Proof. — Assume that a non-trivial group Γ admits a bi-invariant Archimedean order $\preceq$, and let us fix a positive element $f \in \Gamma$. For each $g \in \Gamma$ and each $p \in \mathbb{N}$ let us consider the unique integer $q = q(p)$ such that $f^q \preceq g^p \prec f^{q+1}$.

Claim 1. — *The sequence $q(p)/p$ converges to a real number as p goes to infinite.*

Indeed, if $f^{q(p_1)} \preceq g^{p_1} \prec f^{q(p_1)+1}$ and $f^{q(p_2)} \preceq g^{p_2} \prec f^{q(p_2)+1}$ then

$$f^{q(p_1)+q(p_2)} \preceq g^{p_1+p_2} \prec f^{q(p_1)+q(p_2)+2},$$

and therefore $q(p_1)+q(p_2) \leqslant q(p_1+p_2) \leqslant q(p_1)+q(p_2)+1$. The convergence of the sequence $(q(p)/p)$ to some point in $[-\infty,\infty[$ then follows from a classical lemma on subaditive sequences [40, Page 277]. On the other hand, if we denote by $\phi(g)$ the limit of $q(p)/p$, then for the integer $n \in \mathbb{Z}$ satisfying $f^n \preceq g \prec f^{n+1}$ one has $f^{np} \preceq g^p \prec f^{(n+1)p}$, and therefore

$$n = \lim_{p\to\infty} \frac{np}{p} \leqslant \phi(g) \leqslant \lim_{p\to\infty} \frac{(n+1)p-1}{p} = n+1.$$

Claim 2. — *The map $\phi\colon \Gamma \to (\mathbb{R},+)$ is a group homomorphism.*

Indeed, let g_1, g_2 be arbitrary elements in Γ. Let us suppose that $g_1g_2 \preceq g_2g_1$ (the case where $g_2g_1 \preceq g_1g_2$ is analogous). Since $\preceq$ is bi-invariant, if $f^{q_1} \preceq g_1^p \prec f^{q_1+1}$ and $f^{q_2} \preceq g_2^p \prec f^{q_2+1}$ then

$$f^{q_1+q_2} \preceq g_1^p g_2^p \preceq (g_1g_2)^p \preceq g_2^p g_1^p \prec f^{q_1+q_2+2}.$$

From this one concludes that

$$\phi(g_1)+\phi(g_2) = \lim_{p\to\infty} \frac{q_1+q_2}{p} \leqslant \phi(g_1g_2) \leqslant \lim_{p\to\infty} \frac{q_1+q_2+1}{p} = \phi(g_1)+\phi(g_2),$$

and therefore $\phi(g_1g_2) = \phi(g_1)+\phi(g_2)$.

Claim 3. — *The homomorphism ϕ is one to one.*

Note that ϕ is order preserving, in the sense that if $g_1 \preceq g_2$ then $\phi(g_1) \leqslant \phi(g_2)$. Moreover, $\phi(f) = 1$. Let h be an element in Γ such that $\phi(h) = 0$. Assume that $h \neq \mathrm{id}$. Then there exists $n \in \mathbb{Z}$ such that $h^n \succeq f$. From this one concludes that $0 = n\phi(h) = \phi(h^n) \geqslant \phi(f) = 1$, which is absurd. Therefore, if $\phi(h) = 0$ then $h = \mathrm{id}$, and this concludes the proof. □

If Γ is an infinite group acting freely on the line, then we can fix the order relation introduced in the proof of Proposition 3.2. This order allows us to construct an embedding ϕ from Γ into $(\mathbb{R},+)$. If $\phi(\Gamma)$ is isomorphic to $(\mathbb{Z},+)$ then the action of Γ is conjugate to the action by integer translations. In the other case, the group $\phi(\Gamma)$ is dense in $(\mathrm{R},+)$. For each point x in the line we define

$$\varphi(x) = \sup\big\{\phi(h) \in \mathbb{R} \colon h(0) \leqslant x\big\}.$$

It is easy to see that $\varphi\colon \mathbb{R} \to \mathbb{R}$ is a non-decreasing map. Moreover, it satisfies the equality $\varphi(h(x)) = \varphi(x) + \phi(h)$ for all $x \in \mathbb{R}$ and all $h \in \Gamma$. Finally, φ is continuous, as otherwise $\mathbb{R} \setminus \varphi(\mathbb{R})$ would be a non-empty open set invariant by the translations of $\phi(\Gamma)$, which is impossible.

To summarize, if Γ is a group acting freely on the line, then its action semiconjugates to an action by translations.

3.2. Almost free actions and bi-invariant orders

We will say that the action of a group Γ of orientation-preserving homeomorphisms of the line is *almost free* if for every element $g \in \Gamma$ one has either $g(x) \geqslant x$ for all $x \in \mathbb{R}$ or $g(x) \leqslant x$ for all $x \in \mathbb{R}$. The following proposition gives the algebraic counterpart of this notion.

Proposition 3.4. — *A countable group Γ admits a faithful almost free action on the real line if and only if it is bi-orderable.*

Proof. — If Γ is bi-orderable, then the action on the line of the dynamical realization associated to any of its numberings is almost free. Indeed, if $g \succ \mathrm{id}$ then $gg_i \succ g_i$ for all $g_i \in \Gamma$, and therefore $g(t(g_i)) = t(gg_i) > t(g_i)$. By the construction of the dynamical realization, this implies that $g(x) \geqslant x$ for all $x \in \mathbb{R}$. In an analogous way, for $g \prec \mathrm{id}$ one has $g(x) \leqslant x$ for all $x \in \mathbb{R}$, thus showing that the action is almost free.

Conversely, let Γ be a group of homeomorphisms of the line whose action is almost free. We claim that the order $\preceq$ associated to any dense sequence (x_n) of points in $\mathbb{R}$ is bi-invariant. Indeed, if $f \succeq \mathrm{id}$, then the graph of f does not have any point below the diagonal. Obviously, if g is any element in Γ, then the same is true for the graph of gfg^{-1}. This clearly implies that $gfg^{-1} \succeq \mathrm{id}$, thus proving the bi-invariance of $\preceq$. $\square$

Example 3.5. — Groups of piecewise-linear homeomorphisms of the interval are bi-orderable: It suffices to define $\preceq$ by $f \succ \mathrm{id}$ when $f(x_f + \varepsilon) > x_f + \varepsilon$ for every $\varepsilon > 0$ sufficiently small, where $x_f = \inf\{x \colon f(x) \neq x\}$. As an application of the previous proposition, we obtain for example a non standard action of Thompson's group F on the line. (Compare [50].) A similar construction applies to countable groups of germs at the origin of one dimensional real-analytic diffeomorphisms.

To close this Section, we give a dynamical proof of a fact first remarked by Conrad in [13].

Proposition 3.6. — *Every Archimedean left-invariant total order on a group is bi-invariant.*

Proof. — Let $\{f_1, \ldots, f_k\}$ be any finite family of elements in a group Γ endowed with a total order relation $\preceq$ which is left-invariant and Archimedean. Let us consider some numbering $(h_n)_{n \geqslant 0}$ of the group generated by them, as well as the corresponding dynamical realization. We claim that this action is free. Indeed, if not then there exist $h \in \langle f_1, \ldots, f_k \rangle$ and an interval $]a, b[$ which is not the whole line such that h fixes a and b and has no fixed point in $]a, b[$. By the comments after Proposition 2.1, a moment reflexion shows that such an interval $]a, b[$ can be taken so that $b \neq +\infty$. Moreover, there exists some point of the form $t(h_i)$ inside $]a, b[$, and by conjugating by h_i if necessary, we may assume that $t(\mathrm{id})$ belongs to $]a, b[$. Now since dynamical realizations of non-trivial orderable groups have no global fixed point, there must exist some $\bar{h} \in \langle f, g \rangle$ such that $\bar{h}(t(\mathrm{id})) > b$. We thus have $h^n(t(\mathrm{id})) < b < \bar{h}(t(\mathrm{id}))$ for all $n \in \mathbb{Z}$, which implies that $h^n \prec \bar{h}$ for all $n \in \mathbb{Z}$. Nevertheless, this violates the Archimedean property for $\preceq$.

Now let $f \prec g$ and h be three elements in Γ. Since the dynamical realization associated to the group generated by them is free and $f(t(\mathrm{id})) < g(t(\mathrm{id}))$, one has $f(t(h)) < g(t(h))$, that is, $t(fh) < t(gh)$. By construction, this implies that $fh \prec gh$. Since $f \prec g$ and h were arbitrary elements of Γ, this shows that $\preceq$ is right-invariant. □

3.3. The Conrad property and crossed elements (resilient orbits)

3.3.1. The Conrad property

A left-invariant total order relation $\preceq$ on a group Γ satisfies the *Conrad property* (or it is a Conradian order, or simply a $\mathcal{C}$-order) if for all positive elements f, g there exists $n \in \mathbb{N}$ such that $fg^n \succ g$. If a group admits such an order, then it is said to be Conrad orderable. These notions were introduced in [13], where several characterizations are given (see also [3, 24, 33]). Nevertheless, the following quite simple (and unexpectedly useful) proposition does not seem to appear in the literature.

Proposition 3.7. — *If $\preceq$ is a $\mathcal{C}$-order on a group Γ, then for every positive elements f, g one has $fg^2 \succ g$.*

Proof. — Suppose that two positive elements f, g for an ordering $\preceq'$ on Γ are such that $fg^2 \preceq' g$. Then $(g^{-1}fg)g \preceq'$ id, and since g is a positive element this implies that $g^{-1}fg$ is negative, and therefore $fg \prec' g$. Now for the positive element $h = fg$ and every $n \in \mathbb{N}$ one has

$$\begin{aligned} fh^n = f(fg)^n = f(fg)^{n-2}(fg)(fg) \prec' f(fg)^{n-2}(fg)g \\ = f(fg)^{n-2}fg^2 \preceq' f(fg)^{n-2}g = f(fg)^{n-3}fg^2 \preceq' f(fg)^{n-3}g \preceq' \cdots \\ \preceq' f(fg)g = ffg^2 \preceq' fg = h. \end{aligned}$$

This shows that $\preceq'$ does not satisfy the Conrad property. □

The nice argument of the proof above is due to Jiménez [29]. Latter in §3.3.3 we will see that, in fact, $fg^{n+1} \succ g^n$ for all $n \in \mathbb{N}$. More generally, we will show that if $W(f,g) = f^{m_1}g^{n_1} \cdots f^{m_k}g^{n_k}$ is a word such that $\sum m_i > 0$ and $\sum n_i > 0$, then $W(f,g)$ is a positive element in Γ provided that f and g are both positive. (Notice that $fg^{n+1} \succ g^n$ is equivalent to $g^{-n}fg^{n+1} \succ$ id.) However, we were not able to extend the preceding proof for this, and we will need the dynamical characterization of the Conrad property (or at least its algebraic counterpart, which corresponds to the characterization in terms of convex subgroups: See Remark 3.26).

As a first application of Proposition 3.7 we will show that, for every orderable group, the subset of $\mathcal{O}(\Gamma)$ formed by the Conradian orders is closed. Note that a similar argument to the one given below applies to the (simpler) case of bi-invariant orders. (Compare [59, Proposition 2.1].)

Proposition 3.8. — *If Γ is an orderable group, then the set of $\mathcal{C}$-orders on Γ is closed in $\mathcal{O}(\Gamma)$.*

Proof. — According to Proposition 3.7, an element $\preceq$ of $\mathcal{O}(\Gamma)$ is not Conradian if and only if there exists two elements $f \succ$ id and $g \succ$ id such that $fg^2 \preceq g$, which necessarily implies that $g^{-1}fg^2 \prec$ id. Since the sets $U_{\mathrm{id},f}$, $U_{\mathrm{id},g}$, and $U_{\mathrm{id},g^{-2}f^{-1}g}$, are clopen, the set

$$U(f,g) = U_{\mathrm{id},f} \cap U_{\mathrm{id},g} \cap U_{\mathrm{id},g^{-2}f^{-1}g} = \left\{\preceq : f \succ \mathrm{id}, g \succ \mathrm{id}, g^{-1}fg^2 \prec \mathrm{id}\right\}$$

is open for every f, g in Γ different from the identity. Thus, the union of the $U(f,g)$'s is open, and therefore its complementary set (that is, the set of $\mathcal{C}$-orders) is closed. □

Question 3.9. — What can be said about the topology of the set of Conradian orders? When is the set of Conradian orders open or at least of non-empty interior in $\mathcal{O}(\Gamma)$? (6)

As another application of Proposition 3.7, we give a criterion for Conrad orderability which is similar to those of Proposition 1.4.

PROPOSITION 3.10. — *A group Γ admits a Conradian order if and only if the following condition is satisfied: For every finite family of elements $g_1, \ldots, g_k$ which are different from the identity, there exists a family of exponents $\eta_i \in \{-1, 1\}$ such that* id *does not belong to the smallest semigroup $\langle\langle g_1^{\eta_1}, \ldots, g_k^{\eta_k}\rangle\rangle$ which simultaneously satisfies the following two properties:*

- *It contains all the elements $g_i^{\eta_i}$;*
- *For all f, g in the semigroup, the element $f^{-1}gf^2$ also belongs to it.*

Proof. — The necessity of the condition follows as a direct application of Proposition 3.7 after choosing η_i in such a way that $g_i^{\eta_i}$ is a positive element of Γ. To prove that the condition is sufficient, one proceeds as in the case of Proposition 1.4 by introducing the sets $C\mathcal{X}(g_1 \ldots, g_k; \eta_1, \ldots, \eta_k)$ formed by all the functions sign for which $\mathrm{sign}(g) = +$ and $\mathrm{sign}(g^{-1}) = -$ for each g contained in the semigroup $\langle\langle g_1^{\eta_1}, \ldots, g_k^{\eta_k}\rangle\rangle$. We leave the details to the reader. □

It easily follows from the criterion above that residually Conrad orderable groups are Conrad orderable.(7) As a more interesting application, we give a short proof of a theorem due to Brodskii [6], and independently obtained by Rhemtulla and Rolfsen [54]. For the statement, recall that a group is said to be *locally indicable* if for each non-trivial finitely generated subgroup there exists a non-trivial homomorphism into $(\mathbb{R}, +)$.

PROPOSITION 3.11. — *Every locally indicable group is Conrad orderable.*

Proof. — We need to check that every locally indicable group Γ satisfies the condition of Proposition 3.10. Let $\{g_1, \ldots, g_k\}$ be any finite family of

(6) Added in Proof: This has been partially answered in [55].

(7) Recall that, if P is some group property, then a group Γ is said to be *residually* P if for every $g \in \Gamma \setminus \{\mathrm{id}\}$ there exists a surjective group homomorphism from Γ to a group Γ_g such that the image of g is non-trivial.

elements in Γ which are different from the identity. By hypothesis, there exists a non-trivial homomorphism $\phi_1 \colon \langle g_1, \ldots, g_k \rangle \to (\mathbb{R}, +)$. Let $i_1, \ldots, i_{k'}$ be the indexes (if any) such that $\phi_1(g_{i_j}) = 0$. Again by hypothesis, there exists a non-trivial homomorphism $\phi_2 \colon \langle g_{i_1}, \ldots, g_{i_{k'}} \rangle \to (\mathbb{R}, +)$. Letting $i'_1, \ldots, i'_{k''}$ be the indexes in $\{i_1, \ldots, i_{k'}\}$ for which $\phi_2(g_{i'_j}) = 0$, we may choose a non-trivial homomorphism $\phi_3 \colon \langle g_{i'_1}, \ldots, g_{i'_{k''}} \rangle \to (\mathbb{R}, +)$... Note that this process must finish in a finite number of steps (indeed, it stops in at most k steps). Now for each $i \in \{1, \ldots, k\}$ choose the (unique) index $j(i)$ such that $\phi_{j(i)}$ is defined at g_i and $\phi_{j(i)}(g_i) \neq 0$, and let $\eta_i \in \{-1, 1\}$ be so that $\phi_{j(i)}(g_i^{\eta_i}) > 0$. We claim that this choice of exponents η_i is "compatible". Indeed, for every index j and every f, g for which ϕ_j are defined, one has $\phi_j(f^{-1} g f^2) = \phi_j(f) + \phi_j(g)$. Therefore, $\phi_1(h) \geqslant 0$ for every $h \in \langle\langle g_1^{\eta_1}, \ldots, g_k^{\eta_k} \rangle\rangle$. Moreover, if $\phi_1(h) = 0$, then h actually belongs to $\langle\langle g_{i_1}^{\eta_{i_1}}, \ldots, g_{i_{k'}}^{\eta_{i_{k'}}} \rangle\rangle$. In this case, the preceding argument shows that $\phi_2(h) \geqslant 0$, with equality if and only if $h \in \langle\langle g_{i'_1}^{\eta_{i'_1}}, \ldots, g_{i'_{k''}}^{\eta_{i'_{k''}}} \rangle\rangle$... Continuing in this way, one concludes that $\phi_j(h)$ must be strictly positive for some index j. Thus, the element h cannot be equal to the identity, and this concludes the proof. $\square$

As we will see in §3.3.3, the converse of Proposition 3.11 also holds (*cf.* Proposition 3.16).

3.3.2. Crossed elements, invariant Radon measures, and translation numbers

We say that two orientation-preserving homeomorphisms of the real line are *crossed* on an interval $]a, b[$ if one of them fixes a and b and no other point in $[a, b]$, while the other one sends a or b into $]a, b[$. Here we allow the case where $a = -\infty$ or $b = +\infty$.

If f and g are homeomorphisms of the line which are contained in a group without crossed elements, and if f has a fixed point x_0 which is not fixed by g, then the fixed points of g immediately to the left and to the right of x_0 are also fixed by f. This gives a quite particular combinatorial structure for the dynamics of groups of homeomorphisms of the line without crossed elements. To understand this dynamics better, one can use an extremely useful tool for detecting fixed points of elements, namely the translation number associated to an invariant Radon measure. The Proposition below is originally due to Beklaryan [1]. Here we provide a proof taken from [46, Section 2.1].

Proposition 3.12. — *Let Γ be a finitely generated group of orientation-preserving homeomorphisms of the real line. If Γ has no crossed elements, then Γ preserves a (non-trivial) Radon measure on $\mathbb{R}$ (that is, a measure on the Borelean sets which is finite on the compact subsets of $\mathbb{R}$).*

Proof. — If Γ has global fixed points in $\mathbb{R}$, then the Dirac delta measure on any of such points is invariant by the action. Assume in what follows that the Γ-action on $\mathbb{R}$ has no global fixed point, and take a finite system $\{f_1, \ldots, f_k\}$ of generators for Γ. We first claim that (at least) one of these generators does not have interior fixed points. Indeed, suppose by contradiction that all the maps f_i have interior fixed points, and let $x_1 \in \mathbb{R}$ be any fixed point of f_1. If f_2 fixes x_1, then letting $x_2 = x_1$ we have that x_2 is fixed by both f_1 and f_2. If not, choose a fixed point $x_2 \in \mathbb{R}$ for f_2 such that f_2 does not fix any point between x_1 and x_2. Since f_1 and f_2 are non crossed on any interval, x_2 must be fixed by f_1. Now if x_2 is fixed by f_3, let $x_3 = x_2$; if not, take a fixed point $x_3 \in \mathbb{R}$ for f_3 such that f_3 has no fixed point between x_2 and x_3. The same argument as before shows that x_3 is fixed by f_1, f_2, and f_3. Continuing in this way, we find a common fixed point for all of the generators f_i, and so a global fixed point for the action of Γ, thus giving a contradiction.

Now we claim that there exists a non-empty minimal invariant closed set for the action of Γ on $\mathbb{R}$. To prove this, consider a generator $f = f_i$ without fixed points, fix any point $x_0 \in \mathbb{R}$, and let I be the interval $[x_0, f(x_0)]$ if $f(x_0) > x_0$, and $[f(x_0), x_0]$ if $f(x_0) < x_0$. On the family $\mathcal{F}$ of non-empty closed invariant subsets of $\mathbb{R}$, let us consider the order relation $\preceq$ given by $K_1 \succeq K_2$ if $K_1 \cap I \subset K_2 \cap I$. Since f has no fixed point, every orbit by Γ must intersect the interval I, and so $K \cap I$ is a non-empty compact set for all $K \in \mathcal{F}$. Therefore, we can apply Zorn Lemma to obtain a maximal element for the order $\preceq$, and this element is the intersection with I of a minimal Γ-invariant non-empty closed subset of $\mathbb{R}$.

Consider now the non-empty minimal invariant closed set K obtained above. Note that its boundary ∂K as well as the set of its accumulation points K' are also closed sets invariant by Γ. Because of the minimality of K, there are three possibilities:

Case 1. — $K' = \emptyset$.

In this case, K is discrete, that is, K coincides with the set of points of a sequence $(y_n)_{n\in\mathbb{Z}}$ satisfying $y_n < y_{n+1}$ for all n and without accumulation points inside $\mathbb{R}$. It is then easy to see that the Radon measure $\mu = \sum_{n\in\mathbb{Z}} \delta_{y_n}$ is invariant by Γ.

Case 2. — $\partial K = \emptyset$.

In this case, K coincides with the whole line. We claim that the action of Γ is free. Indeed, if not let $]u,v[$ be an interval strictly contained in $\mathbb{R}$ and for which there exists an element $g \in \Gamma$ fixing $]u,v[$ and with no fixed point inside it. Since the action is minimal, there must be some $h \in \Gamma$ sending a real endpoint of $]u,v[$ inside $]u,v[$; however, this implies that g and h are crossed on $[u,v]$, contradicting our assumption. Now the action of Γ being free, Hölder's theorem implies that Γ is topologically conjugate to a (in this case dense) group of translations. Pulling back the Lebesgue measure by this conjugacy, we obtain an invariant Radon measure for the action of Γ.

Case 3. — $\partial K = K' = K$.

In this case, K is "locally" a Cantor set. Collapsing to a point the closure of each connected component of the complementary set of K, we obtain a topological line on which the original action induces (by semi-conjugacy) an action of Γ. As in the second case, one easily checks that the induced action is free, hence it preserves a Radon measure. Pulling back this measure by the semi-conjugacy, one obtains a Radon measure on $\mathbb{R}$ which is invariant by the original action. $\square$

Recall that for (non necessarily finitely generated) groups of orientation-preserving homeomorphisms of the line preserving a (non-trivial) Radon measure μ, there is an associated *translation number* function $\tau_\mu \colon \Gamma \to \mathbb{R}$ defined by

$$\tau_\mu(g) = \begin{cases} \mu\big([x_0, g(x_0)[\big) & \text{if } g(x_0) > x_0, \\ 0 & \text{if } g(x_0) = x_0, \\ -\mu\big([g(x_0), x_0[\big) & \text{if } g(x_0) < x_0, \end{cases}$$

where x_0 is any point of the line [53]. (One easily checks that this definition is independent of x_0.) The following properties are satisfied (the verification is easy and may be left to the reader):

(i) τ_μ is a group homomorphism;
(ii) $\tau_\mu(g) = 0$ if and only if g has fixed points; in this case, the support of μ is contained in the set of these points;
(iii) τ_μ is trivial if and only if there is no global fixed point for the action of Γ.

Remark 3.13. — For codimension-one foliations, the notion of crossed elements corresponds to that of *resilient leaves* (*feuilles ressort*). In this context, an analogous of Proposition 3.12 holds, but its proof is more difficult and uses completely different ideas (see [20, Théorème E]).

3.3.3. The equivalence

Propositions 3.14 and 3.18 below give the equivalence between the Conrad property and the nonexistence of crossed elements for the actions on the line.

Proposition 3.14. — *Let Γ be a countable group with a $\mathcal{C}$-order $\preceq$. For any numbering $(g_n)_{n\geqslant 0}$ of Γ, the corresponding dynamical realization is a subgroup of $\mathrm{Homeo}_+(\mathbb{R})$ without crossed elements.*

Proof. — The claim is obvious if Γ is trivial; thus, we will assume in the sequel that Γ contains infinitely many elements. Let us suppose that there exist f, g in Γ and an interval $[a,b]$ such that (for their dynamical realizations one has) $\mathrm{Fix}(f) \cap [a,b] = \{a,b\}$ and $g(a) \in]a,b[$ (the case where $g(b)$ belongs to $]a,b[$ is analogous). Changing f by its inverse if necessary, we can suppose that $f(x) < x$ for all $x \in]a,b[$. As we already observed after the proof of Proposition 2.1, there must exist some element $g_i \in \Gamma$ such that $t(g_i)$ belongs to the interval $]a,b[$. Let $j \geqslant 0$ be the index such that $g_j = \mathrm{id}$. By conjugating f and g by the element g_i^{-1} if necessary, we may assume that $t(g_j) = t(\mathrm{id})$ belongs to $]a,b[$. Furthermore, changing g by $f^{-n}g$ for n large enough, we may assume that $g(a) > t(g_j)$. Let us define $c = g(a) \in\,]t(g_j), b[$, and let us fix a point $d \in]c,b[$. Since $gf^n(a) = c$ for all $n \in \mathbb{N}$, and since $gf^n(d)$ converges to $c < d$ as n goes to infinity, for $n \in \mathbb{N}$ sufficiently big the map $h_n = gf^n$ satisfies $h_n(a) > a$, $h_n(d) < d$, $\mathrm{Fix}(h_n) \cap\,]a,d[\, \subset\, [c_n, c'_n]\, \subset\,]c, h_n(d)[$ and $\{c_n, c'_n\} \subset \mathrm{Fix}(h_n)$ for some sequences (c_n) and (c'_n) converging to c by the right. (See Figure 1 below.) Note that each h_n satisfying the preceding properties is positive, because from $h_n\big(t(g_j)\big) > h_n(a) = c > t(g_j)$ one concludes that $t(h_n) > t(\mathrm{id})$, and by the construction of the dynamical realization this implies that $h_n \succ \mathrm{id}$.

Let us fix $m > n$ large enough so that the preceding properties are satisfied for h_m and h_n, and such that $[c_m, c'_m] \subset\,]c, c_n[$. Let us fix $k \in \mathbb{N}$ sufficiently big so that $h_n^k(a) > h_m(c_n)$, and let us define $h = h_n^k$. For each $i \in \mathbb{N}$ one has $h^i\big(t(g_j)\big) \in\,]h_m(c_n), c_n[$, and therefore

$$h_m h^i\big(t(g_j)\big) < h_m(c_n) < h(a) < h\big(t(g_j)\big).$$

Thus, $h_m h^i \prec h$ for each $i \in \mathbb{N}$. Nevertheless, this in contradiction with the Conrad property for the order $\preceq$. $\square$

The reader should note that, for the positive elements h and $\bar{h} = h_m$ that we found, one has $W_1(h,\bar{h}) \prec W_2(h,\bar{h})$ for all reduced words W_1, W_2 in positive powers such that W_1 (resp. W_2) begins with a power of $\bar{h}$ (resp. h). Therefore, the following general characterization for the Conrad property holds: A left-invariant total order relation $\preceq$ on a group Γ is a $\mathcal{C}$-order if and only if for every pair of positive elements f, g in Γ one has $W_1(f,g) \succeq W_2(f,g)$ for some reduced words W_1, W_2 in positive powers such that W_1 (resp. W_2) begins with a power of f (resp. g). This shows in particular that all orderings on an orderable group without free semigroups on two generators are $\mathcal{C}$-orders. (This fact was first proved by Longobardi, Maj, and Rhemtulla in [39].) However, a more transparent argument showing this consists in applying the positive Ping-Pong Lemma to the restrictions of the elements h_m and h to the interval $[c'_m, c_n]$ (see [26], Chapter VII).

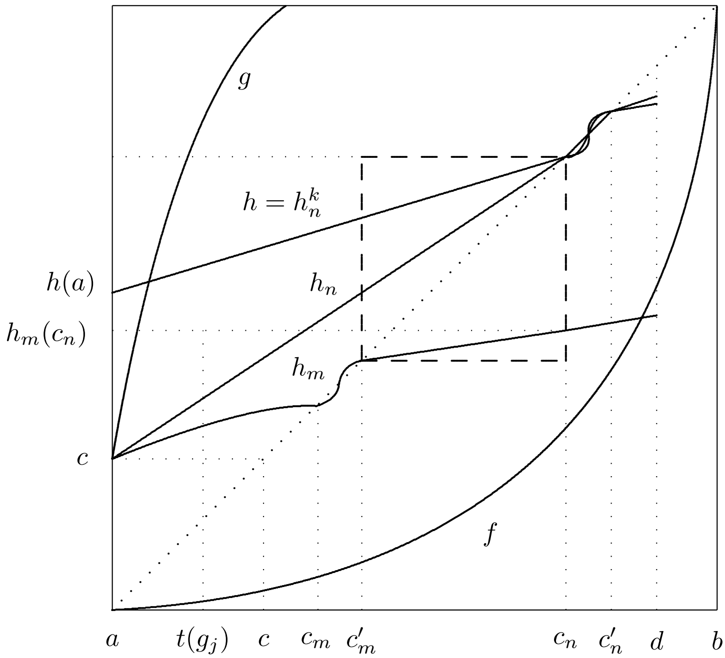

Figure 1

Question 3.15. — What are the orderable groups all of whose orderings are Conradian?

Using Proposition 3.14, one can provide a dynamical proof for the converse of Proposition 3.11. The next proposition is originally due to Conrad [13].

Proposition 3.16. — *Every group admitting a Conradian ordering is locally indicable.*

Proof. — Let Γ be a finitely generated subgroup of a group provided with a Conradian ordering $\preceq$. The restriction of $\preceq$ to Γ is still Conradian. By Proposition 3.14, the dynamical realization of Γ is a group without crossed elements. By Proposition 3.12, this dynamical realization preserves a Radon measure μ. To get a non-trivial homomorphisms from Γ into $(\mathbb{R},+)$, just take the translation number homomorphism associated to μ. $\square$

For another application of Proposition 3.14, recall that, by Thurston's stability theorem, the group $\mathrm{Diff}^1_+([0,1])$ (as well as the group of germs of C^1 diffeomorphisms at the origin) is locally indicable [63]. As a consequence, these groups admit faithful actions on $[0,1]$ without crossed elements.

Remark 3.17. — For interesting obstructions to C^1 smoothing of many actions on the line of some locally indicable groups (as for instance free groups), see [8] and references therein. However, we should point out that the following question remains open: Does there exist a finitely generated locally indicable group having no faithful action by C^1 diffeomorphisms of the interval?[(8)] It is already interesting to know whether surface groups do admit such an action. See also Remark 3.41.

The following is a kind of converse to Proposition 3.14.

Proposition 3.18. — *Let Γ be a subgroup of* $\mathrm{Homeo}_+(\mathbb{R})$ *without crossed elements. If (x_n) is any dense sequence of points in the real line, then the order relation associated to this sequence is a $\mathcal{C}$-order.*

Proof. — Let f and g be two positive elements in Γ, and let Γ_0 be the subgroup generated by them. Let $i \geqslant 0$ and $j \geqslant 0$ be the smallest indexes for which $f(x_i) \neq x_i$ and $g(x_j) \neq x_j$. Assume for instance that $i < j$. (The cases where $i = j$ or $i > j$ are similar and are left to the reader.) Let I be the minimal open interval invariant by Γ_0 and containing x_i. Since Γ does not contain crossed elements, there exists a (non-trivial) Radon measure μ on I which is invariant by Γ_0. Moreover, there is no global fixed point for the action of Γ_0 on it.

By the definition of i and j, one has $f(x_n) = g(x_n) = x_n$ for all $n < i$; moreover, $g(x_i) = x_i$ and $f(x_i) > x_i$. Since f has no fixed point on I, this easily implies that $\tau_\mu(f) > 0$ and $\tau_\mu(g) = 0$. Therefore, $\tau_\mu(g^{-1}fg^2) = \tau_\mu(f) + \tau_\mu(g) = \tau_\mu(f) > 0$, which implies that $g^{-1}fg^2(x) > x$ for all $x \in I$. In particular, $g^{-1}fg^2$ is a positive element of Γ, which shows that $fg^2 \succ g$. $\square$

(8) Added in proof: This has been recently answered by the affirmative in [48].

As an application of the preceding equivalence, we will prove the property concerning positive words in $\mathcal{C}$-ordered groups announced in §3.3.1.

PROPOSITION 3.19. — *Let Γ be any group with a $\mathcal{C}$-order $\preceq$. Let $W(f,g) = f^{m_1}g^{n_1}\cdots f^{m_k}g^{n_k}$ be a word such that $\sum m_i > 0$ and $\sum n_i > 0$. If f and g are positive elements in Γ, then $W(f,g)$ also represents a positive element in Γ.*

Proof. — Let us enumerate the elements of the subgroup Γ_0 generated by f and g, and let us consider the dynamical realization corresponding to this numbering. If τ_μ denotes the translation number function associated to an invariant Radon measure μ, then one has $\tau_\mu(f) \geqslant 0$ and $\tau_\mu(g) \geqslant 0$. Moreover, at least one of these values is strictly greater than zero, as otherwise there would be global fixed points for the dynamical realization. Therefore, denoting $m = \sum m_i > 0$ and $n = \sum n_i > 0$, we have $\tau_\mu(W(f,g)) = m\tau_\mu(f) + n\tau_\mu(g) > 0$, and this implies that $W(f,g)$ is a positive element of Γ. $\square$

Example 3.20. — Dehornoy's ordering is not Conradian (*cf.* Example 1.1). Indeed, for every $i \in \{1,\ldots,n-2\}$ the elements $u = \sigma_i\sigma_{i+1}$ and $v = \sigma_{i+1}$ are positive, but the product

$$\begin{aligned}
u^{-1}v^{-2}u^2v^3 &= \sigma_{i+1}^{-1}\sigma_i^{-1}\sigma_{i+1}^{-2}(\sigma_i\sigma_{i+1}\sigma_i)\sigma_{i+1}\sigma_{i+1}^3\\
&= \sigma_{i+1}^{-1}\sigma_i^{-1}\sigma_{i+1}^{-2}(\sigma_{i+1}\sigma_i\sigma_{i+1})\sigma_{i+1}\sigma_{i+1}^3\\
&= (\sigma_{i+1}^{-1}\sigma_i^{-1}\sigma_{i+1}^{-1})\sigma_i\sigma_{i+1}^5\\
&= (\sigma_i^{-1}\sigma_{i+1}^{-1}\sigma_i^{-1})\sigma_i\sigma_{i+1}^5 = \sigma_i^{-1}\sigma_{i+1}^4
\end{aligned}$$

is negative.

Question 3.21. — Let $W(f,g)$ be a word as in Proposition 3.19. Assume that for an ordering $\preceq$ on a group Γ one has $W(f,g) \succ \mathrm{id}$ for all positive elements f,g. Under what conditions on W one can ensure that $\preceq$ is a $\mathcal{C}$-order? (The reader may easily check that this is for instance the case of $W(f,g) = f^{-1}g^{-1}fgfg$.)

For future reference, we give a slight modification of Proposition 3.18 which involves subgroups of countable groups endowed with a non necessarily Conradian order.

PROPOSITION 3.22. — *Let $\preceq$ be an ordering on a countable group Γ, and let Γ_* be a subgroup of Γ. Let $(g_n)_{n\geqslant 0}$ be any numbering of the elements of Γ starting with $g_0 = \mathrm{id}$. Assume that, for the corresponding dynamical realization of $\preceq$, there exists an interval $]\alpha,\beta[$ containing the* *origin and which is globally fixed by* Γ_*. *If the restriction of* Γ_* *to* $]\alpha,\beta[$ *has no crossed elements, then the order* $\preceq$ *restricted to* Γ_* *is Conradian.*

Proof. — Since for each $g \in \Gamma$ one has $t(g) = g(0)$, for every $g \in \Gamma_*$ the point $t(g)$ must belong to $]\alpha,\beta[$. Moreover, an element $g \in \Gamma$ is positive if and only if $g(0) > 0$. With these facts in mind one may proceed to the proof as in the case of Proposition 3.18. We leave the details to the reader. □

We do not know whether there exists an analogous extension (or modification) of Proposition 3.14. However, in the next Section we will show such an statement under a convexity hypothesis (see Lemma 3.31), and this will be enough for our purposes.

We close this Section with a useful definition.

Definition 3.23. — *Two orientation-preserving homeomorphisms* f, g *of the real line are said to be in transversal position on an interval* $[a,b] \subset \mathbb{R}$ *if* $f(x) < x$ *for all* $x \in]a,b]$ *and* $f(a) = a$, *and* $g(x) > x$ *for all* $x \in [a,b[$ *and* $g(b) = b$.

The reader can easily check that some of the arguments used in the proof of Proposition 3.14 actually show the following.

Proposition 3.24. — *A subgroup of* $\mathrm{Homeo}_+(\mathbb{R})$ *has no crossed elements if and only if it does not contain elements in transversal position.*

3.3.4. The Conradian soul of an order

Let $\preceq$ be a left-invariant total order on a (non necessarily countable) group Γ. A subgroup Γ_* of Γ is said to be *convex* with respect to $\preceq$ (or just $\preceq$-convex) if, for all $f \prec g$ in Γ_*, every element $h \in \Gamma$ satisfying $f \prec h \prec g$ belongs to Γ_*. Equivalently, Γ_* is convex if, for each $f \succ \mathrm{id}$ in Γ_*, every $g \in \Gamma$ such that $\mathrm{id} \prec g \prec f$ belongs to Γ_*.

Example 3.25. — From the definition one easily checks that, for each $n \geqslant 2$ and each $j \in \{1, \ldots, n-1\}$, the subgroup $\langle \sigma_j, \ldots, \sigma_{n-1} \rangle \sim B_{n-j+1}$ of B_n is convex with respect to Dehornoy's ordering (*cf.* Example 1.1).

Note that for every ordering $\preceq$ on a group Γ, the family of $\preceq$-convex subgroups coincides with that of $\overline{\preceq}$-convex ones (*cf.* Remark 1.2). A more important (and also easy to check) fact is that this family is linearly ordered (by inclusion). More precisely, if Γ_0 and Γ_1 are $\preceq$-convex, then either $\Gamma_0 \subset \Gamma_1$ or $\Gamma_1 \subset \Gamma_0$. In particular, the union and the intersection of any family of convex subgroups is a convex subgroup.

Remark 3.26. — Let $\preceq$ be an ordering on a group Γ. For each non-trivial element $g \in \Gamma$ one may define Γ_g (resp. Γ^g) as the largest (resp. smallest) convex subgroup which does not contain g (resp. which contains g). It turns out that $\preceq$ is Conradian if and only if for each $g \neq \mathrm{id}$ the group Γ_g is normal in Γ^g and the order on Γ^g/Γ_g induced by $\preceq$ is Archimedean (and in particular the quotient Γ^g/Γ_g is torsion-free Abelian), see [3, 24, 33]. The reader should note a close relationship between this characterization and the dynamical one given in the previous Section. For instance, a good exercise is to prove Proposition 3.19 using the characterization of $\mathcal{C}$-orders in terms of convex subgroups. (See [29] for more on this.)

We will say that a subgroup Γ_* of Γ is Conradian with respect to an ordering $\preceq$ on Γ (or just $\preceq$-Conradian) if the restriction of $\preceq$ to Γ_* is a $\mathcal{C}$-order. Note that if $\{\Gamma_i\}_{i \in \mathcal{I}}$ is a linearly ordered family of $\preceq$-Conradian subgroups of Γ, then the union $\Gamma_* = \cup_{i \in \mathcal{I}} \Gamma_i$ is still $\preceq$-Conradian. Therefore, the following definition makes sense.

Definition 3.27. — *The* Conradian soul *of* Γ *with respect to* $\preceq$ *(or just the* $\preceq$*-Conradian soul of* Γ*) is the maximal subgroup* $\Gamma^c_{\preceq}$ *of* Γ *which is simultaneously* $\preceq$*-convex and* $\preceq$*-Conradian.*

Example 3.28. — We will see in Example 3.38 that the Conradian soul of B_n with respect to Dehornoy's ordering is the cyclic subgroup generated by σ_{n-1} (*cf.* Examples 1.1 and 3.20).

For the case where Γ is countable, the Conradian soul has a very simple dynamical description. Indeed, fix a numbering $(g_n)_{n \geqslant 0}$ of Γ such that $g_0 = \mathrm{id}$, and for the corresponding dynamical realization define

$$\alpha = \sup\Big\{ b < 0 \colon \text{ there exist } f, g \text{ in } \Gamma \text{ such that } f, g \text{ are crossed on }]a, b[\Big\},$$

$$\beta = \inf\Big\{ a > 0 \colon \text{ there exist } f, g \text{ in } \Gamma \text{ such that } f, g \text{ are crossed on }]a, b[\Big\},$$

where we let $\alpha = -\infty$ (resp. $\beta = +\infty$) if the corresponding set of b's (resp. a's) in $\mathbb{R}$ is empty. Note that the arguments of the proof of Proposition 3.14 show that, in the previous definitions, we can replace "are crossed on $]a, b[$" by "are in transversal position on $[a, b]$" without changing the values of α and β. The following lemma will be implicitly used in what follows, and helps to understand the situation better.

Lemma 3.29. — *The equality* $\alpha = -\infty$ *holds if and and only if* $\beta = +\infty$*. Similarly, one has* $\alpha < 0$ *if and only if* $\beta > 0$*.*

Proof. — Assume that $\beta < +\infty$. Then there exists f, g which are in transversal position on some interval $[a, b]$ satisfying $a \geqslant \beta$. Let $h \in \Gamma$ be such that $h(b) < 0$. Then the elements hfh^{-1} and hgh^{-1} are in transversal position on $[h(a), h(b)]$, and since $h(b) < 0$ this shows that $\alpha > -\infty$. A similar argument shows that the condition $\alpha > -\infty$ implies $\beta < +\infty$.

Now suppose that $\beta = 0$. Then given any $h \succ \mathrm{id}$ there are elements f, g which are in transversal position on an interval $[a, b]$ satisfying $a \in \,]0, t(h)[$. After conjugacy by f^k for $k \in \mathbb{N}$ large enough, we may suppose that the point b also belongs to $]a, t(h)[$. If this is the case, the elements $h^{-1}fh$ and $h^{-1}gh$ are in transversal position on $[h^{-1}(a), h^{-1}(b)] \subset \,]t(h^{-1}), 0[$. Since this construction can be performed for any positive element $h \in \Gamma$, this implies that $\alpha = 0$. A similar argument shows that, if $\alpha = 0$, then $\beta = 0$. □

Note that the equalities $\alpha = -\infty$ and $\beta = +\infty$ hold if and only if $\Gamma^{c}_{\preceq} = \Gamma$, that is, if $\preceq$ is a $\mathcal{C}$-order.

Proposition 3.30. — *With the previous notations, the $\preceq$-Conradian soul of Γ coincides with the stabilizer of the interval $]\alpha, \beta[$.*

To prove this proposition, we will need the following general lemma.

Lemma 3.31. — *Let Γ be a countable group, and let $(g_n)_{n \geqslant 0}$ be a numbering of its elements starting with $g_0 = \mathrm{id}$. Let us consider the dynamical realization associated to an ordering $\preceq$ on Γ and corresponding to this numbering. Suppose that Γ_* is a convex subgroup, and that $]\alpha, \beta[$ is an interval which is fixed by Γ_* and which does not contain any global fixed point of Γ_*. If the restriction of Γ_* to $]\alpha, \beta[$ has crossed elements and $]\alpha, \beta[$ contains the origin, then Γ_* is not $\preceq$-Conradian.*

Proof. — We would like to use similar arguments as those of the proof of Proposition 3.14. Note that those arguments still apply and involve only elements of Γ_*, except perhaps the one concerning the element g_i. More precisely, we need to ensure that an element $g_i \in \Gamma$ such that $t(g_i)$ is in $]a, b[\,\subset\,]\alpha, \beta[$ actually belongs to Γ_*. For this we use the convexity hypothesis. Indeed, since the supermom of the orbit by Γ_* of the origin is a point which is globally fixed by Γ_*, it must coincide with β. In particular, there exists $h_1 \in \Gamma_*$ such that $h_1(0) > t(g_i)$. In an analogous way, one obtains $h_2(0) < t(g_i)$ for some $h_2 \in \Gamma_*$. Now since $h_i(0) = t(h_i)$, this gives $h_2 \prec g_i \prec h_1$. By the convexity of Γ_*, this implies that g_i is contained in Γ_*, thus finishing the proof. □

Now we can pass to the proof of Proposition 3.30. Denote by Γ_* the stabilizer of $]\alpha, \beta[$. We need to verify several facts.

Claim 1. — *The group Γ_* is a $\preceq$-convex subgroup of Γ.*

We first claim that there is no element $h \in \Gamma$ sending α or β into $]\alpha, \beta[$. Indeed, assume that $h(\beta)$ belongs to $]\alpha, \beta[$. (The case $h(\alpha) \in]\alpha, \beta[$ is analogous.) If $h(\beta)$ is in $[0, \beta[$, then let $\varepsilon > 0$ be such that $h([\beta, \beta + \varepsilon]) \subset [0, \beta[$. By the definition of β, there exist $a < b$ and elements f, g in Γ such that $\beta \leqslant a < \beta + \varepsilon$ and such that f, g are in transversal position on $[a, b]$. Changing (if necessary) g by $f^n g f^{-n}$ for n large enough, we may assume that $[a, b]$ is contained in $[\beta, \beta + \varepsilon[$; then changing f by $g^k f g^{-k}$ for k large enough, we may suppose that $[a, b]$ is actually contained in $]\beta, \beta + \varepsilon[$. Now the elements hfh^{-1} and hgh^{-1} are in transversal position on $[h(a), h(b)]$, and since $0 < h(a) < \beta$, this contradicts the definition of β.

When $h(\beta)$ is in $]\alpha, 0[$, the situation is slightly more complicated. Fix $\varepsilon > 0$ such that $h([\beta, \beta + \varepsilon]) \subset \;]\alpha, 0[$. Again by the definition of β, there exist $a < b$ and elements f, g in Γ such that $\beta \leqslant a < \beta + \varepsilon$ and such that f, g are crossed on $]a, b[$, where for concreteness we assume that $Fix(f) \cap [a, b] = \{a, b\}$ and $f(x) < x$ for all $x \in \;]a, b[$. Now refer to Figure 1, where for $m >> n$ big enough the elements h_n and h_m are in transversal position on the interval $[c'_m, c_n]$. Fix $k \in \mathbb{N}$ large enough in such a way $f^k(c_n)$ is near to a so that $h(f^k(c_n)) \in [h(\beta), 0[$. Then the elements $hf^k h_n f^{-k} h^{-1}$ and $hf^k h_m f^{-k} h^{-1}$ are in transversal position on the interval $[hf^k(c'_m), hf^k(c_n)]$, and since $\alpha < h(\beta) < hf^k(c_n) < 0$, this contradicts the definition of α.

Now to conclude the proof of the $\preceq$-convexity of Γ_*, let $h \in \Gamma$ be such that $f \prec h \prec g$ for some elements f, g in Γ_*. We then have $\alpha < t(f) < t(h) < t(g) < \beta$, and therefore $\alpha < h(0) < \beta$. Since both h and h^{-1} do not send neither α nor β into $]\alpha, \beta[$, this easily implies that $h(\alpha) = \alpha$ and $h(\beta) = \beta$. Therefore, h belongs to Γ_*.

Claim 2. — *The restriction of $\preceq$ to Γ_* is Conradian.*

This follows as a direct application of Proposition 3.22.

Claim 3. — *The group Γ_* is a maximal subgroup for the property of being simultaneously $\preceq$-convex and $\preceq$-Conradian.*

Let $\widehat{\Gamma}$ be a convex subgroup of Γ strictly containing Γ_*. Fix a positive element $h \in \widehat{\Gamma} \setminus \Gamma_*$. One has $h(\alpha) \geqslant \beta$, and therefore $h(0) > \beta$. Let $\varepsilon = h(0) - \beta$. As in the proof of Claim 1, there exist f, g in Γ which are in transversal position on an interval $[a, b]$ such that $[a, b] \subset \;]\beta, \beta + \varepsilon[$. We then have

$$t(h) = h(0) = \beta + \varepsilon > t(f) \quad \text{and} \quad t(h) > t(f^{-1}),$$

and similarly $t(h) > t(g)$ and $t(h) > t(g^{-1})$. From the $\preceq$-convexity of $\widehat{\Gamma}$ one easily deduces from this that both elements f and g belong to $\widehat{\Gamma}$. Now the first global fixed point of $\widehat{\Gamma}$ immediately to the right of the origin is to the right of $h(0) \geqslant b$. Therefore, by Lemma 3.31, the subgroup $\widehat{\Gamma}$ is not $\preceq$-Conradian. This proves Claim 3 and finishes the proof of Proposition 3.30.

Remark 3.32. — The reader should have no problem in adapting some of the arguments above to prove that, if Γ is infinite, then $\Gamma^{c}_{\preceq}$ is non-trivial if and only if $\alpha < 0$, which is equivalent to $\beta > 0$.

3.3.5. Extensions of orders and stability of Conradian souls

Let $\preceq$ be an ordering on a group Γ, and let Γ_* be a $\preceq$-convex subgroup of Γ. Let $\preceq_*$ be any (total and left-invariant) order on Γ_*. The *extension of* $\preceq_*$ *by* $\preceq$ is the order relation $\preceq'$ on Γ whose positive cone is $(P^+_{\preceq} \setminus \Gamma_*) \cup P^+_{\preceq_*}$. It is easy to check that $\preceq'$ is also a left-invariant total order relation, and that Γ_* remains convex in Γ (that is, it is a $\preceq'$-convex subgroup of Γ).

Remark 3.33. — With the notations above, one easily checks that the family of $\preceq'$-convex subgroups of Γ is formed by the $\preceq_*$-convex subgroups of Γ_* and the $\preceq$-convex of Γ which contain Γ_*.

The extension procedure is a classical and useful technique which allows for instance to give an alternative approach to the orderings on braid groups introduced by Dubrovina and Dubrovin in [21].

Example 3.34. — Since the cyclic subgroup $\langle\sigma_2\rangle$ is convex in B_3 with respect to Dehornoy's ordering $\preceq_D$ (*cf.* Example 3.25), one can define the order $\preceq_3$ on B_3 as being the extension by $\preceq_D$ of the restriction to $\langle\sigma_2\rangle$ of $\overline{\preceq}_D$ (*cf.* Remark 1.2). We claim that the positive cone of $\preceq_3$ is generated by the elements $u_1 = \sigma_1\sigma_2$ and $u_2 = \sigma_2^{-1}$. Indeed, by definition these elements are positive with respect to $\preceq_3$, and therefore it suffices to show that for every $u \neq \mathrm{id}$ in B_3 either u or u^{-1} belongs to the semigroup $\langle u_1, u_2\rangle^+$ generated by u_1 and u_2. Now if u or u^{-1} is σ_2-positive for Dehornoy's ordering, then there exists an integer $m \neq 0$ such that $u = \sigma_2^m = u_2^{-m}$, and therefore $u \in \langle u_2\rangle^+ \subset \langle u_1, u_2\rangle^+$ if $m < 0$ and $u^{-1} \in \langle u_2\rangle^+ \subset \langle u_1, u_2\rangle^+$ if $m > 0$. If u is σ_1-positive, then for a certain choice of integers $m_1'', \ldots, m_{k''+1}''$ one has

$$u = \sigma_2^{m_1''}\sigma_1\sigma_2^{m_2''}\sigma_1 \cdots \sigma_2^{m_{k''}''}\sigma_1\sigma_2^{m_{k''+1}''}.$$

Using the identity $\sigma_1 = u_1u_2$, this allows us to writte u in the form

$$u = u_2^{m_1'}u_1u_2^{m_2'}u_1 \cdots u_2^{m_{k'}'}u_1u_2^{m_{k'+1}'}$$

for some integers $m'_1, \ldots, m'_{k'+1}$. Now using several times the (easy to check) identity $u_2 u_1^2 u_2 = u_1$, one may express u as a product

$$u = u_2^{m_1} u_1 u_2^{m_2} u_1 \cdots u_2^{m_k} u_1 u_2^{m_{k+1}}$$

in which all the exponents m_i are non negative, and this shows that u belongs to $\langle u_1, u_2 \rangle^+$. Finally, if u^{-1} is σ_1-positive then u^{-1} belongs to $\langle u_1, u_2 \rangle^+$.

Example 3.35. — The generalization of the previous example to all braid groups proceeds inductively as follows. Let us see $B_{n-1} = \langle \tilde{\sigma}_1, \ldots, \tilde{\sigma}_{n-2} \rangle$ as a subgroup of $B_n = \langle \sigma_1, \ldots, \sigma_{n-1} \rangle$ via the monomorphism $\tilde{\sigma}_i \mapsto \sigma_{i+1}$. Via this identification, we obtain from $\preceq_{n-1}$ an order on $\langle \sigma_2, \ldots, \sigma_{n-1} \rangle \subset B_n$, which we still denote by $\preceq_{n-1}$. We then let $\preceq_n$ be the extension of $\overline{\preceq}_{n-1}$ by the Dehornoy's ordering $\preceq_D$. Once again, an important property of $\preceq_n$ is that its positive cone is finitely generated as a semigroup (and therefore, by Proposition 1.8, the ordering $\preceq_n$ is an isolated point of the space of orderings of B_n.) More precisely, letting

$$v_1 = \sigma_1 \sigma_2 \cdots \sigma_{n-1}, \quad v_2 = \sigma_2 \sigma_3 \cdots \sigma_{n-1}, \quad \cdots$$
$$\cdots, \quad v_{n-2} = \sigma_{n-2}\sigma_{n-1}, \quad v_{n-1} = \sigma_{n-1},$$

and $u_i = v_i^{(-1)^{i-1}}$ (where $i \in \{1, \ldots, n-1\}$), the semigroup $P^+_{\preceq_n}$ is generated by the elements $u_1, \ldots, u_{n-1}$. To check this, one proceeds by induction using (as in the case $n = 3$) the remarkable identities

$$\big(u_2 u_3^{-1} \cdots u_{n-1}^{(-1)^{n-1}}\big) u_1^{n-1} \big(u_2 u_3^{-1} \cdots u_{n-1}^{(-1)^{n-1}}\big) = u_1$$

and

$$\big(u_2 u_3^{-1} \cdots u_{n-1}^{(-1)^{n-1}}\big)^2 = u_2^{n-1}.$$

For the sake of clarity, we will denote by $\preceq_{DD}$ the orderings constructed above (called Dubrovina-Dubrovin's orderings in the Introduction).

For countable groups, the extension procedure can be described in pure dynamical terms. Roughly, it corresponds to consider the dynamical realization of $\preceq$, then to change the action of Γ_* on the smallest interval $]\alpha, \beta[$ containing the origin and which is fixed by Γ_* by (a conjugate of) the action associated to a dynamical realization of $\preceq_*$, and then to extend the new action to the whole group Γ in an equivariant way. This approach naturally leads to the following stability type property for Conradian souls: if Γ_* coincides with the $\preceq$-Conradian soul of Γ and $\preceq_*$ is a $\mathcal{C}$-order on Γ_*, then Γ_* also corresponds to the $\preceq'$-Conradian soul of Γ. However, the algebraic presentation of the extension operation being more concise, it allows to give a short proof of this fact which also covers the case of uncountable orderable groups.

Lemma 3.36. — *Let $\preceq$ be an ordering on a group Γ, and let $\preceq_*$ be any left-invariant total order on the $\preceq$-Conradian soul $\Gamma^c_{\preceq}$ of Γ which is still a $\mathcal{C}$-order. If $\preceq'$ denotes the extension of $\preceq_*$ by $\preceq$, then the $\preceq'$-Conradian soul of Γ coincides with $\Gamma^c_{\preceq}$.*

Proof. — Since $\Gamma^c_{\preceq}$ is a convex and Conradian subgroup of Γ with respect to $\preceq'$, we just need to check the maximality property. So let Γ_* be any $\preceq'$-convex subgroup of Γ strictly containing $\Gamma^c_{\preceq}$. We first claim that Γ_* is also $\preceq$-convex. Indeed, assume that $f \prec h \prec g$ for some f, g in Γ_* and $h \in \Gamma$. If either $f^{-1}h$ or $g^{-1}h$ belongs to $\Gamma^c_{\preceq}$ then, since $\Gamma^c_{\preceq}$ is contained in Γ_* and $h = f(f^{-1}h) = g(g^{-1}h)$, the element h belongs to Γ_*. If neither $f^{-1}h$ nor $g^{-1}h$ does belong to $\Gamma^c_{\preceq}$ then, since $\mathrm{id} \prec f^{-1}h$ and $g^{-1}h \prec \mathrm{id}$, one has $\mathrm{id} \prec' f^{-1}h$ and $g^{-1}h \prec' \mathrm{id}$, that is, $f \prec' h \prec' g$. By the $\preceq'$ convexity of Γ_*, this still implies that h is contained in Γ_*, thus showing the $\preceq$-convexity of Γ_*.

Since Γ_* is $\preceq$-convex and strictly contains $\Gamma^c_{\preceq}$, there exist positive elements f, g in Γ_* such that $fg^n \preceq g$ for all $n \in \mathbb{N}$. We claim that g does not belong to $\Gamma^c_{\preceq}$. Indeed, if not then one has $f \notin \Gamma^c_{\preceq}$, and therefore $f^{-1} \prec g$, that is, $fg \succ \mathrm{id}$. Again, since $fg \notin \Gamma^c_{\preceq}$, this implies that $fg \succ g$, which contradicts our choice.

We now claim that, for every $n \geqslant 0$, the element $g^{-1}fg^n$ does not belong to $\Gamma^c_{\preceq}$. Indeed, since g is a positive element not contained in $\Gamma^c_{\preceq}$, if $g^{-1}fg^n$ is in $\Gamma^c_{\preceq}$ then $g \succ (g^{-1}fg^n)^{-1}$, and therefore $g^{-1}fg^{n+1} \succ \mathrm{id}$, contradicting again our choice.

Now we remark that, independently if f does belong or not to $\Gamma^c_{\preceq}$, the element $h = fg$ (is positive and) is not contained in $\Gamma^c_{\preceq}$. Therefore, both g and h are still positive with respect to the ordering $\preceq'$. Moreover, since $g^{-1}fg^n \preceq \mathrm{id}$ and $g^{-1}fg^n \notin \Gamma^c_{\preceq}$ for all $n \geqslant 0$, one necessarily has $g^{-1}hg^n \prec' \mathrm{id}$ for all $n \geqslant 0$. In particular, Γ_* is not a $\preceq'$-Conradian subgroup of Γ. Since this is true for any $\preceq'$-convex subgroup of Γ strictly containing $\Gamma^c_{\preceq}$, this shows that the $\preceq'$-Conradian soul of Γ coincides with $\Gamma^c_{\preceq}$. $\square$

Example 3.37. — The only $\preceq_n$-convex subgroups of B_n are $B^1 = \{\mathrm{id}\}$, $B^2 = \langle u_{n-1} \rangle = \langle \sigma_{n-1} \rangle$, $B^3 = \langle u_{n-2}, u_{n-1} \rangle = \langle \sigma_{n-2}, \sigma_{n-1} \rangle$, $\ldots$, $B^{n-1} = \langle u_2, \ldots, u_{n-1} \rangle = \langle \sigma_2, \ldots, \sigma_{n-1} \rangle$ and $B^n = B_n$. Indeed, suppose that there exists a $\preceq_n$-convex subgroup B of B_n such that $B^i \subsetneq B \subsetneq B^{i+1}$ for some $i \in \{1, \ldots, n-1\}$. Let $\preceq^1$, $\preceq^2$, and $\preceq^3$, be the orderings respectively defined on B^i, B, and B_n, by:

– $\preceq^1$ is the restriction of $\preceq_n$ to B^i,
– $\preceq^2$ is the extension of $\preceq^1$ by the restriction of $\overline{\preceq}_n$ to B,
– $\preceq^3$ is the extension of $\preceq^2$ by $\preceq_n$.

The order $\preceq^3$ is different from $\preceq_n$ (the $\preceq_n$-negative elements in $B \setminus B^i$ are $\preceq^3$-positive), but its positive cone still contains the elements $u_1, \ldots, u_i$, $u_{i+1}, \ldots, u_{n-1}$. Nevertheless, this is impossible, since these elements generate the positive cone of $\preceq_n$.

Note that, by Remark 3.33, the $\preceq_D$-convex subgroups of B_n coincide with the $\preceq_n$-convex subgroups listed above.

Example 3.38. — Since the smallest $\preceq$-convex subgroup strictly containing $\langle \sigma_{n-1} \rangle$ is $\langle \sigma_{n-2}, \sigma_{n-1} \rangle$, and since the restriction of $\preceq_D$ to $\langle \sigma_{n-2}, \sigma_{n-1} \rangle$ is not Conradian (*cf.* Example 3.20), the Conradian soul of B_n with respect to Dehornoy's ordering is the infinite cyclic subgroup generated by σ_{n-1}.

Remark 3.39. — In [58], Short and Wiest study the orderings on braid groups (and more generally on some mapping class groups) which arise from Nielsen's geometrical methods. They define two different families of such orderings, namely those of *finite* and *infinite type*. They distinguish these families by showing that the former ones are *discrete* (that is, there exists a minimal positive element for them), and the latter ones are non discrete. (Dehornoy's ordering belongs to the first family.) It would be nice to pursue a little bit on this point for explicitly determining the Conradian soul in each case.[9]

3.4. Right-recurrent orders

A left-invariant total order relation $\preceq$ on a group Γ is *right-recurrent* if for all positive elements f, g there exists $n \in \mathbb{N}$ such that $gf^n \succ f^n$. Clearly, every such order satisfies the Conrad property, but the converse is not true. Remark that both the sets of $\mathcal{C}$-orders and right-recurrent orders are invariant under the action of Γ by conjugacy.

The property of right-recurrence for left-invariant orders is not so clear as the Conradian property or the bi-invariance. For instance, as the following example shows, there is no analogue of neither Proposition 3.7 nor Proposition 3.8 for right-recurrent orders.

Example 3.40. — Let f be the translation $x \mapsto x + 1$, and let g be any orientation-preserving homeomorphism of the unit interval such that

(9) Added in proof: This has been recently done in [51].

$g(x) > x$ for all $x \in]0,1[$. Fix an increasing sequence (n_i) of non negative integers such that $n_0 = 0$ and such that $n_{2k+1} - n_{2k}$ goes to infinite with k. Extend g into a homeomorphism of the whole line by defining, for $n \in \mathbb{Z}$ and $x \in [n, n+1]$,

$$g(x) = \begin{cases} f^n g f^{-n}(x) & \text{if } n = n_{2k}, \\ f^n g^{-1} f^{-n}(x) & \text{if } n = n_{2k+1}, \\ x & \text{otherwise.} \end{cases}$$

It is not difficult to check that the group Γ generated by f and g is isomorphic to the wreath product $\mathbb{Z} \wr \mathbb{Z}$. For each k let $\preceq_k$ be the order relation on Γ defined by $h_1 \prec_k h_2$ if and only if the minimum integer $i \geqslant n_{2k}$ for which $h_1(i+1/2) \neq h_2(i+1/2)$ is such that $h_1(i+1/2) < h_2(i+1/2)$. One can easily show that each $\preceq_k$ is total, left-invariant, and right-recurrent. (Note that $\preceq_k$ coincides with the image of $\preceq_0$ by $f^{-n_{2k}}$.) Nevertheless, no accumulation point $\preceq$ of the sequence of orders $\preceq_k$ is right-recurrent. Indeed, the elements f and g are positive for all the orders $\preceq_k$. On the other hand, one has $gf^n \prec_k f^n$ for all $n \in \{1, \ldots, n_{2k+1} - n_{2k}\}$, and passing to the limit this gives $gf^n \prec f^n$ for all $n \in \mathbb{N}$.

Although the set of right-recurrent orders is contained in the set of $\mathcal{C}$-orders, it is not necessarily dense therein. (See however Question 3.46.) Indeed, according to [43, Example 4.6], if F is a finite index free subgroup of $\mathrm{SL}(2,\mathbb{Z})$, then the group $\Gamma = F \ltimes \mathbb{Z}^2$ admits no right-recurrent order. However, Γ is locally indicable, and therefore by Proposition 3.11 it admits a $\mathcal{C}$-order. (By Proposition 3.14, it also admits a faithful action on the interval without crossed elements.)

Remark 3.41. — The group Γ above satisfies the relative Kazhdan's property (T) with respect to the *normal* subgroup $\mathbb{Z}^2$. By [45, Théorème A], for no $\varepsilon > 1/2$ this group can act faithfully by $C^{3/2+\varepsilon}$ diffeomorphisms of the interval.[10]

Question 3.42. — Is the property of admitting a rigth-recurrent order a "local" property? (See the comments after the proof of Proposition 1.4.)

Question 3.43. — What are the orderable groups all of whose orderings are right-recurrent? (This should be compared with Question 3.15 as well as Tararin's theorem in §4.1; see also [24, Theorem 6.L])

[10] Added in Proof: This has been recently extended in [63] to actions by C^1 diffeomorphisms.

Somehow related to the preceding question is the following well-known lemma, for which we provide a short proof based on the notion of right-recurrence.

LEMMA 3.44. — *If an orderable group* Γ *admits only finitely many left-invariant total orders, then every element of* $\mathcal{O}(\Gamma)$ *is Conradian.*

Proof. — Since $\mathcal{O}(\Gamma)$ is finite, its points are periodic for the action of every element of Γ. This obviously implies that every order in $\mathcal{O}(\Gamma)$ is right-recurrent, hence Conradian. □

Remark 3.45. — Using Tararin's theorem which describes all orderable groups admitting only finitely many orderings (see §4.1), one can show that every ordering $\preceq$ on such a group satisfies the following: If f is positive and g is any group element, then $fg^2 \succ g^2$. (This should be compared with Proposition 3.7.)

The notion of right-recurrence for left-invariant orders was introduced by Morris-Witte in [43], where he proves that every countable amenable orderable group is locally indicable. Actually, Morris-Witte proves that such a group always admits a right-recurrent ordering. His strategy shows how the dynamical properties of the action of an orderable group on its space of orderings can reveal some of its algebraic properties. His brilliant argument may be summarized as follows:

- Since Γ is amenable and $\mathcal{O}(\Gamma)$ is a compact metric space, the right action of Γ on $\mathcal{O}(\Gamma)$ must preserve a probability measure (see for instance [65]);
- If the right action of a countable orderable group Γ on $\mathcal{O}(\Gamma)$ preserves a probability measure μ, then the set of right-recurrent orderings has full μ-measure, and in particular is non-empty (this follows by applying the Poincaré Recurrence Theorem).

Question 3.46. — If Γ is countable amenable and orderable, is the set of right-recurrent orderings on Γ dense inside the set of $\mathcal{C}$-orders?

Since (countable) amenable groups do not contain free subgroups on two generators, it is natural to ask whether Morris-Witte's theorem is still true under the last (weaker) hypothesis. Partial evidence for an affirmative answer to this question is the result obtained by Linnell in [38]. The (apparently easier) question of the local indicability for orderable groups satisfying a non-trivial law (or identity) is still interesting. For instance, an affirmative answer for this case would allow to conclude that orderable groups satisfying an Engel type identity are locally nilpotent (see [24, Theorem 6.G]).

4. Finitely many or a Cantor set of orders

4.1. The case of Conradian orders

The approximation of Conradian orders is a problem of algebraic nature. In order to deal with it, we will use an elegant result by Tararin [61] (see [33] for a detailed proof). For its statement, recall that a *rational series* for a group Γ is a finite sequence of subgroups

$$\{\mathrm{id}\} = \Gamma^k \subset \Gamma^{k-1} \subset \cdots \subset \Gamma^0 = \Gamma$$

which is *subnormal* (that is, each Γ^i is normal in Γ^{i-1}), and such that each quotient Γ^{i-1}/Γ^i is torsion-free rank-one Abelian. Note that every group admitting a rational series is orderable.

Theorem (Tararin). — *If Γ is a group admitting a rational series*

$$\{\mathrm{id}\} = \Gamma^k \subset \Gamma^{k-1} \subset \cdots \subset \Gamma^0 = \Gamma,$$

then its space of orderings $\mathcal{O}(\Gamma)$ is finite if and only the subgroups Γ^i are normal in Γ and no quotient Γ^{i-2}/Γ^i is bi-orderable. If this is the case, then Γ admits a unique rational series, and for every left-invariant total order on Γ, the convex subgroups are precisely $\Gamma^0, \Gamma^1, \ldots, \Gamma^k$.

Indeed, the number of orderings on a group satisfying the properties above equals 2^k. Moreover, by choosing $g_i \in \Gamma^i \setminus \Gamma^{i-1}$, each of such orderings is uniquely determined by the sequence of *signs* of the elements g_i. Tararin's theorem will be fundamental for establishing the following proposition. (Note that there is no countability hypothesis for the group in the result below.)

Proposition 4.1. — *If Γ is a Conrad orderable group having infinitely many left-invariant total orders, then all neighborhoods in $\mathcal{O}(\Gamma)$ of Conradian orders on Γ do contain homeomorphic copies of the Cantor set.*

To prove this proposition we need to show that, if Γ is an orderable group which admits a Conradian order having a neighborhood in $\mathcal{O}(\Gamma)$ which does not contain any homeomorphic copy of the Cantor set, then Γ admits a rational series as in the statement of Tararin's theorem.

LEMMA 4.2. — *If a $\mathcal{C}$-order $\preceq$ on a group Γ has a neighborhood in $\mathcal{O}(\Gamma)$ which does not contain any homeomorphic copy of the Cantor set, then Γ admits a (finite) subnormal sequence formed by $\preceq$-convex subgroups so that the corresponding successive quotients are torsion-free Abelian.*

Proof. — Since the family of $\preceq$-convex subgroups is completely ordered by inclusion, referring to Remark 3.26 we just need to show that there exist only finitely many distinct subgroups of the form Γ^g. Let $\{f_1,\ldots,f_k\}$ be any finite family of elements of Γ. If there exist infinitely many distinct groups of the form Γ^g, then one may obtain an infinite ascending or descending sequence of these groups Γ^{g_i} in such a way that $f_m^{-1}f_n \notin \Gamma^{g_i} \setminus \Gamma_{g_i}$ for every $m \neq n$ in $\{1,\ldots,k\}$ and every $i \in \mathbb{N}$. Both cases being similar, we will consider only the former one. Following Zenkov [66], for each $i \in \mathbb{N}$ and each $\omega = (\ell_1,\ldots,\ell_i) \in \{0,1\}^i$ let us inductively define the order $\preceq_\omega = \preceq_{(\ell_1,\ldots,\ell_i)}$ on Γ^{g_i} by letting $\preceq_\omega$ be the extension of $\preceq_{(\ell_1,\ldots,\ell_{i-1})}$ by $\preceq$ (resp. by $\overline{\preceq}$) if $\ell_i = 0$ (resp. if $\ell_i = 1$). Passing to the limit, this allows to define a continuous embedding of the Cantor set $\{0,1\}^{\mathbb{N}}$ into the space of orderings of the subgroup $\Gamma_* = \cup_{i\in\mathbb{N}}\Gamma^{g_i}$, which in its turn induces (just extending each resulting order on Γ_* by $\preceq$) a continuous embedding of $\{0,1\}^{\mathbb{N}}$ into $\mathcal{O}(\Gamma)$. Moreover, since $f_m^{-1}f_n \notin \Gamma^{g_i} \setminus \Gamma_{g_i}$ for every $m \neq n$ in $\{1,\ldots,k\}$ and every $i \in \mathbb{N}$, the image of the latter embedding is contained in the neighborhood of $\preceq$ consisting of all orderings which do coincide with $\preceq$ on $\{f_1,\ldots,f_k\}$. Since this finite family of elements was arbitrary, this proves the lemma. □

The lemma below concerns the rank of the quotients Γ^{i-1}/Γ^i.

LEMMA 4.3. — *Let $\preceq$ be a $\mathcal{C}$-order on a group Γ having a neighborhood in $\mathcal{O}(\Gamma)$ which does not contain any homeomorphic copy of the Cantor set. If $\{\mathrm{id}\} = \Gamma^k \subset \Gamma^{k-1} \subset \cdots \subset \Gamma^0 = \Gamma$ is a subnormal sequence of Γ formed by $\preceq$-convex subgroups so that each quotient Γ^{i-1}/Γ^i is torsion-free Abelian, then the rank of each of these quotients equals one.*

Proof. — For the proof we will use an elegant result by Sikora [59] which establishes that $\mathcal{O}(\mathbb{Z}^n)$ has no isolated point (and it is therefore homeomorphic to the Cantor set) for every integer $n \geqslant 2$.

Assume that some of the quotients Γ^{i-1}/Γ^i has rank greater than or equal to 2. We will show that in this case every neighborhood of $\preceq$ contains a homeomorphic copy of the Cantor set. To do this, let $\{f_1,\ldots,f_k\}$ be any finite family of elements of Γ. Denoting by $\pi\colon \Gamma^{i-1} \to \Gamma^{i-1}/\Gamma^i$ the projection map, let Γ_* be a subgroup of Γ^{i-1} containing Γ^i, such that the rank of the quotient Γ_*/Γ^i is finite and greater than or equal to 2, and such that each $f_i^{-1}f_j$ is contained in $\Gamma_* \cup (\Gamma \setminus \Gamma^{i-1})$. Let Γ_{**} be the subgroup of Γ^{i-1} containing Γ^i and such that Γ^{i-1}/Γ^i is the direct sum of Γ_*/Γ^i and Γ_{**}/Γ^i. By Sikora's result, the space of orderings of the quotient Γ_*/Γ^i is homeomorphic to the Cantor set. For each $\preceq'$ in this space we may define an ordering $\preceq^*$ on Γ by letting:

- $\preceq^1$ be the order on Γ^{i-1}/Γ^i defined by $[g_1]+[h_1] \prec^1 [g_2]+[h_2]$ if and only if either $[g_1] \prec' [g_2]$, or $[g_1]=[g_2]$, $[h_1] \neq [h_2]$, and $h_1 \prec h_2$. Here, for $i \in \{1,2\}$ the elements g_i (resp. h_i) belong to Γ_* (resp. Γ_{**}), and $[\cdot]$ stands for their class modulo Γ^i;
- $\preceq^2$ be the order on Γ^{i-1} for which an element g is positive if and only if either $g \in \Gamma^i$ and $g \succ \mathrm{id}$, or $g \notin \Gamma^i$ and $\mathrm{id} \prec^1 [g]$;
- $\preceq^*$ be the extension of $\preceq^2$ by $\preceq$.

The map $\preceq' \mapsto \preceq^*$ is continuous and injective. Therefore, the intersection of its image with the subset of $\mathcal{O}(\Gamma)$ consisting of all orderings which do coincide with $\preceq$ on $\{f_1,\ldots,f_k\}$ corresponds to a homeomorphic copy of the Cantor set inside the corresponding neighborhood of $\preceq$ in $\mathcal{O}(\Gamma)$. Once again, since this finite family of elements was arbitrary, this proves the lemma. $\square$

The next lemma is essentially due to Linnell [36] (see also [66]).

Lemma 4.4. — *Let Γ be a group and Γ^1 a normal subgroup such that Γ^1 and Γ/Γ^1 are torsion-free Abelian of rank one. Let $\preceq$ be a Conradian order on Γ respect to which Γ^1 is a convex subgroup. If Γ is bi-orderable, then every neighborhood of $\preceq$ in $\mathcal{O}(\Gamma)$ contains a homeomorphic copy of the Cantor set.*

Proof. — Let us consider the action by conjugacy $\alpha\colon \Gamma/\Gamma^1 \to \mathrm{Aut}(\Gamma^1)$, namely $\alpha(g\Gamma^1)(h) = ghg^{-1}$, where $g \in \Gamma$ and $h \in \Gamma^1$. If α is trivial then Γ is Abelian and its rank is necessarily greater than or equal to 2. However, this together with the hypothesis is in contradiction with Sikora's theorem. If $\{\mathrm{id}\} \neq \mathrm{Ker}(\alpha) \neq \Gamma/\Gamma^1$ then $(\Gamma/\Gamma^1)/\mathrm{Ker}(\alpha)$ is a non-trivial torsion group, and since the only non-trivial finite order automorphism of Γ^1 is the inversion, there must exist $g \in \Gamma$ such that $ghg^{-1} = h^{-1}$ for every $h \in \Gamma$. This obviously implies that Γ is not bi-orderable. Therefore, $\mathrm{Ker}(\alpha) = \{\mathrm{id}\}$ and $\Gamma/\Gamma^1 \sim (\mathbb{Z},+)$. Viewing Γ^1 as a subgroup of $\mathbb{Q}$, the action of $(\mathbb{Z},+)$ is generated by the multiplication by a non zero rational number q. If q is negative then Γ is still non bi-orderable. It just remains the case where q is positive. Note that in this case Γ embeds in the affine group; more precisely,

Γ can be identified with the group whose elements are of the form

$$(k,a) \sim \begin{pmatrix} q^k & a \\ 0 & 1 \end{pmatrix},$$

where $a \in \Gamma^1$ and $k \in (\mathbb{Z},+)$. Let $(k_1,a_1),\ldots,(k_n,a_n)$ be an arbitrary family of positive elements of Γ indexed in such a way that $k_1 = k_2 = \cdots = k_r = 0$ and $k_{r+1} \neq 0,\ldots,k_n \neq 0$ for some $r \in \{1,\ldots,n\}$. Four cases are possible:

$$\begin{array}{rlcl}
\text{(i)} & a_1 > 0,\ldots,a_r > 0 & \text{and} & k_{r+1} > 0,\ldots,k_n > 0,\\
\text{(ii)} & a_1 < 0,\ldots,a_r < 0 & \text{and} & k_{r+1} > 0,\ldots,k_n > 0,\\
\text{(iii)} & a_1 > 0,\ldots,a_r > 0 & \text{and} & k_{r+1} < 0,\ldots,k_n < 0,\\
\text{(iv)} & a_1 < 0,\ldots,a_r < 0 & \text{and} & k_{r+1} < 0,\ldots,k_n < 0.
\end{array}$$

As in Example 2.2, for each irrational number ε let us consider the ordering $\preceq_\varepsilon$ on Γ whose positive cone is

$$P_{\preceq_\varepsilon} = \big\{(k,a)\colon q^k + \varepsilon a > 1\big\}.$$

Note that if $\varepsilon_1 \neq \varepsilon_2$ then $\preceq_{\varepsilon_1}$ is different from $\preceq_{\varepsilon_2}$. (Remark also that no order $\preceq_\varepsilon$ is Conradian.) Now in case (i), for ε positive and very small the order $\preceq_\varepsilon$ is different from $\preceq$ but still makes all the elements (k_i,a_i) positive. The same is true in case (ii) for ε negative and near zero. In case (iii) this still holds for the order $\overline{\preceq_\varepsilon}$ when ε is negative and near zero. Finally, in case (iv) one needs to consider again the order $\overline{\preceq_\varepsilon}$ but for ε positive and small. Now letting ε vary over a Cantor set formed by irrational numbers[(11)] very near to 0 (and which are positive or negative according to the case), this shows that the neighborhood of $\preceq$ consisting of the orderings on Γ which make all of the elements (k_i,a_i) positive contains a homeomorphic copy of the Cantor set. Since the finite family of elements (k_i,a_i) which are positive for $\preceq$ was arbitrary, this proves the lemma. $\square$

We may now pass to the proof of Proposition 4.1. By Lemmas 4.2 and 4.3, every countable group $\overline{\Gamma}$ admitting a $\mathcal{C}$-order $\preceq'$ having a neighborhood in $\mathcal{O}(\overline{\Gamma})$ which does not contain any homeomorphic copy of the Cantor set admits a rational series

$$\{\mathrm{id}\} = \overline{\Gamma}^k \subset \overline{\Gamma}^{k-1} \subset \cdots \subset \overline{\Gamma}^1 \subset \overline{\Gamma}^0 = \overline{\Gamma}$$

formed by $\preceq'$-convex subgroups. Assume by contradiction that the family $\mathcal{F}$ of these groups $\overline{\Gamma}$ having an infinite space of orderings is non-empty. For each $\overline{\Gamma}$ in $\mathcal{F}$ let $k(\overline{\Gamma}) \in \mathbb{N}$ be the minimum possible length for a rational

(11) Take for instance the set of numbers of the form $\sum_{i\geqslant 1} \frac{i_k}{4^k}$, where $i_k \in \{0,1\}$, and translate it by $\sum_{j\geqslant 1} \frac{2}{4^{j^2}}$.

series formed by $\preceq'$-convex subgroups with respect to some $\mathcal{C}$-order $\preceq'$ having a neighborhood in $\mathcal{O}(\overline{\Gamma})$ which does not contain any homeomorphic copy of the Cantor set. Let k the minimum of $k(\overline{\Gamma})$ for $\overline{\Gamma}$ ranging over all groups in $\mathcal{F}$, and let Γ and $\preceq$ be respectively a countable group in $\mathcal{F}$ and a $\mathcal{C}$-order on it realizing this value k. Clearly, one has $k \neq 0$ and $k \neq 1$. Moreover, Lemma 4.4 together with Tararin's theorem implies that $k \neq 2$.

To get a contradiction in the other cases, we fist claim that all the corresponding subgroups Γ^i are normal in Γ. Indeed, the restriction of $\preceq$ to Γ^1 is Conradian, and it clearly has a neighborhood in $\mathcal{O}(\Gamma^1)$ which does not contain any homeomorphic image of the Cantor set. Since

$$\{\mathrm{id}\} = \Gamma^k \subset \Gamma^{k-1} \subset \cdots \subset \Gamma^1$$

is a rational series of length $k-1$ formed by $\preceq$-convex subgroups of Γ^1, the minimality of the index k implies that $\mathcal{O}(\Gamma^1)$ is finite. By Tararin's theorem, the rational series for Γ^1 is unique. Therefore, since Γ^1 is already normal in Γ, for every $g \in \Gamma$ the rational series for Γ^1 given by

$$\{\mathrm{id}\} = g\Gamma^k g^{-1} \subset g\Gamma^{k-1} g^{-1} \subset \cdots \subset g\Gamma^1 g^{-1} = \Gamma^1$$

must coincide with the original one. Since the element $g \in \Gamma$ was arbitrary, this shows that all the subgroups Γ^i are normal in Γ.

We now claim that no quotient Γ^{i-2}/Γ^i is bi-orderable. Indeed, for the normal sequence

$$\{\mathrm{id}\} = \Gamma^i/\Gamma^i \subset \Gamma^{i-1}/\Gamma^i \subset \Gamma^{i-2}/\Gamma^i$$

the groups Γ^{i-1}/Γ^i and

$$(\Gamma^{i-2}/\Gamma^i)/(\Gamma^{i-1}/\Gamma^i) \sim \Gamma^{i-2}/\Gamma^{i-1}$$

are torsion-free rank-one Abelian. Moreover, $\preceq$ induces a Conradian order $\preceq'$ on the quotient Γ^{i-2}/Γ^i respect to which Γ^{i-1}/Γ^i is convex. Since $\preceq$ has a neighborhood in $\mathcal{O}(\Gamma)$ which does not contain any homeomorphic copy of the Cantor set, an extension type argument shows that a similar property holds for $\preceq'$ inside $\mathcal{O}(\Gamma^{i-2}/\Gamma^i)$. The fact that Γ^{i-2}/Γ^i is not bi-orderable then follows from Lemma 4.4.

We already know that each Γ^i is normal in Γ and no quotient Γ^{i-2}/Γ^i is bi-orderable. As another application of Tararin's theorem we obtain that the space of orders $\mathcal{O}(\Gamma)$ is finite, thus finishing the proof of Proposition 4.1.

Proof of Theorem B. — An easy consequence of Tararin's theorem is that a non-trivial torsion-free nilpotent group which admit only finitely many orderings is rank-one Abelian. By the comments just before Figure 1, every ordering on an orderable group without free semigroups on two generators (and therefore, every ordering on a torsion-free nilpotent group) is Conradian. It follows from Proposition 4.1 that if Γ is a non-trivial torsion-free nilpotent group which is not rank-one Abelian, then $\mathcal{O}(\Gamma)$ has no isolated point. As a consequence, if Γ is countable, then $\mathcal{O}(\Gamma)$ is a totally disconnected compact metric space without isolated points, and therefore homeomorphic to the Cantor set (see [27, Theorem 2-80])). This proves the first claim of Theorem B. The second claim of the theorem follows directly from the first one and Proposition 1.8. □

Remark 4.5. — The main property used in the proof above is that every ordering on a torsion-free nilpotent group is Conradian. This holds more generally for orderable groups without free semigroups on two generators. Actually, the conclusion of Theorem B applies to all these groups, provided they are countable and orderable. A relevant example, namely Grigorchuk-Maki's group of intermediate growth, was extensively studied in [46].

4.2. The case of orders with trivial Conradian soul

In the "pure non Conradian case" (that is, when the Conradian soul is trivial), our method for approximating a given ordering on a (countable infinite) group will consist in taking conjugates of it. More precisely, given a countable orderable group Γ and an element $\preceq$ of $\mathcal{O}(\Gamma)$, we will denote by $\mathrm{orb}(\preceq)$ the orbit of $\preceq$ by the right action of Γ. We begin by noting that, if $\preceq$ is non isolated in $\mathrm{orb}(\preceq)$, then the closure $\overline{\mathrm{orb}(\preceq)}$ is a Γ-invariant closed subset of $\mathcal{O}(\Gamma)$ without isolated points, and therefore homeomorphic to the Cantor set (because $\mathcal{O}(\Gamma)$ is metrizable and totally disconnected). To show that a particular order is non isolated inside its orbit (that is, it may be approximated by its conjugates), the following elementary lemma will be very useful.

Lemma 4.6. — *Let $\preceq$ be an ordering on a countable group Γ. Assume that the following property holds for the dynamical realization of $\preceq$ associated to a numbering $(g_i)_{i\geqslant 0}$ of Γ such that $g_0 = \mathrm{id}$: For every $\varepsilon > 0$ there exists $g \succ \mathrm{id}$ and $x \in [-\varepsilon, \varepsilon]$ such that $g(x) < x$. Then $\preceq$ is a non isolated point of $\mathrm{orb}(\preceq)$.*

Proof. — Fix a complete exhaustion $\mathcal{G}_0 \subset \mathcal{G}_1 \subset \cdots$ of Γ by symmetric finite sets. We need to show that for all fixed $n \in \mathbb{N}$ there exists $\preceq_n$ in $\mathrm{orb}(\preceq)$ different from $\preceq$ such that an element $g \in \mathcal{G}_n$ satisfies $g \succ_n \mathrm{id}$ if and only if $g \succ \mathrm{id}$. Now recall that, for all $h \in \Gamma$, the value of $h(0) =$

$h(t(\mathrm{id})) = t(h)$ is positive (resp. negative) if and only if $h \succ \mathrm{id}$ (resp. $h \prec \mathrm{id}$). For each $h \succ \mathrm{id}$ denote $\varepsilon(h) = \inf\{|x| \colon h(x) \leqslant x\}$. (We remark that $\varepsilon(h)$ is strictly positive, perhaps equal to infinite.) Now let

$$\varepsilon_n = \min\big\{\varepsilon(g) \colon g \succ d,\ g \in \mathcal{G}_n\big\}.$$

By the "transversality" hypothesis, there exists an element $g_n \succ \mathrm{id}$ in Γ such that $g_n(x_n) < x_n$ for some $x_n \in \,]-\varepsilon_n, \varepsilon_n[$. Moreover, according to the comments after Proposition 2.1, such a point x_n may be taken equal to $t(h_n^{-1})$ for some element $h_n \in \Gamma$. Now consider the order relation $\preceq_n = h_n(\preceq)$, that is, $g \succ_n \mathrm{id}$ if and only if $g(x_n) > x_n$. The equivalence between the conditions $g \succ \mathrm{id}$ and $g \succ_n \mathrm{id}$ holds for every $g \in \mathcal{G}_n$ by the definition of ε_n. On the other hand, one has $g_n \succ \mathrm{id}$ and $g_n \prec_n \mathrm{id}$, thus showing that $\preceq$ and $\preceq_n$ are different. □

The transversality hypothesis does not hold for all dynamical realizations. Indeed, according to §3.2, if the order $\preceq$ is bi-invariant then (for the associated dynamical realization) the graph of no element crosses the diagonal. It seems also difficult to apply directly the previous argument for general $\mathcal{C}$-orders. However, according to §3.3.4, the transversality condition clearly holds when the Conradian soul of $\preceq$ is trivial. As a consequence, we obtain the following proposition.

Proposition 4.7. — *If an ordering $\preceq$ on a non-trivial countable group Γ has trivial Conradian soul, then $\preceq$ is an accumulation point of its set of conjugates. In particular, the closure of the orbit of $\preceq$ under the right action of Γ is homeomorphic to the Cantor set.*

Question 4.8. — Does there exist a pure algebraic characterization of the elements of $\mathcal{O}(\Gamma)$ which are not accumulation points of their orbits by the action of Γ (equivalently, of the orderings which are non approximable by their conjugates)?

4.3. The general case

For Conrad orderable groups, Theorem C follows immediately from Proposition 4.1. If Γ has an ordering $\preceq$ having a Conradian soul $\Gamma_{\preceq}^{c}$ admitting infinitely many orders, then $\mathcal{O}(\Gamma_{\preceq}^{c})$ contains a homeomorphic copy of the Cantor set. Therefore, extending by $\preceq$ all the orderings on $\Gamma_{\preceq}^{c}$ to the whole group Γ, we obtain a homeomorphic copy of the Cantor set inside $\mathcal{O}(\Gamma)$.

Since for the case of trivial Conradian soul Proposition 4.7 applies, it just remains the case of a non Conradian ordering $\preceq$ whose Conradian soul is non-trivial but admits only finitely many orderings. Let $\preceq_1, \ldots, \preceq_{2^k}$ be all of the elements of $\mathcal{O}(\Gamma^c_\preceq)$. For $j \in \{1, \ldots, 2^k\}$ denote by $\preceq^j$ the extension of $\preceq_j$ by $\preceq$. Note that, by Lemmas 3.36 and 3.44, the subgroup $\Gamma^c_\preceq$ coincides with the Conradian soul of Γ with respect to all of the orderings $\preceq^j$. To finish the proof of Theorem C, it suffices to show the following.

Proposition 4.9. — *With the notations above, at least one of the orderings $\preceq^j$ is an accumulation point of its orbit.*

For the proof of this proposition, fix a numbering $(g_i)_{i \geqslant 0}$ of the elements of Γ such that $g_0 = \mathrm{id}$, and denote by $\alpha < 0$ and $\beta > 0$ the constants appearing in the corresponding dynamical realization of $\preceq$ associated to the Conradian soul $\Gamma^c_\preceq$ (*cf.* Proposition 3.30).

Claim 1. — *For every $\varepsilon > 0$ there exist $f_\varepsilon, g_\varepsilon$ in Γ and $a_\varepsilon, b_\varepsilon$ in $]\beta, \beta+\varepsilon[$ such that $f_\varepsilon, g_\varepsilon$ are in transversal position on $[a_\varepsilon, b_\varepsilon]$.*

Indeed, by the definition of β, there exist elements f, g in Γ which are in transversal position on some interval $[a, b]$ such that $\beta \leqslant a < \beta + \varepsilon$. Changing g by $f^n g f^{-n}$ for $n \in \mathbb{N}$ large enough, we may suppose that $b < \beta + \varepsilon$. Similarly, changing f by gfg^{-1} if necessary, we may also assume that $a > \beta$.

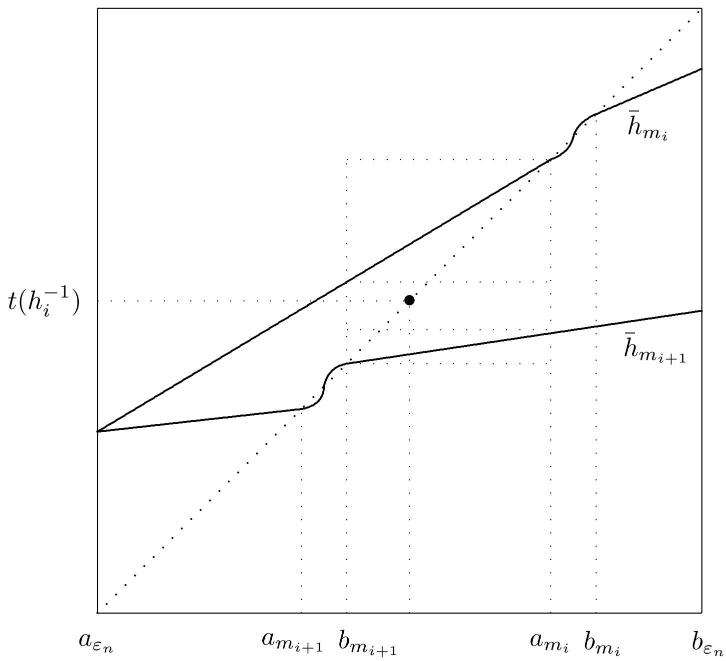

Figure 2

For $g \in \Gamma \setminus \Gamma^c_\preceq$ such that $g \succ \mathrm{id}$, let $\varepsilon(g) > 0$ be the positive number defined by $\varepsilon(g) = g(0) - \beta$. Let $\mathcal{G}_0 \subset \mathcal{G}_1 \subset \cdots$ be a complete exhaustion of Γ by finite sets. Given $n \in \mathbb{N}$ let ε_n be the (positive) number defined by

$$\varepsilon_n = \min\big\{\varepsilon(g) \colon g \succ \mathrm{id}, g \in \mathcal{G}_n \setminus \Gamma^c_\preceq\big\}. \tag{4.1}$$

Put $\bar{f} = f_{\varepsilon_n}$ and $\bar{g} = g_{\varepsilon_n}$. For $m \geqslant 1$ let a_m (resp. b_m) be the first (resp. the last) fixed point of the element $\bar{h}_m = \bar{g}\bar{f}^m$ in $]a_{\varepsilon_n}, b_{\varepsilon_n}[$. It is not difficult to check that, choosing an appropriate subsequence (m_i), we may ensure that for each $i \in \mathbb{N}$ the following hold (see Figure 2):

- $a_{m_i} > b_{m_{i+1}}$,
- $\bar{h}_{m_{i+1}}(a_{m_i}) < \bar{h}_{m_i}(b_{m_{i+1}})$,
- there exists $h_i \in \Gamma$ such that $t(h_i^{-1})$ belongs to the interval $]\bar{h}_{m_{i+1}}(a_{m_i}),\ \bar{h}_{m_i}(b_{m_{i+1}})[$.

Claim 2. — *For each $i \in \mathbb{N}$ and each $j \in \{1, \ldots, 2^k\}$, an element in $\mathcal{G}_n \setminus \Gamma^c_\preceq$ belongs to the positive cone of $(\preceq^j)_{h_i}$ if and only if it belongs to the positive cone of $\preceq$.*

Indeed, for any element $h \in \mathcal{G}_n \setminus \Gamma^c_\preceq$ which is positive with respect to $\preceq$ one has

$$t(hh_i^{-1}) = h\big(t(h_i^{-1})\big) > h(0) \geqslant \beta + \varepsilon_n > a_{m_{i-1}} > t(h_i^{-1}).$$

This implies that $hh_i^{-1} \succ h_i^{-1}$, and therefore $h_i h h_i^{-1} \succ \mathrm{id}$. If we show that the element $h_i h h_i^{-1}$ is not contained in $\Gamma^c_\preceq$, then this would give $h_i h h_i^{-1} \succ^j \mathrm{id}$, that is, h is positive with respect to $(\succ^j)_{h_i}$. Now, if $h_i h h_i^{-1}$ was equal to some element $\bar{h} \in \Gamma^c_\preceq$, then the interval

$$h_i\left(\big[t(h_i^{-1}), t(hh_i^{-1})\big]\right) = [0, t(\bar{h})] \subset\ ]\alpha, \beta[$$

would contain in its interior the interval $[h_i(b_{m_i}), h_i(a_{m_{i-1}})]$ over which the elements $h_i\bar{h}_{m_i}h_i^{-1}$ and $h_i\bar{h}_{m_{i-1}}h_i^{-1}$ are crossed. However, this contradicts the definition of the interval $]\alpha, \beta[$.

If $h \in \mathcal{G}_n \setminus \Gamma^c_\preceq$ is negative with respect to $\preceq$, the above argument shows that h^{-1} is positive with respect to $(\succ^j)_{h_i}$, and therefore h is negative with respect to this ordering as well. This finishes the proof of Claim 2.

Claim 3. — *For each fixed $j \in \{1, \ldots, 2^k\}$ the orders $(\preceq^j)_{h_i}$ are two-by-two distinct (for $i \in \mathbb{N}$).*

It easily follows from the construction that the inequality $\bar{h}_{m_\ell}(t(h_i^{-1})) > t(h_i^{-1})$ holds if and only if $\ell \leqslant i$. If this is the case, then $\bar{h}_{m_\ell}(t(h_i^{-1})) > \bar{h}_{m_i}(b_{m_{i+1}})$. Therefore, for $n' >> n$ large enough, the elements $f_{n'} = \bar{h}_{m_i}\bar{h}^{n'}_{m_{i+1}}$ and $f_n = \bar{h}_{m_i}\bar{h}^{n}_{m_{i+1}}$ are in transversal position on some closed interval $[a,b]$ contained in $]t(h_i^{-1}), \bar{h}_{m_i}(t(h_i^{-1}))[$ (see Figure 3 below). We claim that this implies that the element $h_i \bar{h}_{m_\ell} h_i^{-1}$ does not belong to $\Gamma^c_{\preceq}$ for all $\ell \leqslant i$. Indeed, if $h_i \bar{h}_{m_\ell} h_i^{-1}$ was equal to some element $\bar{h} \in \Gamma^c_{\preceq}$ then, since $a > t(h_i^{-1})$ and $b < t(\bar{h}_{m_i} h_i^{-1}) \leqslant t(\bar{h}_{m_\ell} h_i^{-1})$, the interval

$$\big[0, t(\bar{h})\big] = \big[0, t(h_i \bar{h}_{m_\ell} h_i^{-1})\big] = h_i\left(\big[t(h_i^{-1}), t(\bar{h}_{m_\ell} h_i^{-1})\big]\right)$$

would be contained in $[0,\beta]$ and would contain in its interior the interval $[h_i(a), h_i(b)]$. However, on the last interval the elements $h_i f_{n'} h_i^{-1}$ and $h_i f_n h_i^{-1}$ are in transversal position, and this contradicts the definition of the interval $]\alpha,\beta[$.

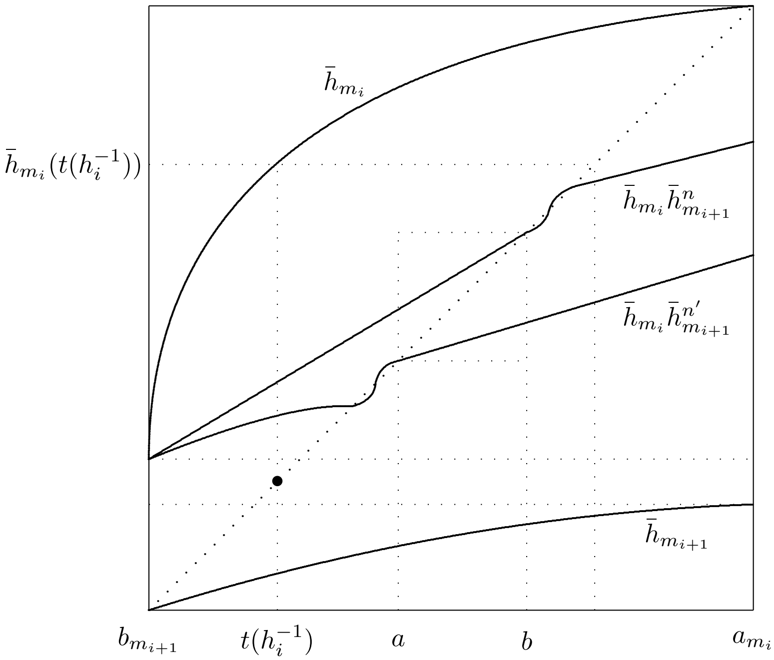

Figure 3

Now since $h_i \bar{h}_{m_\ell} h_i^{-1} \succ \mathrm{id}$ for all $\ell \leqslant i$, one also has $h_i \bar{h}_{m_\ell} h_i^{-1} \succ^j \mathrm{id}$ for all $j \in \{1,\dots,2^k\}$. In other words, the element $\bar{h}_{m_\ell}$ is positive with respect to $(\succ^j)_{h_i}$ for every $\ell \leqslant i$. In an analogous way, one proves that $\bar{h}_{m_\ell}$ is negative with respect to $(\succ^j)_{h_i}$ for all $\ell > i$. These two facts together obviously imply that the orders $(\preceq^j)_{h_i}$ are two-by-two different.

Proof of Proposition 4.9. — Let (ε_m) be the decreasing sequence of positive numbers converging to 0 defined by (4.1). With respect to this sequence we may perform the construction given in Claim 1. By Claim 2, for each $m \in \mathbb{N}$ we may then fix an element $g_m \in \Gamma$ such that, for each $j \in \{1, \ldots, 2^k\}$, an element in $\mathcal{G}_m \setminus \Gamma_{\preceq}^c$ belongs to the positive cone of $(\preceq^j)_{g_m}$ if and only if it belongs to the positive cone of $\preceq$. Moreover, by Claim 3, the sequence (g_m) may be taken in such a way that, for each fixed $j \in \{1, \ldots, 2^k\}$, the orderings $(\preceq^j)_{g_m}$ are two-by-two different. Passing to a subsequence if necessary, Claim 2 allows to ensure that each sequence of orderings $(\preceq^j)_{g_m}$ converges to some ordering of the form $\preceq^{j'}$. Thus, $\preceq^{j'}$ belongs to the set of accumulation points $\mathrm{acc}(\mathrm{orb}(\preceq^j))$ of the orbit of $\preceq^j$. Let us fix $j_0 \in \{1, \ldots, 2^k\}$. By the above one has $\preceq^{j_1} \in \mathrm{acc}(\mathrm{orb}(\preceq^{j_0}))$ for some $j_1 \in \{1, \ldots, 2^k\}$. If $j_0 = j_1$ then we are done. If not, then for a certain $j_2 \in \{1, \ldots, 2^k\}$ one has $\preceq^{j_2} \in \mathrm{acc}(\mathrm{orb}(\preceq^{j_1}))$, and therefore $\preceq^{j_2} \in \mathrm{acc}(\mathrm{orb}(\preceq^{j_0}))$. If j_2 equals j_0 or j_1 then we are done. If not, we continue the process... Clearly, in no more than 2^k steps we will find an index j such that $\preceq^j \in \mathrm{acc}(\mathrm{orb}(\preceq^j))$, and this concludes the proof. □

Although very natural, our proof of Theorem C in the case of an ordering having a non-trivial Conradian soul with finitely many orders is quite elaborate. However, an affirmative answer to the following question would allow to reduce the general case to those of Propositions 4.1 and 4.7.

Question 4.10. — Let Γ be a countable orderable group. If Γ admits a non Conradian ordering, is it necessarily true that Γ admits an ordering having trivial Conradian soul?

4.4. An application to braid groups

For the proof of Theorem D we first consider the case of the braid group B_3. According to Examples 3.34, 3.35, and 3.38, the Conradian soul of Dehornoy's ordering coincides with the cyclic subgroup generated by σ_2. Since this subgroup admits finitely many (namely, two) different orderings, we are under the hypothesis of Proposition 4.9 for the orderings $\preceq^1 = \preceq_D$ and $\preceq^2 = \preceq_{DD}$. Now the conjugates of $\preceq_D$ cannot approximate $\preceq_{DD}$, because the latter ordering is isolated in $\mathcal{O}(B_3)$. Therefore, according to the proof of Proposition 4.9, there exists a sequence of elements $g_m \in B_3$ such that both sequences of orderings $(\preceq_D)_{g_m}$ and $(\preceq_{DD})_{g_m}$ converge to $\preceq_D$.

Now, for the case of general braid groups B_n, recall that the subgroup $\langle \sigma_{n-2}, \sigma_{n-1} \rangle$ is isomorphic to B_3 via the map $\sigma_{n-2} \mapsto \sigma_1$, $\sigma_{n-1} \mapsto \sigma_2$, which respects Dehornoy's orderings. By the argument above, there exists a sequence of elements g_m in $\langle \sigma_{n-2}, \sigma_{n-1} \rangle$ such that the restrictions to $\langle \sigma_{n-2}, \sigma_{n-1} \rangle$ of the orderings $(\preceq_D)_{g_m}$ converge to the restriction of $\preceq_D$ to the same subgroup. We claim that actually $(\preceq_D)_{g_m}$ converges to $\preceq_D$ over the whole group B_n. Indeed, if g belongs to $B_3 \setminus \langle \sigma_{n-2}, \sigma_{n-1} \rangle$ and $h \in B_n$ is σ_i-positive (resp. σ_i-negative) for some $i \in \{1, \ldots, n-3\}$, then each of the elements $g_m h g_m^{-1}$ is still σ_i-positive (resp. σ_i-negative). Since the orderings $(\preceq_D)_{g_m}$ are two-by-two distinct, this finishes the proof of Theorem D.

Remark 4.11. — It would be interesting to obtain a proof of Theorem A using the methods of that of Theorem D.

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Andrés NAVAS
Univ. de Santiago de Chile
Alameda 3363
Est. Central Santiago (Chile)
andres.navas@usach.cl