Groups Geom. Dyn. 4 (2010), xx–xx

Groups, Geometry, and Dynamics

Describing all bi-orderings on Thompson's group *F*

Andrés Navas and Cristóbal Rivas

Abstract. We describe all possible ways of bi-ordering Thompson's group F: its space of bi-orderings is made up of eight isolated points and four canonical copies of the Cantor set.

Mathematics Subject Classification (2010). 06F15, 20F60, 22F50.

Keywords. Group orderings, Conrad's property, Thompson's group F.

Introduction

In recent years, the well developed theory of orderable groups has re-emerged, mainly due to its connections with many different branches of mathematics. One of the aspects which has been emphasized is that, in general, orderable groups actually admit many invariant total order relations. This makes natural the problem of searching for an ordering satisfying a nice property implying a relevant algebraic (or dynamical) property of the underlying group. This issue has been successfully exploited for instance by Witte Morris in his beautiful proof of the local indicability for left-orderable amenable groups [10]. The reader is referred to [11] for other applications of this approach.

A closely related problem concerns the description of *all* (invariant) orderings on particular classes of groups. In this direction, Tararin's concise classification of groups admitting only finitely many left-orderings corresponds to a relevant piece of the theory [6]. Another significant (and easier) result is the description of all possible orderings on torsion-free finite rank Abelian groups [14], [15], [17].

Although the description of *all* orderings seems to be out of reach for general orderable groups, one may address the weaker question of the description of the corresponding space of orderings from a topological viewpoint (recall that the space of orderings on any space corresponds to the projective limit of the orders on finite sets, and hence carries the structure of a compact topological space). For instance, ruling out the existence of *isolated points* in this space (that is, orderings which are completely determined by finitely many inequalities) appears to be a fundamental question. This has been done for example for the spaces of left-orderings of finitely generated torsion-free nilpotent groups which are not rank 1 Abelian [11], [15]. For the free group F_n (where $n \geq 2$), it is known that there is no isolated point in the corresponding space of left-orderings [7], [11], [16]. The similar question for the space of bi-orderings on F_n remains open, and though it is not treated here, it inspires much of this work.

In this article, we focus on a remarkable bi-orderable group, namely Thompson's group F, and we provide a complete description of all its possible bi-orderings. Recall that F is the group of orientation-preserving piecewise-linear homeomorphisms f of the interval $[0,1]$ such that:

- the derivative of f on each linearity interval is an integer power of 2,
- f induces a bijection of the set of dyadic rational numbers in $[0,1]$.

For each non-trivial $f \in F$ we will denote by x_f^- (resp. x_f^+) the leftmost point x^- (resp. the rightmost point x^+) for which $f'_+(x^-) \neq 1$ (resp. $f'_-(x^+) \neq 1$), where f'_+ and f'_- stand for the corresponding lateral derivatives. One can then immediately visualize four different bi-orderings on (each subgroup of) F, namely:

- the bi-ordering $\preceq^+_{x^-}$ for which $f \succ \mathrm{id}$ if and only if $f'_+(x_f^-) > 1$,
- the bi-ordering $\preceq^-_{x^-}$ for which $f \succ \mathrm{id}$ if and only if $f'_+(x_f^-) < 1$,
- the bi-ordering $\preceq^+_{x^+}$ for which $f \succ \mathrm{id}$ if and only if $f'_-(x_f^+) < 1$,
- the bi-ordering $\preceq^-_{x^+}$ for which $f \succ \mathrm{id}$ if and only if $f'_-(x_f^+) > 1$.

Although F admits many more bi-orderings than these, the case of its derived subgroup F' is quite different.

Theorem (V. Dlab). *The only bi-orderings on F' are $\preceq^+_{x^-}$, $\preceq^-_{x^-}$, $\preceq^+_{x^+}$ and $\preceq^-_{x^+}$.*

Dlab's arguments apply to many other (in general, non finitely generated) groups of piecewise-affine homeomorphisms of the line. Some of them appear to be non-Abelian, though having only two different bi-orderings (compare Remark 1.6). We refer to the original reference [5] for all of this (see also [6], [8], [9], [18]). Here we provide a new proof using an argument which allows us to obtain the complete classification of all the bi-orderings on F.

Note that there are also four other "exotic" bi-orderings on F, namely:

- the bi-ordering $\preceq^{+,-}_{0,x^-}$ for which $f \succ \mathrm{id}$ if and only if either $x_f^- = 0$ and $f'_+(0) > 1$, or $x_f^- \neq 0$ and $f'_+(x_f^-) < 1$,
- the bi-ordering $\preceq^{-,+}_{0,x^-}$ for which $f \succ \mathrm{id}$ if and only if either $x_f^- = 0$ and $f'_+(0) < 1$, or $x_f^- \neq 0$ and $f'_+(x_f^-) > 1$,
- the bi-ordering $\preceq^{+,-}_{1,x^+}$ for which $f \succ \mathrm{id}$ if and only if either $x_f^+ = 1$ and $f'_-(1) < 1$, or $x_f^+ \neq 1$ and $f'_-(x_f^+) > 1$,
- the bi-ordering $\preceq^{-,+}_{1,x^+}$ for which $f \succ \mathrm{id}$ if and only if either $x_f^+ = 1$ and $f'_-(1) > 1$, or $x_f^+ \neq 1$ and $f'_-(x_f^+) < 1$.

Remark that, when restricted to F', the bi-ordering $\preceq_{0,x-}^{+,-}$ (resp. $\preceq_{0,x-}^{-,+}$, $\preceq_{1,x+}^{+,-}$, and $\preceq_{1,x+}^{-,+}$) coincides with $\preceq_{x-}^{-}$ (resp. $\preceq_{x-}^{+}$, $\preceq_{x+}^{-}$, and $\preceq_{x+}^{+}$). Let us denote the set of the previous eight bi-orderings on F by $\mathcal{BO}_{\mathrm{Isol}}(F)$.

There is another natural procedure for creating bi-orderings on F. For this, recall the well-known (and easy to check) fact that F' coincides with the subgroup of F formed by the elements f satisfying $f'_+(0) = f'_-(1) = 1$. Now let $\preceq_{\mathbb{Z}^2}$ be any bi-ordering on $\mathbb{Z}^2$, and let $\preceq_{F'}$ be any bi-ordering on F'. It readily follows from Dlab's theorem that $\preceq_{F'}$ is invariant under conjugacy by elements in F. Hence, one may define a bi-ordering $\preceq$ on F by declaring that $f \succ \mathrm{id}$ if and only if either $f \notin F'$ and $(\log_2(f'_+(0)), \log_2(f'_-(1))) \succ_{\mathbb{Z}^2} (0,0)$, or $f \in F'$ and $f \succ_{F'} \mathrm{id}$.

All possible ways of ordering finite rank Abelian groups have been described in [14], [15], [17]. In particular, when the rank is greater than 1, the corresponding spaces of bi-orderings are homeomorphic to the Cantor set. Since there are only four possibilities for the bi-ordering $\preceq_{F'}$, the preceding procedure gives four natural copies (which we will coherently denote by Λ_{x-}^{+}, Λ_{x-}^{-}, Λ_{x+}^{+}, and Λ_{x+}^{-}) of the Cantor set in the space of bi-orderings of F. The main result of this work establishes that these bi-orderings, together with the special eight bi-orderings previously introduced, fill out the list of all possible bi-orderings on F.

Theorem. *The space of bi-orderings of F is the disjoint union of the finite set $\mathcal{BO}_{\mathrm{Isol}}(F)$ (whose elements are isolated bi-orderings) and the copies of the Cantor set Λ_{x-}^{+}, Λ_{x-}^{-}, Λ_{x+}^{+}, and Λ_{x+}^{-}.*

The first ingredient of the proof of this result comes from the theory of Conradian orderings [4]. Indeed, since F is finitely generated, every bi-ordering $\preceq$ on it admits a maximal proper convex subgroup $F_{\preceq}^{\max}$. More importantly, this subgroup may be detected as the kernel of a non-trivial, non-decreasing group homomorphism into $(\mathbb{R}, +)$. Since F' is simple (see for instance [2]) and non-Abelian, it must be contained in $F_{\preceq}^{\max}$. The case of coincidence is more or less transparent: the bi-ordering on F is contained in one of the four canonical copies of the Cantor set, and the corresponding bi-ordering on $\mathbb{Z}^2$ is of *irrational type* (i.e., its positive elements are those which are in one of the two half-planes determined by a line of irrational slope passing through the origin). The case where F' is strictly contained in $F_{\preceq}^{\max}$ is more complicated. The bi-ordering may still be contained in one of the four canonical copies of the Cantor set, but the corresponding bi-ordering on $\mathbb{Z}^2$ must be of *rational type* (e.g., a lexicographic ordering). However, it may also coincide with one of the eight special bi-orderings listed above. Distinguishing these two possibilities is the hardest part of the proof. For this, we strongly use the internal structure of F, in particular the fact that the subgroup consisting of elements whose support is contained in a prescribed closed dyadic interval is isomorphic to F itself.

Acknowledgments. The first author would like to thank José Burillo for his explanations on the group of outer automorphisms of F, as well as Adam Clay and Dale

Rolfsen for helpful discussions on orderable groups. This work was partially funded by the PBCT-Conicyt via the Research Network on Low Dimensional Dynamical Systems. The second-named author was also funded by the Conicyt PhD Fellowship 21080054.

1. Some background

1.1. On group orderings. Throughout this work, the word *left-ordering* (resp. *bi-ordering*) stands for a total order relation on a group which is invariant by left multiplication (resp. by left and right multiplication simultaneously). An element f is said to be *positive* (resp. *negative*) with respect to some left-ordering $\preceq$ if $f \succ \mathrm{id}$ (resp. $f \prec \mathrm{id}$). The set of positive elements forms a semigroup $P^+_{\preceq}$, which is called the *positive cone* of $\preceq$, and the whole group equals the disjoint union of $P^+_{\preceq}$ together with $P^-_{\preceq} = \{f \mid f^{-1} \in P^+_{\preceq}\}$ and $\{\mathrm{id}\}$. Conversely, given a subsemigroup P^+ of a group Γ such that Γ equals the disjoint union of P^+ together with $P^- = \{f \mid f^{-1} \in P^+\}$ and $\{\mathrm{id}\}$, one may realize P^+ as the positive cone of a left-ordering $\preceq$: it suffices to declare that $f \succ g$ if and only if $g^{-1}f$ belongs to P^+. The resulting ordering will be bi-invariant if and only if P^+ is a normal subsemigroup, that is, if $gfg^{-1} \in P^+$ for all $f \in P^+$ and all $g \in \Gamma$.

Every left-ordering (resp. bi-ordering) $\preceq$ on a group Γ comes together with an associated (*reverse*) left-ordering (resp. bi-ordering) $\overline{\preceq}$ whose positive cone coincides with $P^-_{\preceq}$. Clearly, the map $\preceq \,\mapsto\, \overline{\preceq}$ is an involution of the set of left-orderings (resp. bi-orderings).

Example 1.1. Clearly, there are only two bi-orderings on $\mathbb{Z}$. The case of $\mathbb{Z}^2$ is more interesting. According to [14], [15], [17], there are two different types of bi-orderings on $\mathbb{Z}^2$. Bi-orderings of *irrational type* are completely determined by an irrational number λ: for such an order $\preceq_\lambda$ an element (m,n) is positive if and only if $\lambda m + n$ is a positive real number. Bi-orderings of rational type are characterized by two data, namely a pair $(a,b) \in \mathbb{Q}^2$ up to multiplication by a positive real number, and the choice of one of the two possible bi-orderings on the subgroup $\{(m,n) \mid am + bn = 0\} \sim \mathbb{Z}$. Thus an element $(m,n) \in \mathbb{Z}^2$ is positive if and only if either $am+bn$ is a positive real number, or $am+bn = 0$ and (m,n) is positive with respect to the chosen bi-ordering on the kernel line (isomorphic to $\mathbb{Z}$). The description of all bi-orderings on $\mathbb{Z}^n$ for bigger n continues inductively. (A good exercise is to show all of this by using Conrad's theorem from §1.3.)

1.2. On spaces of orderings. Given a left-orderable group Γ (of arbitrary cardinality), we denote by $\mathcal{LO}(\Gamma)$ the set of all left-orderings on Γ. This set has a natural topology: a basis of neighborhoods of $\preceq$ in $\mathcal{LO}(\Gamma)$ is the family of the sets $U_{g_0,\ldots,g_k}$ of all left-orderings $\preceq'$ on Γ which coincide with $\preceq$ on $\{g_0,\ldots,g_k\}$, where $\{g_0,\ldots,g_k\}$

runs over all finite subsets of Γ. Endowed with this topology, $\mathcal{LO}(\Gamma)$ is totally disconnected, and by (an easy application of) the Tychonov Theorem, it is compact. The (perhaps empty) subspace $\mathcal{BO}(\Gamma)$ of bi-orderings on Γ is closed inside $\mathcal{LO}(\Gamma)$, and hence is also compact.

If Γ is countable, then the above topology is metrizable: given an exhaustion $\Gamma_0 \subset \Gamma_1 \subset \cdots$ of Γ by finite sets, for different $\preceq$ and $\preceq'$ one may define $\mathrm{dist}(\preceq, \preceq') = 1/2^n$, where n is the first integer such that $\preceq$ and $\preceq'$ do not coincide on Γ_n. If Γ is finitely generated, one may take Γ_n as being the ball of radius n with respect to some fixed finite system of generators. (The metrics arising from two different finite systems of generators are Hölder equivalent.)

By definition, an isolated point $\preceq$ in $\mathcal{LO}(\Gamma)$ corresponds to an ordering for which there exist $g_0, \ldots, g_k$ in Γ such that $U_{g_0,\ldots,g_k}$ reduces to $\{\preceq\}$. This is the case for example if $g_1, \ldots, g_k$ generate the positive cone of $\preceq$ as a semigroup and $g_0 = \mathrm{id}$: see [11], Proposition 1.8. Analogously, $\preceq$ is an isolated point of $\mathcal{BO}(\Gamma)$ if $U_{g_0,\ldots,g_k} \cap \mathcal{BO}(\Gamma)$ reduces to $\{\preceq\}$ for some $g_0, \ldots, g_k$ in Γ. According to the (obvious) proposition below, this happens for instance if $g_1, \ldots, g_k$ generate the positive cone of $\preceq$ as a *normal* semigroup and $g_0 = \mathrm{id}$ (recall that a subset S of a normal subsemigroup P of a group Γ generates P as a normal semigroup if P coincides with the smallest normal subsemigroup $\langle S \rangle_N^+$ of Γ containing S): see Questions 2.2 and 3.1 on this.

Proposition 1.2. *Suppose that the positive cone of a bi-ordering $\preceq$ on a group Γ is generated as a normal semigroup by elements $g_1, \ldots, g_k$. Then $\preceq$ is the unique bi-ordering on Γ for which all of these elements are positive.*

As has been observed by many people (see for example [11]), the group of automorphisms $\mathrm{Aut}(\Gamma)$ of a left-orderable group Γ acts by homeomorphisms of $\mathcal{LO}(\Gamma)$: given $\gamma \in \mathrm{Aut}(\Gamma)$ and $\preceq$ in $\mathcal{LO}(\Gamma)$, the image of $\preceq$ by γ is the left-ordering $\preceq_\gamma$ whose positive cone is the preimage under γ of the positive cone of $\preceq$. If Γ is bi-orderable, then this action restricted to $\mathcal{BO}(\Gamma)$ factors through the group of outer automorphisms $\mathrm{Out}(\Gamma)$.

The dynamical properties of the preceding action for general bi-orderable groups seem interesting. For instance, the action of $\mathrm{GL}(2,\mathbb{Z})$ on $\mathcal{BO}(\mathbb{Z}^2)$ is transitive on the set of bi-orderings of rational type, while the set of bi-orderings of irrational type decomposes into uncountably many orbits (cf. Example 1.1).

In a similar direction, the action of $\mathrm{Out}(F_n)$ could be useful for understanding $\mathcal{BO}(F_n)$. Nevertheless, in the case of Thompson's group F, the action of $\mathrm{Out}(F)$ on $\mathcal{BO}(F)$ is almost trivial. Indeed, according to [1], the group $\mathrm{Out}(F)$ contains an index-two subgroup $\mathrm{Out}_+(F)$ whose elements are (equivalence classes of) conjugacies by certain orientation preserving homeomorphisms of the interval $[0,1]$. Although these homeomorphisms are dyadically piecewise-affine on $]0,1[$, the points of discontinuity of their derivatives may accumulate at 0 and/or 1, but in some "periodically coherent" way. It turns out that the conjugacies by these homeomorphisms preserve the derivatives of non-trivial elements $f \in F$ at the points x_f^- and x_f^+: this is obvious when these points are different from 0 and 1, and in the other case this follows from the explicit description of $\mathrm{Out}(F)$ given in [1]. According to our main theorem, this implies that the action of $\mathrm{Out}_+(F)$ on $\mathcal{BO}(F)$ is trivial.

The set $\mathrm{Out}(F)\backslash\mathrm{Out}_+(F)$ corresponds to the class of the order-two automorphism σ induced by the conjugacy by the map $x \mapsto 1-x$. One can easily check that

$$(\preceq^+_{x^-})_\sigma = \preceq^-_{x^+},\ (\preceq^-_{x^-})_\sigma = \preceq^+_{x^+},\ (\preceq^{+,-}_{0,x^-})_\sigma = \preceq^{-,+}_{1,x^+},\ \text{and}\ (\preceq^{-,+}_{0,x^-})_\sigma = \preceq^{+,-}_{1,x^+}.$$

Moreover, $\sigma(\Lambda^+_{x^-}) = \Lambda^-_{x^+}$ and $\sigma(\Lambda^-_{x^-}) = \Lambda^+_{x^+}$, and the action on the bi-orderings of the $\mathbb{Z}^2$-fiber can be easily described. We leave the details to the reader.

Remark 1.3. As in the case of σ, the dynamics of the involution $\preceq \mapsto \overline{\preceq}$ can be also easily described. However, in the case of F, this involution does not occur as the action of any group automorphism.

1.3. On Conradian orderings. Besides $\mathcal{BO}(\Gamma)$, for a left-orderable group Γ there is another relevant (perhaps empty) closed subset of $\mathcal{LO}(\Gamma)$, namely the subset $\mathcal{CO}(\Gamma)$ formed by the left-orderings $\preceq$ such that $g^{-1}fg^2 \succ \mathrm{id}$ for all positive elements f, g (see for instance [4], [11]). A left-ordering satisfying this property is said to be a $\mathcal{C}$-ordering or *Conradian* ordering, and a group admitting such a left-ordering is called *Conrad-orderable* or simply $\mathcal{C}$-orderable. Notice that every bi-invariant ordering is Conradian.

In [4], a structure theory for Conradian orderings is given. (An alternative dynamical approach appears in [11], [12].) This is summarized in the theorem below. To state it properly, recall that a subgroup Γ_0 of a group Γ endowed with a left-ordering $\preceq$ is said to be $\preceq$*-convex* if every $g \in \Gamma$ satisfying $g_1 \preceq g \preceq g_2$ for some g_1, g_2 in Γ_0 actually belongs to Γ_0. Equivalently, every $h \in \Gamma$ satisfying $\mathrm{id} \preceq h \preceq g$ for some $g \in \Gamma_0$ is contained in Γ_0. Notice that given any two $\preceq$-convex subgroups of Γ, one of them is necessarily contained in the other. Consequently, the union and the intersection of groups in an arbitrary family of $\preceq$-convex subgroups is also $\preceq$-convex.

Theorem (P. Conrad). *Let Γ be a group endowed with a $\mathcal{C}$-ordering. Given $g \in \Gamma$, denote by Γ_g (resp. Γ^g) the maximal (resp. minimal) convex subgroup which does not contain (which contains) g. Then Γ_g is normal in Γ^g, and there exists a non-decreasing group homomorphism $\tau^g_\preceq \colon \Gamma \to (\mathbb{R}, +)$ whose kernel coincides with Γ_g. This homomorphism is unique up to multiplication by a positive real number.*

Moreover, if Γ is finitely generated, then it contains a (unique) maximal proper $\preceq$-convex subgroup $\Gamma^{\max} = \Gamma^{\max}_\preceq$. This subgroup coincides with the kernel of a (unique up to multiplication by a positive real number) non-decreasing group homomorphism $\tau_\preceq \colon \Gamma \to (\mathbb{R}, +)$.

A direct consequence of this theorem is that Conrad-orderable groups are locally indicable, that is, their non-trivial finitely generated subgroups admit non-trivial group homomorphisms into $(\mathbb{R}, +)$. Actually, the converse is also true (see [11] and references therein).

The study of the topological properties of $\mathcal{CO}(\Gamma)$ is much simpler than those of $\mathcal{BO}(\Gamma)$. Indeed, in most of the cases, $\mathcal{CO}(\Gamma)$ has no isolated point (and hence it is homeomorphic to the Cantor set if the group is countable). To show a result in this direction, we need to recall the *extension procedure* for creating group orderings.

Let $\preceq$ be a left-ordering on a group Γ, let Γ_0 be a $\preceq$-convex subgroup of Γ, and let $\preceq_0$ be a left-ordering on Γ_0. The extension of $\preceq_0$ by $\preceq$ is the left-ordering $\preceq^*$ on Γ obtained by "changing" $\preceq$ into $\preceq_0$ on Γ_0, and "keeping it" outside. More precisely, the positive cone of $\preceq^*$ is $P^+_{\preceq_0} \cup (P^+_{\preceq} \setminus \Gamma_0)$. One can easily check that Γ_0 remains $\preceq^*$-convex. Moreover, if $\preceq$ and $\preceq_0$ are Conradian, then the resulting $\preceq^*$ is also a $\mathcal{C}$-ordering. Unfortunately (or perhaps fortunately), the bi-invariance of both $\preceq$ and $\preceq_0$ does not guarantee the bi-invariance of $\preceq^*$: to ensure this, we also need to assume that the positive cone of $\preceq_0$ is invariant under conjugacies by elements in Γ. Finally, it is not difficult to check that if Γ_0 is a $\preceq$-convex *normal* subgroup of Γ, then $\preceq$ induces a left-ordering on the quotient Γ / Γ_0, which is a bi-ordering if $\preceq$ is bi-invariant.

Example 1.4. For simplicity, let us denote by $\preceq$ the bi-ordering $\preceq^+_{x^-}$ on F. Then, for a non-trivial element $g \in F$, the subgroups F_g and F^g coincide with $\{f \in F \mid \mathrm{supp}(f) \subset]x_g^-, 1]\}$ and $\{f \in F \mid \mathrm{supp}(f) \subset [x_g^-, 1]\}$, respectively, where $\mathrm{supp}(f) = \overline{\{x \mid f(x) \neq x\}}$ is the *support* of f. The quotient Γ^g / Γ_g is order isomorphic to $\mathbb{Z}$ via the homomorphism $f\Gamma_g \mapsto \log_2(f'_+(x_g^-))$. A curious $\mathcal{C}$-ordering $\preceq'$ on F (which is not bi-invariant!) is obtained as follows: take the extension $\preceq^*$ of the restriction of $\preceq$ to Γ_g by the restriction of $\bar{\preceq}$ to Γ^g, and then extend $\preceq^*$ by $\preceq$. This left-ordering obeys the following rule: a non-trivial element $f \in F$ is positive with respect to $\preceq'$ if and only if either $x_f^- \neq x_g^-$ and $f'_+(x_f^-) > 1$, or $x_f^- = x_g^-$ and $f'_+(x_f^-) < 1$.

Example 1.5. As the reader can easily check, the bi-ordering $\preceq^{+,-}_{0,x^-}$ appears as the extension by $\preceq^+_{x^-}$ of the restriction of its conjugate $\bar{\preceq}^+_{x^-}$ (which coincides with $\preceq^-_{x^-}$) to the maximal proper $\preceq^+_{x^-}$-convex subgroup $F^{\max} = \{f \in F \mid f'_+(0) = 1\}$. The bi-orderings $\preceq^{-,+}_{0,x^-}$, $\preceq^{+,-}_{1,x^+}$, and $\preceq^{-,+}_{1,x^+}$ may be obtained in the same way starting from $\preceq^-_{x^-}$, $\preceq^+_{x^+}$, and $\preceq^-_{x^+}$, respectively.

Remark 1.6. In general, if Γ is a finitely generated (non-trivial) group endowed with a bi-ordering $\preceq$, one can easily check that the ordering $\preceq^*$ obtained as the extension by $\preceq$ of $\bar{\preceq}$ restricted to $\Gamma^{\max}_{\preceq}$ is bi-invariant. This bi-ordering (resp. its conjugate $\bar{\preceq}_*$) is always different from $\bar{\preceq}$ (resp. from $\preceq$), and it coincides with $\preceq$ (resp. with $\bar{\preceq}$) if and only if the only proper $\preceq$-convex subgroup is the trivial one; by Conrad's theorem, Γ is necessarily Abelian in this case. We thus conclude that every non-Abelian finitely generated bi-orderable group admits at least four different bi-orderings. Moreover, (non-trivial) torsion-free Abelian groups having only two bi-orderings are those of rank one (in higher rank one may consider lexicographic type orderings).

Proposition 1.7. *If* Γ *is a non-solvable Conrad-orderable group, then* $\mathcal{CO}(\Gamma)$ *contains no isolated point.*

Proof. Throughout the proof, fix a $\mathcal{C}$-ordering $\preceq$ on Γ. We will first show that if there are infinitely many subgroups of the form Γ_g, then $\preceq$ is not isolated inside $\mathcal{CO}(\Gamma)$. Indeed, given finitely many distinct elements $g_1, \ldots, g_k$ in Γ, consider the elements $f_{i,j}$ of the form $g_i^{-1} g_j$. We need to produce a $\mathcal{C}$-ordering $\preceq^*$ on Γ different from $\preceq$ but for which the "signs" of the elements $f_{i,j}$ are the same. To do this, choose $g \in \Gamma$ such that Γ_g is different from all of the subgroups $\Gamma_{f_{i,j}}$. This condition implies that the corresponding Γ^g is different from all of the $\Gamma^{f_{i,j}}$. Now define $\preceq'$ as being the extension by $\preceq$ of the extension of the restriction of $\preceq$ to Γ_g by the restriction of $\overline{\preceq}$ to Γ^g. One can easily show that $\preceq'$ verifies all the desired properties.

Suppose now that, for some integer $n \geq 1$, there are precisely n subgroups of the form Γ_g. We claim that Γ is solvable with solvability length at most n. Indeed, If Γ_{g_1} denotes the maximal proper $\preceq$-convex subgroup of Γ then, by Conrad's theorem, Γ_{g_1} is normal in Γ, and the quotient Γ / Γ_{g_1} is Abelian. Hence, Γ' is contained in Γ_{g_1}. Since Γ_{g_1} contains at most $n-1$ subgroups of the form Γ_g, we may repeat this argument... In at most n steps all the n-commutators in Γ will appear to be trivial, which concludes the proof. $\square$

Left-orderable solvable groups are Conrad-orderable [3], [10]. Moreover, according to [11], if a group Γ has infinitely many left-orderings, then no Conradian ordering on Γ is isolated in $\mathcal{LO}(\Gamma)$. It would be then interesting to classify left-orderable solvable groups Γ for which $\mathcal{CO}(\Gamma)$ has isolated points.[1]

2. Bi-orderings on F'

For each dyadic (open, half-open, or closed) interval I, we will denote by F_I the subgroup of F formed by the elements whose supports are contained in I. Notice that if I is closed, then F_I is isomorphic to F. Therefore, for every closed dyadic interval $I \subset]0,1[$, every bi-ordering $\preceq^*$ on F' gives rise to a bi-ordering on $F \sim F_I$. Moreover, if we fix such an I, then the induced bi-ordering on F_I completely determines $\preceq^*$ (this is due to the invariance by conjugacy). The content of Dlab's theorem consists of the assertion that only a few (namely four) bi-orderings on F_I may be extended to bi-orderings on F'. To reprove this result, we will first focus on a general property of bi-orderings on F.

Let $\preceq$ be a bi-ordering on F. Since bi-invariant orderings are Conradian and F is finitely generated, Conrad's theorem provides us with a (unique up to positive

[1]Added in proof: this has been recently done in [13].

scalar factor) non-decreasing group homomorphism $\tau_{\preceq} \colon F \to (\mathbb{R}, +)$ whose kernel coincides with the maximal proper $\preceq$-convex subgroup of F. Since F' is a non-Abelian simple group [2], this homomorphism factors through $F/F' \sim \mathbb{Z}^2$, where the last isomorphism is given by $fF' \mapsto (\log_2(f'_+(0)), \log_2(f'_-(1)))$. Hence, we may write (each representative of the class of) τ in the form

$$\tau_{\preceq}(f) = a \log_2(f'_+(0)) + b \log_2(f'_-(1)).$$

A canonical representative is obtained by taking a, b so that $a^2 + b^2 = 1$. We will call this the *normalized Conrad homomorphism* associated to $\preceq$. In many cases, we will consider this homomorphism as defined on $\mathbb{Z}^2 \sim F/F'$, so that $\tau_{\preceq}((m,n)) = am + bn$, and we will identify $\tau_{\preceq}$ to the pair (a, b).

Now let $\preceq^*$ be a bi-ordering on F'. For each closed dyadic interval $I \subset \,]0,1[$ let us consider the induced bi-ordering on $F \sim F_I$. Since all the subgroups F_I for different closed dyadic intervals are conjugate by elements in F', this induced bi-ordering on F –which we will just denote by $\preceq$– does not depend on I, and hence it is inherent to $\preceq^*$. For each such an I, let us consider the corresponding normalized Conrad homomorphism $\tau_{\preceq, I}$.

Lemma 2.1. *If $\tau_{\preceq}$ corresponds to the pair (a,b), then either $a = 0$ or $b = 0$.*

Proof. Assume by contradiction that $a > 0$ and $b > 0$ (all the other cases are analogous). Fix $f \in F_{[1/2, 3/4]}$ such that $f'_+(1/2) > 1$ and $f'_-(3/4) < 1$, and denote $I_1 = [1/4, 3/4]$ and $I_2 = [1/2, 7/8]$. Viewing f as an element in $F_{I_1} \sim F$ we have

$$\tau_{\preceq, I_1}(f) = b \log_2(f'_-(3/4)) < 0.$$

Since Conrad's homomorphism is non-decreasing, this implies that f is negative with respect to the restriction of $\preceq^*$ to F_{I_1}, and therefore $f \prec^* \mathrm{id}$. Now viewing f as an element in $F_{I_2} \sim F$ we have

$$\tau_{\preceq, I_2}(f) = a \log_2(f'_+(1/2)) > 0,$$

which implies that $f \succ^* \mathrm{id}$, thus giving a contradiction. □

We may now pass to the proof of Dlab's theorem. Indeed, assume that for the Conrad's homomorphism above one has $a > 0$ and $b = 0$. We claim that $\preceq^*$ then coincides with $\preceq^+_{x^-}$. To show this, we need to show that a non-trivial element $f \in F'$ is positive with respect to $\preceq^*$ if and only if $f'_+(x_f^-) > 1$. Now such an f may be seen as an element in $F_{[x_f^-, x_f^+]}$, and viewed in this way Conrad's homomorphism yields

$$\tau_{\preceq, [x_f^-, x_f^+]}(f) = a \log_2(f'_+(x_f^-)).$$

Since $a > 0$, if $f'_+(x_f^-) > 1$ then the right-hand member in this equality is positive. Conrad's homomorphism being non-decreasing, this implies that f is positive with respect to $\preceq^*$. Analogously, if $f'_+(x_f^-) < 1$ then f is negative with respect to $\preceq^*$.

Similar arguments show that the case $a < 0$, $b = 0$ (resp. $a = 0$, $b > 0$, and $a = 0, b < 0$) necessarily corresponds to the bi-ordering $\preceq_{x^-}^-$ (resp. $\preceq_{x^+}^-$, and $\preceq_{x^+}^+$), which concludes the proof.

Question 2.2. According to Proposition 1.2, a bi-ordering whose positive cone is finitely generated as a normal semigroup is completely determined by finitely many inequalities. This makes it natural to ask whether this is the case for the restrictions to F' of $\preceq_{x^-}^+$, $\preceq_{x^-}^-$, $\preceq_{x^+}^+$, and $\preceq_{x^+}^-$. A more sophisticated question is the existence of generators f, g of F' such that

- $f'_+(x_f^-) > 1$, $g'_+(x_g^-) > 1$, $f'_-(x_f^+) < 1$, and $g'_-(x_g^+) > 1$,
- $F' \setminus \{\mathrm{id}\}$ is the disjoint union of $\langle\{f, g\}\rangle_N^+$ and $\langle\{f^{-1}, g^{-1}\}\rangle_N^+$,
- $F' \setminus \{\mathrm{id}\}$ is also the disjoint union of $\langle\{f^{-1}, g\}\rangle_N^+$ and $\langle\{f, g^{-1}\}\rangle_N^+$.

A positive answer to this question would immediately imply Dlab's theorem. Indeed, any bi-ordering $\preceq$ on F' would be completely determined by the signs of f and g. For instance, if $f \succ \mathrm{id}$ and $g \succ \mathrm{id}$ then $P_{\preceq}^+$ would necessarily contain $\langle\{f, g\}\rangle_N^+$, and by the second property above this would imply that $\preceq$ coincides with $\preceq_{x^-}^+$.

3. Bi-orderings on F

3.1. Isolated bi-orderings on F. Before classifying all bi-orderings on F, we will first give a proof of the fact that the eight elements in $\mathcal{BO}_{\mathrm{Isol}}(F)$ are isolated in $\mathcal{BO}(F)$. As in the case of F', this proof strongly uses Conrad's homomorphism.

We just need to consider the cases of $\preceq_{x^-}^+$ and $\preceq_{0,x^-}^{+,-}$. Indeed, all the other elements in $\mathcal{BO}_{\mathrm{Isol}}(F)$ are obtained from these by the action of the (finite Klein's) group generated by the involutions $\preceq \mapsto \overline{\preceq}$ and $\preceq \mapsto \preceq_\sigma$.

Let us first deal with $\preceq_{x^-}^+$, denoted $\preceq$ for simplicity. Let $(\preceq_k)$ be a sequence in $\mathcal{BO}(F)$ converging to $\preceq$, and let $\tau_k \sim (a_k, b_k)$ be the normalized Conrad's homomorphism for $\preceq_k$ (so that $\tau_k(m,n) = a_k m + b_k n$ and $a_k^2 + b_k^2 = 1$).

Claim 1. For k large enough one has $b_k = 0$.

Indeed, let f, g be two elements in $F_{]1/2,1]}$ which are positive with respect to $\preceq$ and such that $f'_-(1) = 1/2$ and $g'_-(1) = 2$. For k large enough, these elements must be positive also with respect to $\preceq_k$. Now notice that

$$\tau_k(f) = -b_k \quad \text{and} \quad \tau_k(g) = b_k.$$

Thus, if $b_k \neq 0$ then either $f \prec_k \mathrm{id}$ or $g \prec_k \mathrm{id}$, which is a contradiction. Therefore, $b_k = 0$ for k large enough.

Let us now consider the bi-ordering $\preceq^*$ on $F \sim F_{[1/2,1]}$ obtained as the restriction of $\preceq$. Let $\tau^* \sim (a^*, b^*)$ be the corresponding normalized Conrad's homomorphism.

Claim 2. One has $b^* = 0$.

Indeed, for the elements f, g in $F_{]1/2,1]}$ above we have

$$\tau^*(f) = -b^* \quad \text{and} \quad \tau^*(g) = b^*.$$

If $b^* \neq 0$ this would imply that one of these elements is negative with respect to $\preceq^*$, and hence with respect to $\preceq$, which is a contradiction. Thus, $b^* = 0$.

Denote now by $\preceq_k^*$ the restriction of $\preceq_k$ to $F_{[1/2,1]}$, and let $\tau_k^* \sim (a_k^*, b_k^*)$ be the corresponding normalized Conrad's homomorphism.

Claim 3. For k large enough one has $b_k^* = 0$.

Indeed, the sequence $(\preceq_k^*)$ clearly converges to $\preceq^*$. Knowing also that $b^* = 0$, the proof of this claim is similar to that of Claim 1.

Claim 4. For k large enough one has $a_k > 0$ and $a_k^* > 0$.

Since Conrad's homomorphism is non-trivial, both a_k and a_k^* are nonzero. Take any $f \in F$ such that $f'_+(0) = 2$. We have $\tau_k(f) = a_k$. Hence, if $a_k < 0$ then $f \prec_k \mathrm{id}$, while $f \succ \mathrm{id}$. Analogously, if $a_k^* < 0$ then one would have $g \prec_k \mathrm{id}$ and $g \succ \mathrm{id}$ for any $g \in F_{[1/2,1]}$ satisfying $g'(1/2) = 2$.

Claim 5. If a_k and a_k^* are positive and b_k and b_k^* are zero, then $\preceq_k$ coincides with $\preceq$.

Given $f \in F$ such that $f \succ \mathrm{id}$, we need to show that f is positive also with respect to $\preceq_k$. If $x_f^- = 0$ then $f'_+(0) > 1$, and since $a_k > 0$ this gives $\tau_k(f) = a_k \log_2(f'_+(0)) > 0$, and thus $f \succ_k \mathrm{id}$. If $x_f^- \neq 0$ then $f'_+(x_f^-) > 1$, and since $a_k^* > 0$ this gives $\tau_k^*(f) = a_k^* \log_2(f'_+(x_f^-)) > 0$, and therefore one still has $f \succ_k \mathrm{id}$.

The proof for $\preceq_{0,x^-}^{+,-}$ is similar to the above one. Indeed, Claims 1, 2, and 3, still hold. Concerning Claim 4, one now has that $a_k > 0$ and $a_k^* < 0$ for k large enough. Having this in mind, one easily concludes that $\preceq_k$ coincides with $\preceq_{0,x^-}^{+,-}$ for k very large.

Question 3.1. It would be nice to know whether the positive cone of each element in $\mathcal{BO}_{\mathrm{Isol}}(F)$ is finitely generated as a normal semigroup. Notice however that these bi-orderings cannot be completely determined by the signs of finitely many elements, since $\mathcal{BO}(F)$ is infinite (compare Question 2.2).

3.2. Classifying all bi-orderings on *F*. To simplify, we will denote by Λ the union of $\Lambda_{x^-}^+$, $\Lambda_{x^-}^-$, $\Lambda_{x^+}^+$, and $\Lambda_{x^+}^-$. To prove our main result, fix a bi-ordering $\preceq$ on F, and let $\tau_\preceq \colon F \to (\mathbb{R}, +)$ be the corresponding normalized Conrad's homomorphism.

Since $\tau_{\preceq} \sim (a,b)$ is non-trivial and factors through $\mathbb{Z}^2 \sim F/F'$, there are two different cases to be considered.

Case I. The image $\tau_{\preceq}(\mathbb{Z}^2)$ has rank two.

This case appears when the quotient a/b is irrational. In this case, $\preceq$ induces the bi-ordering of irrational type $\preceq_{a/b}$ on $\mathbb{Z}^2$ viewed as F/F' (cf. Example 1.1). Indeed, for each $f \in F \setminus F'$ the value of $\tau_{\preceq}(f)$ is nonzero, and hence it is positive if and only if $f \succ \mathrm{id}$.

The kernel of $\tau_{\preceq}$ coincides with F'. By Dlab's theorem, the restriction of $\preceq$ to F' must coincide with one of the bi-orderings $\preceq^{+}_{x-}$, $\preceq^{-}_{x-}$, $\preceq^{+}_{x+}$, or $\preceq^{-}_{x+}$. Therefore, $\preceq$ is contained in Λ, and the bi-ordering induced on the $\mathbb{Z}^2$-fiber is of irrational type.

Case II. The image $\tau_{\preceq}(\mathbb{Z}^2)$ has rank one.

This is the difficult case: it appears when either a/b is rational or $b = 0$. There are two sub-cases.

Subcase 1. Either $a = 0$ or $b = 0$.

Assume first that $b = 0$. Denote by $\preceq^*$ the bi-ordering induced on $F_{[1/2,1]}$, and let $\tau_{\preceq^*} \sim (a^*, b^*)$ be its normalized Conrad's homomorphism. We claim that either a^* or b^* is equal to zero. Indeed, suppose for instance that $a^* > 0$ and $b^* > 0$ (all the other cases are analogous). Let m, n be integers such that $n > 0$ and $a^*m - b^*n > 0$, and let f be an element in $F_{[3/4,1]}$ such that $f'_+(3/4) = 2^m$ and $f'_-(1) = 2^{-n}$. Then $\tau_{\preceq^*}(f) = -b^*n < 0$, and hence $f \prec \mathrm{id}$. On the other hand, taking $h \in F$ such that $h(3/4) = 1/2$, we get that $h^{-1}fh \in F_{[1/2,1]}$, and

$$\tau_{\preceq^*}(h^{-1}fh) = a^* \log_2((h^{-1}fh)'_+(1/2)) + b^* \log_2((h^{-1}fh)'_-(1)) = am - bn > 0.$$

But this implies that $h^{-1}fh$, and hence f, is positive with respect to $\preceq$, which is a contradiction.

(i) If $a > 0$ and $a^* > 0$: We claim that $\preceq$ coincides with $\preceq^{+}_{x-}$ in this case. Indeed, let $f \in F$ be an element which is positive with respect to $\preceq^{+}_{x-}$. We need to show that $f \succ \mathrm{id}$. Now, since $a > 0$, if $x^-_f = 0$ then

$$\tau_{\preceq}(f) = a \log_2(f'_+(0)) > 0,$$

and hence $f \succ \mathrm{id}$. If $x^-_f \neq 0$ then taking $h \in F$ such that $h(x^-_f) = 1/2$ we obtain that $h^{-1}fh \in F_{[1/2,1]}$, and

$$\tau_{\preceq^*}(h^{-1}fh) = a^* \log_2((h^{-1}fh)'(1/2)) = a^* \log_2(f'(x^-_f)).$$

Since $a^* > 0$, the value of the last expression is positive, which implies that $h^{-1}fh$, and hence f, is positive with respect to $\preceq$.

(ii) If $a > 0$ and $a^* < 0$: Similar arguments to those of (i) above show that $\preceq$ coincides with $\preceq^{+,-}_{0,x-}$ in this case.

(iii) If $a > 0$ and $b^* > 0$: We claim that $\preceq$ belongs to Λ, and that the induced bi-ordering on the $\mathbb{Z}^2$-fiber is the lexicographic one. To show this, we first observe that if $f \in F \setminus F'$ is positive then either $f'_+(0) > 1$, or $f'_+(0) = 1$ and $f'_-(1) > 1$. Indeed, if $f'_+(0) \neq 1$ then the value of $\tau_{\preceq}(f) = a \log_2(f'_+(0)) \neq 0$ must be positive, since Conrad's homomorphism is non-decreasing. If $f'_+(0) = 1$ we take $h \in F$ such that $h(1/2) = x_f^-$. Then $h^{-1} f h$ belongs to $F_{[1/2,1]}$, and the value of

$$\tau_{\preceq^*}(h^{-1} f h) = b^* \log_2((h^{-1} f h)'_-(1)) = b^* \log_2(f'_-(1)) \neq 0$$

must be positive, since f (and hence $h^{-1} f h$) is a positive element of F.

To show that $\preceq$ induces a bi-ordering on $\mathbb{Z}^2$, we need to check that F' is $\preceq$-convex. Let $g \in F'$ and $h \in F$ be such that $\mathrm{id} \preceq h \preceq g$. If h was not contained in F', then hg^{-1} would be a negative element in $F \setminus F'$. But since

$$(hg^{-1})'_+(0) = h'_+(0) \quad \text{and} \quad (hg^{-1})'_-(1) = h'_-(1),$$

this would contradict the remark above. Therefore, h belongs to F', which shows the $\preceq$-convexity of F'. Again, the remark above shows that the induced bi-ordering on $\mathbb{Z}^2$ is the lexicographic one.

(iv) If $a > 0$ and $b^* < 0$: As in (iii) above, $\preceq$ belongs to Λ, and the induced bi-ordering $\preceq_{\mathbb{Z}^2}$ on the $\mathbb{Z}^2$-fiber is the one for which $(m,n) \succ_{\mathbb{Z}^2} (0,0)$ if and only if either $m > 0$, or $m = 0$ and $n < 0$.

(v) If $a < 0$ and $a^* > 0$: As in (i) above, $\preceq$ coincides with $\preceq^{-,+}_{0,x^-}$ in this case.

(vi) If $a < 0$ and $a^* < 0$: As in (i) above, $\preceq$ coincides with $\preceq^{-}_{x^-}$ in this case.

(vii) If $a < 0$ and $b^* > 0$: As in (iii) above, $\preceq$ belongs to Λ, and the induced bi-ordering $\preceq_{\mathbb{Z}^2}$ on the $\mathbb{Z}^2$-fiber is the one for which $(m,n) \succ_{\mathbb{Z}^2} (0,0)$ if and only if either $m < 0$, or $m = 0$ and $n > 0$.

(viii) If $a < 0$ and $b^* < 0$: As in (iii) above, $\preceq$ belongs to Λ, and the induced bi-ordering $\preceq_{\mathbb{Z}^2}$ on the $\mathbb{Z}^2$-fiber is the one for which $(m,n) \succ_{\mathbb{Z}^2} (0,0)$ if and only if either $m < 0$, or $m = 0$ and $n < 0$.

The case $a = 0$ is analogous to the preceding one. Letting now $\preceq^*$ be the restriction of $\preceq$ to $F_{[0,1/2]}$, for the normalized Conrad's homomorphism $\tau_{\preceq^*} \sim (a^*, b^*)$ one may check that either $a^* = 0$ or $b^* = 0$.

Assume that $b > 0$. In the case $b^* > 0$ (resp. $b^* < 0$), the bi-ordering $\preceq$ coincides with $\preceq^{-}_{x^+}$ (resp. $\preceq^{-,+}_{1,x^+}$). If $a^* > 0$ (resp. $a^* < 0$), then $\preceq$ corresponds to a point in Λ whose induced bi-ordering $\preceq_{\mathbb{Z}^2}$ on the $\mathbb{Z}^2$-fiber is the one for which $(m,n) \succ_{\mathbb{Z}^2} (0,0)$ if and only if either $n > 0$, or $n = 0$ and $m > 0$ (resp. either $n > 0$, or $n = 0$ and $m < 0$).

Assume now that $b < 0$. In the case $b^* > 0$ (resp. $b^* < 0$), the bi-ordering $\preceq$ coincides with $\preceq^{+,-}_{1,x^+}$ (resp. $\preceq^{+}_{x^+}$). If $a^* > 0$ (resp. $a^* < 0$), then $\preceq$ corresponds to a point in Λ whose induced bi-ordering $\preceq_{\mathbb{Z}^2}$ on the $\mathbb{Z}^2$-fiber is the one for which $(m,n) \succ_{\mathbb{Z}^2} (0,0)$ if and only if either $n < 0$, or $n = 0$ and $m > 0$ (resp. either $n < 0$, or $n = 0$ and $m < 0$).

Subcase 2. Both a and b are nonzero.

The main issue here is to show that F' is necessarily $\preceq$-convex in F. Now since $\ker(\tau_{\preceq})$ is already $\preceq$-convex in F, to prove this it suffices to show that F' is $\preceq$-convex in $\ker(\tau_{\preceq})$. Assume by contradiction that f is a positive element in $\ker(\tau_{\preceq}) \setminus F'$ that is smaller than some $h \in F'$. Suppose first that $\preceq$ restricted to F' coincides with either $\preceq^{+}_{x^-}$ or $\preceq^{-}_{x^-}$, and denote by p the leftmost fixed point of f in $]0,1]$. We claim that f is smaller than any positive element $g \in F_{]0,p[}$. Indeed, since $\preceq$ coincides with either $\preceq^{+}_{x^-}$ or $\preceq^{-}_{x^-}$ on F', the element f is smaller than any positive $\bar{h} \in F_{]0,p[}$ such that $x^{+}_{\bar{h}}$ is to the left of x^{-}_{h}; taking $n \in \mathbb{Z}$ such that $f^{-n}(x^{-}_{h})$ is to the right of x^{-}_{g}, this gives $f = f^{-n} f f^{n} \prec f^{-n} \bar{h} f^{n} \prec g$.

Now take a positive element $h_0 \in F_{]0,p[}$ such that for $\bar{f} = h_0 f$ there is no fixed point in $]0,p[$ (it suffices to consider a positive $h_0 \in F_{[\frac{p}{4},\frac{3p}{4}]}$ whose graph is very close to the diagonal). Then $\mathrm{id} \prec \bar{f} \prec h_0 g$ for every positive $g \in F_{]0,p[}$. The argument above then shows that $\bar{f}$ is smaller than every positive element in $F_{]0,p[}$. In particular, since $h_0 = \bar{f} f^{-1}$ is in $F_{]0,p[}$ and is positive, this implies that $\bar{f} \prec \bar{f} f^{-1}$, and hence $f \prec \mathrm{id}$, which is a contradiction.

If the restriction of $\preceq$ to F' coincides with either $\preceq^{+}_{x^+}$ or $\preceq^{-}_{x^+}$, one proceeds similarly but working on the interval $[q,1]$ instead of $[0,p]$, where q denotes the rightmost fixed point of f in $[0,1[$. This concludes the proof of the $\preceq$-convexity of F', and hence that of our main result.

Remark 3.2. Our arguments may be easily modified to show that the subgroup $F_- = \{f \in F \mid f'_+(0) = 1\}$ has six different bi-orderings, namely (the restrictions of) $\preceq^{+}_{x^-}$, $\preceq^{-}_{x^-}$, $\preceq^{+}_{x^+}$, $\preceq^{-}_{x^+}$, $\preceq^{+,-}_{1,x^+}$, and $\preceq^{-,+}_{1,x^+}$. An analogous statement holds for $F_+ = \{f \in F \mid f'_-(1) = 1\}$. Finally, the group of piecewise-affine orientation-preserving dyadic homeomorphisms of the real line whose support is bounded from the right (resp. from the left) admits only two bi-orderings, namely (the natural analogues of) $\preceq^{+}_{x^+}$ and $\preceq^{-}_{x^+}$ (resp. $\preceq^{+}_{x^-}$ and $\preceq^{-}_{x^-}$). Notice however that this last result is already contained in Dlab's work [5] (compare Remark 1.6).

References

[1] M. G. Brin, The chameleon groups of Richard J. Thompson: automorphisms and dynamics. *Inst. Hautes Études Sci. Publ. Math.* **84** (1996), 5–33. Zbl 0891.57037 MR 1441005

[2] J. W. Cannon, W. J. Floyd, and W. R. Parry, Introductory notes on Richard Thompson's groups. *Enseign. Math.* (2) **42** (1996), 215–256. Zbl 0880.20027 MR 1426438

[3] I. M. Chiswell and P. H. Kropholler, Soluble right orderable groups are locally indicable. *Canad. Math. Bull.* **36** (1993), 22–29. Zbl 0802.20035 MR 1205890

[4] P. Conrad, Right-ordered groups. *Michigan Math. J.* **6** (1959), 267–275. Zbl 0099.01703 MR 0106954

[5] V. Dlab, On a family of simple ordered groups. *J. Austral. Math. Soc.* **8** (1968), 591–608. Zbl 0165.34502 MR 0228397

[6] V. M. Kopytov and N. Y. Medvedev, *Right-ordered groups*. Siberian School of Algebra and Logic, Plenum Publ. Corp., New York 1996. Zbl 0852.06005 MR 1393199

[7] S. H. McCleary, An even better representation for free lattice-ordered groups. *Trans. Amer. Math. Soc.* **290** (1985), 81–100. Zbl 0546.06014 MR 787956

[8] N. Ya. Medvedev, Maximal orders on groups of piecewise linear functions of the real line. In *Ordered algebraic structures*. Algebra Logic Appl. 16, Gordon and Breach, Amsterdam 2001, 77–89. Zbl 1038.06508 MR 1938920

[9] N. Ya. Medvedev, Partial orders on Dlab groups. *Algebra i Logika* **40** (2001), 135–157; English transl. *Algebra and Logic* **40** (2001), 75–86. Zbl 0974.06012 MR 1850443

[10] D. W. Morris, Amenable groups that act on the line. *Algebr. Geom. Topol.* **6** (2006), 2509–2518. Zbl 05118602 MR 2286034

[11] A. Navas, On the dynamics of (left) orderable groups. *Ann. Inst. Fourier* (*Grenoble*), to appear.

[12] A. Navas and C. Rivas, A new characterization of Conrad's property for group orderings, with applications. *Algebr. Geom. Topol.* **9** (2009), 2079–2100.

[13] C. Rivas, On spaces of Conradian group orderings. *J. Group Theory*, to appear.

[14] L. Robbiano, Term orderings on the polynomial ring. In *EUROCAL* '85, vol. 2 (Linz, 1985), Lecture Notes in Comput. Sci. 204, Springer-Verlag, Berlin 1985, 513–517. Zbl 0584.13016 MR 0826583

[15] A. S. Sikora, Topology on the spaces of orderings of groups. *Bull. London Math. Soc.* **36** (2004), 519–526. Zbl 1057.06006 MR 2069015

[16] L. H. Smith, On ordering free groups. *J. Symbolic Comput.* **40** (2005), 1285–1290; Corrigendum (with A. Clay). *Ibid.* **44** (2009), 1529–1532. Zbl 1124.20024 MR 2178087; Zbl 05603621

[17] H.-H. Teh, Construction of orders in Abelian groups. *Proc. Cambridge Philos. Soc.* **57** (1961), 476–482. Zbl 0104.24603 MR 0123625

[18] A. V. Zenkov and N. Ya. Medvedev, On Dlab groups. *Algebra i Logika* **38** (1999), 531–548; English transl. *Algebra and Logic* **38** (1999), 289–298. Zbl 0937.06016 MR 1766701

Received February 20, 2009; revised March 24, 2009

A. Navas, Departamento de Matemática y Ciencia de la Computación, Universidad de Santiago de Chile, Alameda 3363 – Estación Central, Santiago, Chile

E-mail: andres.navas@usach.cl

C. Rivas, Departamento de Matemática y Ciencia de la Computación, Universidad de Chile, Las Palmeras 3425, Ñuñoa, Santiago, Chile

E-mail: cristobalrivas@u.uchile.cl