MOSCOW MATHEMATICAL JOURNAL
Volume 9, Number 2, April–June 2009, Pages 263–303

ON THE QUESTION OF ERGODICITY FOR MINIMAL GROUP ACTIONS ON THE CIRCLE

BERTRAND DEROIN, VICTOR KLEPTSYN, AND ANDRÉS NAVAS

ABSTRACT. This work is devoted to the study of minimal, smooth actions of finitely generated groups on the circle. We provide a sufficient condition for such an action to be ergodic (with respect to the Lebesgue measure), and we illustrate this condition by studying two relevant examples. Under an analogous hypothesis, we also deal with the problem of the zero Lebesgue measure for exceptional minimal sets. This hypothesis leads to many other interesting conclusions, mainly concerning the stationary and conformal measures. Moreover, several questions are left open. The methods work as well for codimension-one foliations, though the results for this case are not explicitly stated.

2000 MATH. SUBJ. CLASS. Primary: 37C85; Secondary: 37A50, 37D25, 37E10, 37F15.

KEY WORDS AND PHRASES. Ergodic theory, group actions, circle diffeomorphisms, Lyapunov exponents, random dynamical systems, stationary measures.

1. INTRODUCTION

1.1. Minimality and ergodicity. An invariant probability measure μ (for a map, or for a group action) is said to be *ergodic* if every invariant measurable set is either of zero or full μ-measure. This is equivalent to the fact that every invariant measurable function is constant μ-a.e., and also to the fact that μ is an extremal point of the compact, convex set formed by the invariant probability measures.

The definition of ergodicity can be naturally extended to non necessarily invariant measures μ which are at least *quasi-invariant*, that is, such that $g_*\mu$ is absolutely continuous with respect to μ for every element g in the acting group. (To simplify the exposition, all the measures in this article are supposed to be probability measures.)

Definition 1.1. Let μ be a measure on a measurable space X which is quasi-invariant by the action of a group G. We say that μ is *ergodic* if for every measurable G-invariant subset $A \subset X$ either $\mu(A) = 0$ or $\mu(X \setminus A) = 0$.

Received June 8, 2008.

Supported in part by the Swiss National Science Foundation and the PBCT-Conicyt Research Project ADI-17. The second named author supported in part by the RFBR grants 7-01-00017-a and CNRS-L_a 05-01-02801.

Notice that the definition of ergodicity concerns both the action and the measure. However, for several actions an invariant (or quasi-invariant) measure is naturally defined. For instance, symplectic maps or Hamiltonian flows and their restrictions to fixed energy levels have natural invariant measures, and for any C^1 diffeomorphism the Lebesgue measure is quasi-invariant. In these situations, one focuses on the action itself, and the ergodicity is always meant with respect to this natural measure.

Ergodicity can be thought of as a property involving some complexity for the orbits of the action coming from the fact that this action is irreducible from a measurable point of view. The topological counterpart to this notion corresponds to *minimality*:

Definition 1.2. A continuous action of a group G on a topological space X is said to be *minimal* if for every G-invariant closed subset $A \subset X$ either $A = \varnothing$ or $A = X$. Equivalently, an action is minimal if all of its orbits are dense.

It is natural to ask to what extend the properties of ergodicity and minimality are related. In one direction, it is easy to see that, in general, the former does not imply the latter. Indeed, ergodicity concerns the behavior of almost every point, and not of *all* the points. Actually, one can easily construct examples of ergodic group actions having global fixed points. The question in the opposite direction is more interesting:

Question 1.3. Under what conditions a smooth minimal action of a group on a compact manifold is necessarily Lebesgue-ergodic?

The following widely known conjecture concerns the one-dimensional case of this question. The main result of this work, namely Theorem A later on, allows to solve it by the affirmative under some additional assumptions that seem to us interesting and sufficiently mild.

Conjecture 1.4. *Every minimal smooth action of a finitely generated group on the circle is ergodic with respect to the Lebesgue measure.*[1]

Conjecture 1.4 has been answered by the affirmative in many cases using the exponential expansion strategy (going back to D. Sullivan). We will recall this strategy in Section 2. Here we content ourselves in recalling the definition of the Lyapunov expansion exponent in order to state the main result which is known in this direction. For simplicity of the exposition, from now on we assume that the diffeomorphisms in our group G preserve the orientation. This assumption is non-restrictive, as otherwise one can pass (without loosing the minimality) to the index-two subgroup formed by the orientation preserving elements.

Definition 1.5. Let G be a finitely generated group of circle diffeomorphisms. Let $\mathcal{F} = \{g_1, \ldots, g_k\}$ be a finite set of elements generating G as a semigroup, and let

[1] One may also ask about ergodicity for smooth actions of finitely generated groups on the circle or the interval having a dense orbit. However, we will not deal with this more general question in this work.

$\|\cdot\|_{\mathcal{F}}$ be the corresponding word-length norm. The *Lyapunov expansion exponent* of G at a point $x \in S^1$ is

$$\lambda_{\exp}(x; \mathcal{F}) := \limsup_{n\to\infty} \max_{\|g\|_{\mathcal{F}} \leqslant n} \frac{1}{n} \log(g'(x)).$$

Notice that the value of the Lyapunov expansion exponent $\lambda_{\exp}(x; \mathcal{F})$ depends on the choice of the finite system of generators $\mathcal{F}$. However, the fact that this number is equal to zero or is positive does not depend on this choice. Thus, relations of the form $\lambda_{\exp}(x) = 0$ or $\lambda_{\exp}(x) > 0$ make sense, although the number $\lambda_{\exp}(x)$ is not well-defined without referring to $\mathcal{F}$.

Theorem 1.6 (S. Hurder [16])**.** *If G is a finitely generated subgroup of $\mathrm{Diff}_+^{1+\alpha}(S^1)$ acting minimally, then the Lyapunov expansion exponent is constant Lebesgue-almost everywhere. If this constant is positive, then the action is ergodic.*

This constant is called the Lyapunov expansion exponent $\lambda_{\exp}(G; \mathcal{F})$ of the action. As before, it depends on the particular choice of the system of generators $\mathcal{F}$, though the relations of the form $\lambda_{\exp}(G) > 0$ or $\lambda_{\exp}(G) = 0$ make sense without referring to $\mathcal{F}$.

A very simple compactness type argument shows that $\lambda_{\exp}(G) > 0$ in the case where for all $x \in S^1$ there exists $g \in G$ such that $g'(x) > 1$. Actually, the ergodicity of minimal $C^{1+\alpha}$ actions satisfying the latter condition was proved earlier in [23].

All of this serves as a good motivation for the following

Definition 1.7. A point $x \in S^1$ is said to be *non-expandable* if for all $g \in G$ one has $g'(x) \leqslant 1$.

One should immediately point out that the presence of non-expandable points does not contradict the minimality. For instance, the canonical action of the modular group $\mathrm{PSL}_2(\mathbb{Z})$ on $S^1 = \mathbb{P}(\mathbb{R}^2)$ is minimal, but the points $(0 : 1)$ and $(1 : 0)$ are non-expandable (see Section 5.2). Another example is provided by the smooth actions of the Thompson group T constructed by É. Ghys and V. Sergiescu (see Section 5.1). However, the non-expandable points represent a (potential) obstacle for performing the exponential expansion strategy.

Denote the set of non-expandable points by $\mathrm{NE} = \mathrm{NE}(G)$. Notice that this set depends on the coordinates chosen on the circle. Thus, we suppose a metric on the circle to be fixed, and we will discuss the dependence of the NE-set on the metric later (see Corollary 1.10). The assumption of our main result is given by the next

Definition 1.8. The group G satisfies *property* $(\star)$ if it acts minimally and for every $x \in \mathrm{NE}(G)$ there exist g_+, g_- in G such that $g_+(x) = g_-(x) = x$ and g_+ (respectively, g_-) has no other fixed point in some interval $(x, x+\varepsilon)$ (respectively, $(x-\varepsilon, x)$).

Remark 1.9. Notice that property $(\star)$ holds if for every $x \in \mathrm{NE}$ there exists an element $g \in G$ such that x is an isolated fixed point of g. In particular, if G is a group of real-analytic diffeomorphisms, then property $(\star)$ is equivalent to:

$$\text{for all } x \in \mathrm{NE}(G) \text{ there exists } g \in G \text{ such that } g \neq \mathrm{id} \text{ and } g(x) = x.$$

Indeed, every fixed point of a nontrivial analytic diffeomorphism is isolated.

We are now ready to state the main result of this paper. In order to simplify our discussion[2], we will only deal with finitely generated groups of circle diffeomorphisms that are of class C^2.

Theorem A. *If G is a finitely generated subgroup of $\mathrm{Diff}^2_+(S^1)$ satisfying property $(\star)$, then the following hold:*

(1) $\mathrm{NE}(G)$ *is finite.*
(2) *For every point $x \in S^1$ either the set of derivatives $\{g'(x) \colon g \in G\}$ is unbounded, or x belongs to the orbit of some non-expandable point.*
(3) *G is ergodic with respect to the Lebesgue measure.*

The second conclusion of Theorem A allows deducing the following

Corollary 1.10. *For finitely generated groups of C^2 circle diffeomorphisms, property $(\star)$ does not depend on the choice of the Riemannian metric on the circle.*

The assumption in Theorem A is well illustrated by two fundamental examples. The first one appears in [11], where É. Ghys and V. Sergiescu showed that the canonical (and actually unique up to semiconjugacy, as was proved by Ghys [12] and later by Liousse [21] using different ideas) action of the Thompson group T on the circle is topologically conjugate to an action by C^∞ diffeomorphisms.[3] For this (minimal) action (that we recall in Section 5.1) we prove the following

Theorem B. *The* NE*-set for the minimal Ghys–Sergiescu's action of the Thompson group T consists of a single point, which is an isolated fixed point of an element. (Therefore, this action satisfies property $(\star)$.) However, the Lyapunov expansion exponent of the action is zero.*

It was pointed out to us by É. Ghys that there exist smooth actions of the group T (still satisfying property $(\star)$ and with zero Lyapunov expansion exponent) having more than one non-expandable point: see Remark 5.1.

The second example of a minimal group action with a non-empty set of non-expandable points is the already mentioned action of $\mathrm{PSL}_2(\mathbb{Z})$. (Notice that, since this group is discrete inside $\mathrm{PSL}_2(\mathbb{R})$, the rotation number of each of its elements is rational: compare footnote 3.) For this case we have the following

Theorem C. *The only non-expandable points of the canonical action of* $\mathrm{PSL}_2(\mathbb{Z})$ *(in the standard affine chart) are* 0 *and* ∞*. Both of them are isolated fixed points of certain elements (namely, $x \mapsto x/(x+1)$ and $x \mapsto x+1$, respectively), and therefore the action satisfies property $(\star)$. However, its Lyapunov expansion exponent equals zero.*

The fact that the action of $\mathrm{PSL}_2(\mathbb{Z})$ is ergodic is well-known. Indeed, one way to show this consists in extending this action inside the Poincaré disc $\mathbb{D}$. If a

[2]The reader will easily check that, as it is usual in the subject, the methods and results in this work also apply to groups of diffeomorphisms of class C^1 having Lipschitz derivative.

[3]They also constructed a C^∞-action of T with a minimal invariant Cantor set which is semi-conjugate to the standard one, obtaining as a corollary the rationality of the rotation number for each element of T. As a consequence, the ergodicity for the smooth minimal actions of T on the circle cannot be deduced from Katok-Herman's result discussed in Section 2.1.

measurable invariant set A existed, then the solution of the Dirichlet problem with $\mathbf{1}_A$ as boundary value would be an invariant harmonic function. This function would then descend to the quotient $\mathbb{D}/\mathrm{PSL}_2(\mathbb{Z})$, which is the modular surface. However, there exists no non-constant, bounded, harmonic function on the modular surface, which gives a contradiction. Despite this very simple argument, it is interesting to notice that the ergodicity can be also deduced from the property $(\star)$.

1.2. Exceptional minimal sets. Recall that for every group of circle homeomorphisms without finite orbits and whose action is not minimal, there exists a minimal closed invariant set which is homeomorphic to the Cantor set (and which is commonly called an *exceptional minimal set*): see for instance [12]. The following conjecture, stated by G. Hector (as far as we know, in 1977–78), appears to be fundamental in this context:

Conjecture 1.11 (G. Hector)**.** *If a finitely generated group of C^2 circle diffeomorphisms admits an exceptional minimal set Λ, then the Lebesgue measure of Λ is zero.*

In this work, we deal with this conjecture under a condition which is analogous to property $(\star)$:

Definition 1.12. Let G be a finitely generated group of circle diffeomorphisms admitting an exceptional minimal set Λ. We say that G satisfies *property* $(\Lambda\star)$, if for every $x \in \Lambda \cap NE$ there exist g_-, g_+ in G such that $g_+(x) = g_-(x) = x$ and g_+ (respectively, g_-) has no other fixed point in some interval $(x, x+\varepsilon)$ (respectively, $(x-\varepsilon, x)$).[4]

The theorem below is a natural analogue of our Theorem A for exceptional minimal sets. As their proofs are also completely analogous, we leave to the reader the task of adapting the arguments of the proof of Theorem A to this case.

Theorem D. *Let G be a finitely generated group of C^2 circle diffeomorphisms having an exceptional minimal set Λ. If the action satisfies the property $(\Lambda\star)$, then:*

(1) *The set $\Lambda \cap \mathrm{NE}(G)$ is finite.*
(2) *For each $x \in \Lambda$ not contained in the orbit of $\mathrm{NE}(G)$, the set of derivatives $\{g'(x) \colon g \in G\}$ is unbounded.*
(3) *The Lebesgue measure of Λ is zero.*

As in the case of minimal actions, the second conclusion of the theorem above implies the following

Corollary 1.13. *For finitely generated groups of C^2 circle diffeomorphisms having an exceptional minimal set Λ, property $(\Lambda\star)$ does not depend on the choice of the Riemannian metric on the circle.*

[4]Actually, for the case where x is isolated in Λ from one side this condition may be weakened, only asking for an element having x as an isolated fixed point from the side where it is an accumulation point of Λ.

1.3. Conformal measures. For conformal (in particular, for one-dimensional, smooth) maps, a fundamental property of the Lebesgue measure is that its infinitesimal change at a point is given by the derivative to the power of the dimension of the underlying space. This property was generalized by D. Sullivan (see [29]), who introduced the concept of *conformal measure* as a powerful tool for studying the dynamics on exceptional minimal sets.

Definition 1.14. Let G be a group of conformal transformations. A measure μ on the underlying space is said to be *conformal with exponent* δ (or simply δ-*conformal*), if for every Borel set B and for every map $g \in G$ one has

$$\mu(g(B)) = \int_B |g'(x)|^\delta \, d\mu(x),$$

where $|g'|$ stands for the scalar part of the (conformal) derivative of g. Equivalently, for μ-almost every point x, the Radon–Nikodym derivative of $g_*\mu$ w.r.t. μ equals

$$\frac{dg_*\mu}{d\mu}(x) = \frac{1}{|g'(g^{-1}(x))|^\delta}.$$

For the case of the Lebesgue measure, the conformal exponent equals the dimension. Nevertheless, in presence of a proper closed invariant set, one can ask for the existence of a conformal measure (perhaps with a different exponent) supported on it. For the case of a subgroup of $\mathrm{PSL}_2(\mathbb{C})$ acting conformally on the Riemann sphere with a proper minimal closed invariant set different from a finite orbit, such a measure was constructed by D. Sullivan in [28].

It is unclear whether for every finitely generated group of circle diffeomorphisms having an exceptional minimal set Λ, there exists a conformal measure supported on Λ. This is the case for groups of real-analytic diffeomorphisms. Indeed, in this case the group acts discretely on the complement of Λ (see [14]). Then, the arguments used by D. Sullivan in [28] apply to prove the existence of an exponent $0 < \delta \leqslant 1$ for which a δ-conformal measure exists.

Conformal measures are expected to be ergodic as the Lebesgue measure is supposed to be in the minimal case. Although we are not able to settle this problem in its full generality here, we are able to deal with a weaker property, namely *conservativity*:

Definition 1.15. An action of a group G on a measurable space X that quasi-preserves a measure μ is said to be *conservative*, if for every measurable set A with $\mu(A) > 0$, there exists an element $g \in G \setminus \{e\}$ such that $\mu(A \cap g(A)) > 0$.

A non-conservative action is indeed automatically non-ergodic, since for each measurable subset $B \subset A$ of intermediate measure $0 < \mu(B) < \mu(A)$, the set $G(B) = \bigcup_{g \in G} g(B)$ is invariant and has intermediate measure $0 < \mu(G(B)) < 1$.

The following result, due to D. Sullivan, can be viewed as a partial evidence supporting Conjecture 1.4:

Theorem 1.16 (D. Sullivan [29])**.** *Let G be a finitely generated group of C^2 circle diffeomorphisms. If the action of G is minimal, then it is conservative* (*with respect to the Lebesgue measure*).

By adapting D. Sullivan's arguments, we obtain the following

Theorem E. *Let G be a finitely generated group of C^2 circle diffeomorphisms. Then any conformal measure of exponent $\delta \leqslant 1$ without atomic part and which is supported on an infinite minimal invariant set is conservative.*

The ergodicity (and uniqueness) of a conformal measure on a minimal set Λ was proved by D. Sullivan when the group G has the *expansion property*, i.e., G has no finite orbits, and $\mathrm{NE}(G)$ does not intersect Λ:

Theorem 1.17 (D. Sullivan [28])**.** *Let G be a group of C^2 circle diffeomorphisms acting with the expansion property. Then there is a unique conformal measure supported on the minimal set. If the action is minimal, this is the Lebesgue measure. If there is an exceptional minimal set Λ, this is the corresponding (normalized) Hausdorff measure, which is then non-vanishing and finite. In the latter case, the conformal exponent equals to the Hausdorff dimension of Λ, which verifies $0 < HD(\Lambda) < 1$.*

It is unclear whether in the general case the Hausdorff measure on the minimal set is finite and nonzero. (If this is the case, it would be a conformal measure.) However, using the control of distortion technique for the expansion, one can obtain the following uniqueness result for the conformal measure:

Theorem F. *Let G be a finitely generated group of C^2 circle diffeomorphisms.*

(1) *If G (acts minimally and) satisfies property $(\star)$, then the Lebesgue measure is the unique conformal measure which does not charge the orbit $G(\mathrm{NE})$ of the set NE of non-expandable points. Moreover, all the other (atomic) conformal measures (if any) are supported on $G(\mathrm{NE})$, and their conformal exponents are strictly greater than 1.*

(2) *If the action of G carries an exceptional minimal set Λ for which the property $(\Lambda\star)$ holds, then there exists at most one conformal measure supported on Λ and not charging the G-orbit of $\Lambda \cap \mathrm{NE}$. If such a measure exists, then its exponent belongs to the interval $(0, 1)$.*

(3) *In particular, in both cases (1) and (2) above, the non-atomic conformal measures supported on the minimal invariant set (the whole circle or Λ, respectively), are ergodic.*

As a final remark let us point out that, quite surprisingly, there exist examples of conformal measures on the circle whose conformal exponent exceeds one (that is, the dimension of the circle). These examples illustrate the restrictions imposed in Theorem F:

Example 1.18. There exists a Ghys–Sergiescu's non-minimal C^2 action on the circle of the Thompson group T such that, for every $\delta \geqslant 1$, there exists a conformal measure of exponent δ concentrated on the endpoints of the complementary intervals of the minimal set.

Example 1.19. For the (minimal) standard $\mathrm{PSL}_2(\mathbb{Z})$-action, for every exponent $\delta > 1$ there exists a δ-conformal measure concentrated on the orbit of the non-expandable point $(0 : 1)$.

1.4. Stationary measures. Another concept related to the study of group actions is that of random dynamics. Namely, if in addition to an action of a group G on a compact space X is given a measure m on G, then one can consider the left random walk on G generated by m, and the corresponding random process on X. In other words, one deals with *random* sequences of compositions

$$\mathrm{id},\ g_1,\ g_2 \circ g_1,\ \ldots,\ g_n \circ g_{n-1} \circ \cdots \circ g_1,\ \ldots$$

where all the $g_j \in G$ are chosen independently and are distributed with respect to m. The images $x_k = g_k \cdots g_1(x_0)$ of a given initial point $x_0 \in X$ can be then considered as its "random iterates", as one has $x_{k+1} = g_{k+1}(x_k)$. Associated to this concept there is the following

Definition 1.20. A measure ν on X is *stationary with respect to* m if it coincides with the average of its images, that is, for every Borel set $B \subset X$,

$$\nu(B) = \int_G (g_*\nu)(B)\, dm(g).$$

This widely studied notion is in some sense analogous to that of an invariant measure for single maps. For instance, the existence of a stationary measure may be deduced by the classical Krylov–Bogolubov procedure of time averaging; moreover, an analogue of the Birkhoff Ergodic Theorem holds... We recall more details on this in Section 2.

For a diffeomorphism g of the circle we let

$$\|g\|_{\mathrm{Diff}^1} := \max\bigl\{\sup_x g'(x),\, \sup_x (g^{-1})'(x)\bigr\}.$$

This allows to define the *first* Diff^1*-moment*[5] of a measure m on G by

$$\int_G \log^+(\|g\|_{\mathrm{Diff}^1})\, dm(g).$$

A related notion is the *first word-moment* defined by

$$\int_G \|g\|_{\mathcal{F}}\, dm(g),$$

where $\mathcal{F} \subset G$ is a prescribed generating set and $\|\cdot\|_{\mathcal{F}}$ denotes the word-norm with respect to $\mathcal{F}$. These two notions are related, but not equivalent: we comment this difference at the end of this section.

If for an action on the circle the first Diff^1-moment of the measure m is finite, one can define a *random Lyapunov exponent* with respect to an ergodic stationary measure. If there is no common invariant measure for the elements in the support supp(m) of m, then a result of P. Baxendale [2] ensures the existence of an ergodic stationary measure whose random Lyapunov exponent is negative (which corresponds to some kind of a local contraction by random compositions).

[5]Formally speaking, it would be more exact to call it the first log-Diff^1-moment, as we are taking the logarithm of the Diff^1-norm, in order to deal with a composition-subadditive number. However, we prefer shortening the terminology.

Using the negativity of the random Lyapunov exponent, and somehow reversing the time of the dynamics, we obtain the following result relating the stationary measures to the Lyapunov expansion exponent of individual points.

Theorem G. *Let G be a group of C^1 circle diffeomorphisms, and let m be a measure on G having finite first word-moment. Assume that there is no measure on the circle which is invariant by all the elements in* $\mathrm{supp}(m)$*. Then there exists a m-stationary measure ν such that the Lyapunov expansion exponent is positive at ν-almost every point.*

This result turns out to be interesting for the study of the question of the regularity of the stationary measure that we explain below.

Namely, quite often the stationary measure turns out to be unique. More precisely, the local contraction, coming from the negativity of the random Lyapunov exponent of some stationary measure ν, enables to prove the uniqueness of the stationary measure in the basin of attraction of ν (see [1], [6]). This provides the uniqueness of the stationary measure in the case where, in addition to the absence of a $\mathrm{supp}(m)$-invariant measure, either the support $\mathrm{supp}(m)$ generates the whole group G, or the action on the circle of the semigroup generated by $\mathrm{supp}(m)$ is minimal. We recall the precise definitions and statements of these results later in Section 2.2.

Now, in the case of uniqueness of the stationary measure, the latter is either absolutely continuous, or singular with respect to the Lebesgue measure (as both the absolutely continuous and the singular parts would be stationary). This dichotomy is at the origin of very interesting problems; in particular, there is the following conjecture, that we learned from Y. Guivarc'h, V. Kaimanovich, and F. Ledrappier:

Conjecture 1.21. *For any finitely supported measure m on a lattice $\Gamma < \mathrm{PSL}_2(\mathbb{R})$ whose support generates Γ, the corresponding stationary measure on the circle is singular.*

This was proven by Y. Guivarc'h and Y. Le Jan in [13] for non-cocompact lattices (i.e., the quotient of the hyperbolic disc by the action has at least one cusp). Moreover, their result still holds if the measure m on Γ has finite first word-moment.

In this direction, our Theorem G provides the following

Corollary 1.22. *Assume that the Lyapunov expansion exponent for a finitely generated group G of C^2 circle diffeomorphisms is equal to zero. Then for any measure m on G such that the corresponding stationary measure ν is unique and non* $\mathrm{supp}(m)$*-invariant, ν is singular with respect to the Lebesgue one.*

In particular, this holds for measures m with finite first word-moment, without $\mathrm{supp}(m)$*-common invariant measures, and such that either the support* $\mathrm{supp}(m)$ *generates G, or the semigroup generated by this support acts minimally on the circle.*

Remark 1.23. Corollary 1.22 applies (under the same conditions on the measure m) for the minimal actions of the Thompson group T and of $\mathrm{PSL}_2(\mathbb{Z})$ that we mentioned earlier.

Remark 1.24. According to the definition, the Lyapunov expansion exponent corresponds to an upper limit. Therefore, the "exponentially expanding" compositions for ν-almost every point in the conclusion of Theorem G are proved to exist only for an infinite subsequence of lengths n_k. However, under some more restrictive assumptions on the moments (that are satisfied, for instance, if the measure m is finitely supported), one can prove that for ν-almost every point the compositions with exponentially big derivative exist for *every* n.

To end this paragraph, we would like to notice a subtle and interesting difference, concerning the question of the finiteness of the first moment; this difference was pointed out to us by V. Kaimanovich. Namely, the first moment for a measure on a lattice $\Gamma < \mathrm{PSL}_2(\mathbb{R})$ can be measured in two different ways: in the sense of (the logarithm of) the Diff^1-norm[6], and in the sense of the word-norm. Since

$$\log^+ \|f \circ g\|_{\mathrm{Diff}^1} \leqslant \log^+ \|f\|_{\mathrm{Diff}^1} + \log^+ \|g\|_{\mathrm{Diff}^1},$$

finiteness of the first word-moment implies the finiteness of the first Diff^1-moment. However, the converse does not hold. Indeed, the Furstenberg discretization procedure [10] allows to find a measure m on $PLS_2(\mathbb{Z})$ such that the m-stationary measure is the Lebesgue one. Due to the result of Y. Guivarc'h and Y. Le Jan [13] (or due to our Corollary 1.22), such a measure m cannot be of finite first word-moment. Nevertheless (see [18] and references therein), it can be chosen with a finite Diff^1-moment!

1.5. Structure of the paper. In Section 2, we give some background on the problems that we consider and we recall several facts that will be used later. Section 3 is devoted to the open questions. Some of these questions were widely known before this work, and some other naturally appeared when writing this paper. Section 4 is devoted to the description of one of the main tools in one-dimensional dynamics, namely the control of distortion technique. We also give the proof of Theorem E therein. In Section 5, we study two examples discussed before, namely the $\mathrm{PSL}_2(\mathbb{Z})$ and the Thompson group actions, and prove Theorems B and C. Finally, Section 6 is devoted to the remaining proofs, i.e., those of Theorems A, F, and G.

2. Background

2.1. Minimality, ergodicity, and exceptional minimal sets. We begin by pointing out that all the assumptions of Conjecture 1.4 are essential, and none of them can be omitted. Concerning the dimension, a celebrated construction by H. Furstenberg [8] leads to examples of area preserving diffeomorphisms of the torus $\mathbb{T}^2$ which are minimal but not ergodic. These diffeomorphisms are skew-products over irrational rotations, that is, maps of the form $F : (x,\, y) \mapsto (x+\alpha,\, y+$

[6]In fact, the value of $\log \|g\|_{\mathrm{Diff}^1}$ is equivalent to that of $\mathrm{Var}_{S^1}(\log g')$. A direct computation shows for $g \in \mathrm{PSL}_2(\mathbb{R})$ that $\mathrm{Var}_{S^1}(\log g') = 4\,\mathrm{dist}_{hyp}(g(0),\, 0)$, where the circle $S^1 \subset \mathbb{C}$ is the unit circle, and the map g is naturally extended to the interior of the hyperbolic disc $\mathbb{D} \ni 0$. On the other hand, $\mathrm{PSL}_2(\mathbb{R})$ can be thought as the unit tangent bundle $T_1(\mathbb{D})$ (which is hence a hyperbolic metric space), so the latter value is equivalent to the hyperbolic distance from g to the identity.

$\varphi(x)$). Assume that the angle α is Liouvillian, and that the cocycle corresponding to the function $\varphi(x)$ is measurably trivial but nontrivial in the continuous category. Then the map F appears to be measurably conjugate to an horizontal rotation $(x, y) \mapsto (x+\alpha, y)$, and thus non-ergodic; however, the absence of a continuous conjugacy allows to show that it is minimal.

For the remaining hypotheses, following the general approach of [6], it is convenient to distinguish two cases according to the existence or nonexistence of an invariant

The hypothesis concerning smoothness is very subtle. Indeed, it is not difficult to construct minimal circle homeomorphisms that are non-ergodic. However, the construction of C^1 diffeomorphisms with these properties is quite technical and much more difficult: see [25]. (It is very plausible that, by refining the methods from [25], one may actually provide examples of $C^{1+\alpha}$ such diffeomorphisms for any $0 < \alpha < 1$.) For a non measure preserving example, one may follow (an easy extension of) the construction in [11] starting with a slight modification of the expanding map constructed by Quas in [26] (so that it becomes tangent to the identity at the endpoints). For $n \geqslant 10$, this provides us with examples of C^1 actions of the n-adic Thompson groups (which are finitely presented) that are minimal but not ergodic. (We point out however that these examples seem to be non $C^{1+\alpha}$ smoothable for any $\alpha > 0$.)

Finally, to illustrate the finite generation hypothesis, one may construct an example of an Abelian group action via a sequence of actions of $G_n = \mathbb{Z}/2^n\mathbb{Z}$, where the n^{th} action is obtained from the previous one by specifying a particular choice of a "square root" of the generator. Such a choice is equivalent to the choice of a two-fold covering $S^1/G_n \to S^1/G_{n+1}$. It may be checked that with a well chosen sequence of actions, one can ensure that the resulting action of the group $G = \bigcup_n G_n$ is minimal, but it is non-ergodic and even non-conservative: there is a set of positive measure which is disjoint from all of its images by nontrivial elements of G. Although this construction seems to be well known, we didn't find it in the literature, and for the reader's convenience we provide more details at the end of Section 4. We point out, however, that there exists a simpler example (due to D. Sullivan [29]) of a non-ergodic, minimal, smooth group action on the circle without invariant measure. Namely, fixing a Cantor set $\Lambda \subset S^1$ of positive Lebesgue measure, for each connected component I of $S^1 \setminus \Lambda$ one chooses an hyperbolic reflection g_I with respect to the geodesic joining the endpoints of I. Then the action of the group generated by the g_I's is minimal (this can be checked using the fact that every orbit intersects all the complementary intervals of Λ, and thus accumulates everywhere on Λ). Nevertheless, it is non-ergodic (and even non-conservative), since the set Λ of positive measure is disjoint from all of its images by nontrivial elements.

Let us now consider the case of a subgroup $G \subset \mathrm{Diff}^2(S^1)$ satisfying the hypotheses of Conjecture 1.4. We first point out that the ergodicity is a nontrivial issue even when G is generated by a single diffeomorphism. Indeed, Poincaré's theorem implies that every minimal circle homeomorphism is topologically conjugate to an irrational rotation. However, for "an essential part" of the set of minimal diffeomorphisms, the conjugating map appears to be singular, and therefore the ergodicity with respect to the Lebesgue measure after conjugacy does not imply the ergodicity with respect to the Lebesgue measure before it. Nevertheless, the conjecture for this case has been settled independently by A. Katok for C^{1+bv} diffeomorphisms (see for instance [19]) and by M. Herman for C^{1+lip} diffeomorphisms (see [15]). Actually, Katok's proof uses arguments of control of distortion for the iterations that are based on the existence of decompositions of the circle into arcs which are almost permuted by the dynamics (and which come from the good rational approximations of the rotation number).

If G has no element with irrational rotation number, the result above cannot be applied. However, in this case the dynamics has some hyperbolic behaviour. To show the ergodicity one then would like to apply the exponential expansion strategy. This classical procedure consists in expanding very small intervals which concentrate a good proportion of some invariant set, in such a way that the distortion (see a precise definition in Section 4) of the compositions remains controlled. More precisely, the scheme works as follows. Let $A \subset S^1$ be an invariant measurable subset of positive Lebesgue measure. By Lebesgue's theorem, almost every point $x \in A$ is a density (or Lebesgue) point, that is,

$$\frac{\mu_L(U_\delta(x) \cap A)}{\mu_L(U_\delta(x))} \to 1 \quad \text{as } \delta \to 0,$$

where $U_\delta(x)$ denotes the δ-neighborhood of x. Now take $\delta > 0$ such that the proportion of points of A in $U_\delta(x)$ is very close to 1. If one can expand this interval keeping a uniform bound for the distortion, then each one of the "expanded" intervals also has a proportion of points in A very close to 1. On the other hand, since their length stay bounded away from zero, after passing to the limit along a sequence $\delta_n \to 0$ what we see is an interval in which the points in A form a subset of full relative measure. If the action is minimal, this implies that A is a subset of full measure in the circle.

The arguments that we have just cited were used by the third author in [23] and by S. Hurder in [16] for establishing their ergodicity results that we mentioned in the Introduction: roughly speaking, if the expansion can be done "sufficiently quickly", then minimal actions are necessarily ergodic.

For exceptional minimal sets, the zero Lebesgue measure conjecture was proven by the third author for the case where for each $x \in \Lambda$ there exists $g \in G$ such that $g'(x) > 1$ (see [23]). Later on, S. Hurder showed in [16] that the Lebesgue measure of the intersection $\Lambda \cap (S^1 \setminus \{x \colon \lambda_{\exp}(x) = 0\})$ is equal to zero. The conjecture has been proved by Cantwell and Conlon also for the case where the dynamics is "Markovian" (see [4]).

Once again, both hypotheses of G. Hector's Conjecture 1.11 are essential, and one can construct counter-examples in the case where they are not satisfied (see for instance [3] and [15] for the hypothesis concerning smoothness).

2.2. Random dynamics. Random dynamical systems have been studied for a long time, and we are certainly unable to recall here all (and even the main) achievements of this theory. We shall then restrict ourselves to those that will be necessary for the exposition.

First, we would like to recall that a random dynamics can be modeled in terms of a single map. Indeed, consider the map

$$F\colon X \times G^{\mathbb{N}} \to X \times G^{\mathbb{N}}, \quad F(x, (g_i)_{i=1}^{\infty}) = (g_1(x), (g_{i+1})_{i=1}^{\infty}),$$

which is a skew-product over the left shift on $G^{\mathbb{N}}$. In terms of this map, instead of saying that we consider random compositions of maps, we can say that we take a random point in $G^{\mathbb{N}}$, distributed with respect to $m^{\mathbb{N}}$, and then we consider the iterations of F on the fiber over this point. A direct computation then shows that a measure ν is m-stationary if and only if the measure $\nu \times m^{\mathbb{N}}$ is F-invariant.

The latter remark allows to apply to the random dynamics all the arsenal of techniques from Ergodic Theory — Krylov–Bogolubov procedure (implying the existence of stationary measures), Birkhoff Ergodic Theorem (ensuring the convergence of random time averages), etc.

In particular, one can define Lyapunov exponents for a smooth random dynamics on a compact manifold, provided that the first Diff^1-moment of m is finite. We will not do this in a general situation, and we will restrict ourselves to the case of the dynamics on the circle. In this case, the *random Lyapunov exponent* corresponding to a point $x \in S^1$ and to a sequence $(g_i) \in G^{\mathbb{N}}$ is defined as the limit

$$\lim_{n\to\infty} \frac{1}{n} \log(g_n \circ \dots g_1)'(x). \tag{1}$$

Simple arguments show that, for a given measure m on G with finite first Diff^1-moment, and for any m-stationary ergodic measure ν, the limit (1) is constant (in particular, it exists) almost everywhere w.r.t. the measure $\nu \times m^{\mathbb{N}}$. By Birkhoff Ergodic Theorem, this constant equals

$$\int_{S^1} \int_G \log g'(x)\, dm(g)\, d\nu(x),$$

and we denote it by $\lambda_{RD}(m; \nu)$.

Now, according to a general principle in one-dimensional dynamics which has been developed in the work of many authors, if a random dynamics on the circle that does not preserve any measure then "random compositions contract". In other words, under certain general assumptions (for instance, the system should be supposed to be non-factorizable) one can conclude that a long composition, most probably, will map almost all the circle (except for a small interval) into a small interval. Equivalently, for any two given points of the circle, most probably their orbits along the same random sequence of compositions will approach each other.

We would like to recall here the following results illustrating this principle. In his seminal work [9], H. Furstenberg proved the contraction statement for projective dynamics in arbitrary dimension (in particular, on the circle). The work of V.A. Antonov [1] established the contraction for any minimal and inverse-minimal non-factorizable random dynamics on the circle (one can also find an exposition of this work in [20], [24]). In his excellent work, P. Baxendale [2] studied the sum of the Lyapunov exponents for a smooth random dynamics on a compact manifold of any dimension.

More precisely, P. Baxendale proved that for such a dynamics, if the first Diff^1-moment is finite (so that the random Lyapunov exponents are well-defined) and there is no common invariant measure, then there exists an ergodic stationary measure such that the sum of its Lyapunov exponents (which can be thought of as the exponential rate of volume change) is negative. In particular, for the circle (as it is one-dimensional), this implies the negativity of the Lyapunov exponent, which, in its turn (due to the distortion control arguments) implies local contraction by the random dynamics.

Together with the results of V.A. Antonov, the above result becomes a powerful tool for studying group dynamics on the circle. In particular, this was exploited by the authors in [6], where they proved the (global) contraction property for a symmetric measure m (in fact, the same arguments work if the support $\mathrm{supp}(m)$ generates the acting group as a semigroup).

The global contraction property implies the uniqueness of the stationary measure (see [1], [6]). Moreover, if the contraction property holds only locally, then the stationary measure is still unique provided that the system is minimal.

3. Open Questions

We must point out that the actions of the Thompson group T and of $\mathrm{PSL}_2(\mathbb{Z})$ that we deal with in this article are (up to some easy modifications) the only minimal, smooth actions of non-Abelian groups on the circle for which we know that $\mathrm{NE} \neq \varnothing$. This motivates the following

Question 3.1. Does every (sufficiently smooth) minimal action on the circle of a non-Abelian finitely generated group satisfy property $(\star)$?

Both the positive or negative answer to this question would be interesting: the positive one would lead to an interesting general property of minimal actions on the circle, and the negative one would give an example of a "monster", certainly having very strange properties.

According to Theorems B and C, for the actions of T and $\mathrm{PSL}_2(\mathbb{Z})$ the corresponding Lyapunov expansion exponents are zero. Therefore, the following question makes sense:

Question 3.2. Let G be a finitely generated non-Abelian group (perhaps having property $(\star)$) of (sufficiently smooth) circle diffeomorphisms. If the action of G is minimal and $\mathrm{NE}(G) \neq \varnothing$, is it necessarily true that $\lambda_{\mathrm{exp}}(G) = 0$?

Once again, both the positive or negative answer to this question are interesting, the positive one leading to a general property, and the negative one providing us with an interesting and perhaps strange action.

A particular case of Question 3.2, closely related to Conjecture 1.21, is the following one:

Question 3.3. Is it true that for every non-cocompact lattice $\Gamma < \mathrm{PSL}_2(\mathbb{R})$, the Lyapunov expansion exponent of its action on the circle is zero?

A positive answer to this question looks very plausible. By joining it to our Theorem G, it would give another proof to the singularity theorem of Y. Guivarc'h and Y. Le Jan cited in the Introduction.

In fact, adapting the arguments of the proof of Theorem C (and using some techniques from Riemann Surfaces Theory), one can show that, for every lattice as above, its action on the circle satisfies the property $(\star)$. Moreover, the NE-set turns out to be non-empty and corresponds in some precise sense to the set of cusps in the quotient surface. The proofs of these facts are, however, rather technical, and we do not give them here since this would overload the paper.

For the study of conformal measures, the results we stated in the Introduction lead to many other questions that seem interesting to us. First, the fact that for a minimal dynamics the only non-atomic conformal measure is the Lebesgue one, was proven only under the assumption of property $(\star)$. In would be interesting to answer this question in general:

Question 3.4. Is it true that for any minimal smooth action of a finitely generated group on the circle the only non-atomic conformal measure is the Lebesgue one?

By Theorem F, a positive answer to Question 3.1 would automatically imply a positive answer to Question 3.4, but certainly the latter question can be attacked independently (and perhaps will be simpler to handle via some other way).

Analogous questions, as well as several new ones, are interesting in presence of an exceptional minimal set: does every finitely generated action with an exceptional minimal set satisfy property $(\Lambda\star)$? Is it true that for every finitely generated group action (not necessarily satisfying property $(\Lambda\star)$) with an exceptional minimal set Λ, there exists at most one non-atomic conformal measure supported on it? Does such a measure always (or under the assumption of property $(\Lambda\star)$) exist? If yes, does it coincide with the normalized Hausdorff measure (which then will be non-vanishing and finite)? In the case of existence of such a measure, does its conformal exponent coincide with the the Hausdorff dimension of the minimal set? Is it true that, in the general (i.e., minimal dynamics or exceptional minimal set) situation, a conformal measure with exponent greater than one is atomic?

To conclude this section, we would like to state a question due to É. Ghys concerning the dichotomy between absolute continuity and singularity for stationary measures. To motivate this question, first notice that, in the examples of singular stationary measure for minimal actions that we have already mentioned (Thompson's group T, non-cocompact lattices in $\mathrm{PSL}_2(\mathbb{R})$), the corresponding groups are generated by maps that are "far" from the identity.

Moreover, singular stationary measures naturally appear for (expanding) actions of fundamental groups of closed genus $g > 1$ surfaces. Indeed, to each conformal structure on such a surface corresponds an action of its fundamental group on the circle (viewed as the boundary of its universal cover, i.e., the Poincaré disc). Though the actions corresponding to different complex structures are topologically conjugate, the conjugating map is always singular. Thus, among the stationary measures corresponding to different structures (for the same probability distribution on the group), at most one is absolutely continuous. Notice however that, once again, these groups are generated by maps that are "far" from the identity.

In another direction, a result due to J. Rebelo [27] asserts that topological conjugacies between non-solvable groups of circle diffeomorphisms generated by elements *near the identity* are absolutely continuous. Thus, the above methods for obtaining a singular stationary measure stop working if we restrict ourselves to such actions.

Due to all of this, it is interesting to find out whether there are examples of singular stationary measures for actions generated by maps close to the identity:

Question 3.5 (É. Ghys)**.** Let G be a non-Abelian group of C^2 circle diffeomorphisms without finite orbits and generated as a semigroup by finitely many elements close to rotations (in the sense of [23]). If m is any measure supported on this system of generators, is it necessarily true that the associated stationary measure on S^1 is equivalent to the Lebesgue measure? Is this true under the additional assumption that the set of generators and the distribution m are symmetric with respect to inversion?

It is interesting to notice that, under some assumptions, for analytic perturbations of the trivial system the equation for the density of the stationary measure admits at least a formal solution as a power series in the parameter.

4. Control of Distortion and Conservativity

We begin this section by recalling several lemmas concerning control of distortion which are classical in the context of smooth one-dimensional dynamics. A more detailed discussion may be found, for instance, in [6], and in many of the references therein. Therefore, we will not enter into the technical details of their proofs here, and we will just briefly describe the ideas. We begin with a general definition.

Definition 4.1. Given two intervals I, J and a C^1 map $F\colon I \to J$ which is a diffeomorphism onto its image, we define the *distortion coefficient* of F on I by

$$\varkappa(F;\, I) := \log\Big(\frac{\max_I F'}{\min_I F'}\Big),$$

and its *distortion norm* by

$$\eta(F;\, I) := \sup_{\{x,y\}\subset I} \frac{\log\big(\frac{F'(x)}{F'(y)}\big)}{|F(x)-F(y)|} = \max_J \left|\big(\log((F^{-1})')\big)'\right|.$$

It is easy to check that the distortion coefficient is subadditive under composition. Moreover, by the Lagrange Theorem, one has $\varkappa(F,\, I) \leqslant C_F|I|,$ where the constant C_F depends only on the Diff^2-norm of F (indeed, one can take C_F as being the maximum of the absolute value of the derivative of the function $\log(F')$). This implies immediately the following

Proposition 4.2. *Let $\mathcal{F}$ be a subset of $\mathrm{Diff}^2_+(S^1)$ which is bounded with respect to the Diff^2-norm. If I is an interval on the circle and $f_1, \ldots, f_n$ are finitely many elements chosen from $\mathcal{F}$, then*

$$\varkappa(f_n \circ \cdots \circ f_1;\, I) \leqslant C_{\mathcal{F}} \sum_{i=0}^{n-1} |f_i \circ \cdots \circ f_1(I)|,$$

where the constant $C_{\mathcal{F}}$ depends only on the set $\mathcal{F}$.

In other words, if we compose several “relatively simple” maps, then a bound for the sum of the lengths of the successive images of an interval I provides a control for the distortion of the whole composition over I. Using this fact one can show the following

Corollary 4.3. *Under the assumptions of Proposition 4.2, let us fix a point $x_0 \in I$, and let us denote $F_i := f_i \circ \cdots \circ f_1$, $I_i := F_i(I)$, and $x_i := F_i(x_0)$. Then the following inequalities hold*:

$$\exp\Big(-C_{\mathcal{F}} \sum_{j=0}^{i-1} |I_j|\Big) \cdot \frac{|I_i|}{|I|} \leqslant F_i'(x_0) \leqslant \exp\Big(C_{\mathcal{F}} \sum_{j=0}^{i-1} |I_j|\Big) \cdot \frac{|I_i|}{|I|}, \tag{2}$$

$$\sum_{i=0}^{n} |I_i| \leqslant |I| \exp\Big(C_{\mathcal{F}} \sum_{i=0}^{n-1} |I_i|\Big) \sum_{i=0}^{n} F_i'(x_0). \tag{3}$$

Notice that the sum in the exponential in (3) goes up to $i = n-1$, while the sum of the lengths in the left hand side expression goes up to n. Using an induction argument, this seemingly innocuous remark appears to be fundamental for establishing the following

Proposition 4.4. *Under the assumptions of Proposition 4.2, given a point $x_0 \in S^1$ let us denote $S := \sum_{i=0}^{n-1} F_i'(x_0)$. Then for every $\delta \leqslant \log(2)/2C_{\mathcal{F}}S$ one has $\varkappa(F_n, U_{\delta/2}(x_0)) \leqslant 2C_{\mathcal{F}}S\delta$.*

As a consequence, if the sum of the derivatives is not too big, then up to a multiplicative constant one can approximate the length of the image interval in Proposition 4.2 by the length of the initial interval I times the derivative of the composition at a given point in I. This simple fact allows us already to prove the conservativity of conformal measures.

Proof of Theorem E.. Let $\mathcal{F}$ be a finite family of generators of G as a semi-group. Suppose that there exists a Borel subset A of the circle such that $\mu(A) > 0$, and $\mu(A \cap g(A)) = 0$ for every nontrivial element $g \in G$. This immediately yields $\mu(g(A) \cap h(A)) = 0$ for every $g \neq h$ in G, which gives

$$1 \geqslant \mu\Big(\bigcup_{g\in G} g(A)\Big) = \sum_{g\in G} \mu\big(g(A)\big) = \sum_{g \in G} \int_A g'(x)^\delta \, d\mu(x) = \int_A \Big(\sum_{g\in G} g'(x)^\delta\Big) d\mu(x).$$

Therefore, for μ-almost every point $x \in A$, the sum $\sum_{g\in G} g'(x)^\delta$ converges, and since $\delta \leqslant 1$, the same holds for the sum $S(x) := \sum_{g\in G} g'(x)$. Let us fix one of these points x_0, also belonging to the minimal set Λ (we can do this, as the measure μ is concentrated on Λ), and let I be an open neighborhood centered at x_0 having length strictly smaller than $\log(2)/2C_{\mathcal{F}}S(x_0)$. We claim that for every $g \in G$, and every $x \in I$, one has $g'(x) \leqslant 2g'(x_0)$. Indeed, this follows directly from Proposition 4.4 by writing g as a product of generators. Now the above implies that the μ-measure of the set $B := \bigcup_{g\in G} g(I)$ is smaller than or equal to

$$2^\delta \mu(I) \sum_{g\in G} g'(x_0)^\delta.$$

Since μ is non-atomic, if I is chosen small enough, then the value of this expression is strictly smaller than 1. If this is the case, the complementary set of B is of positive μ-measure, and hence intersects Λ. On the other hand, B is an open G-invariant set containing $x_0 \in \Lambda$. Therefore, $\Lambda \cap (S^1 \setminus B)$ is a nonempty closed invariant set, strictly containted in Λ, and this contradicts the minimality of Λ. $\square$

As we have seen in the Introduction, D. Sullivan's Theorem 1.16 is no longer true for non finitely generated groups of circle diffeomorphisms acting minimaly. For the sake of completeness, we provide below the details of the already mentioned example of a non finitely generated Abelian group of circle diffeomorphisms whose action is minimal but non-conservative.

The construction works by induction. Fix a dense sequence of points x_n in S^1. Let g_1 the Euclidean rotation of order 2, and assume that for an integer $n \geqslant 2$ a generator g_{n-1} of G_{n-1} has been already constructed. Let $p_{n-1}\colon S^1 \to S^1$ be the $(n-1)$-fold covering map induced by g_{n-1}. For $\varepsilon_n > 0$ small enough, the set $p_{n-1}^{-1}(p_{n-1}(U_{\varepsilon_n}(x_n))$ is formed by 2^{n-1} disjoint intervals, and the lengths of these intervals tend to zero as ε_n goes to zero. Let us enumerate these intervals (modulo 2^{n-1} and respecting their cyclic order on S^1) by $I_{n-1}^1, \ldots, I_{n-1}^{2^{n-1}}$, and let us denote by J_{n-1}^i the maximal open interval to the right of I_{n-1}^i contained in the complementary set of the union of the I_{n-1}^j's. Now choose a generator g_n of G_n sending each I_{n-1}^i (resp. J_{n-1}^i) into J_{n-1}^i (resp. I_{n-1}^{i+1}), by appropriately lifting from the quotient S^1/G_{n-1} a diffeomorphism that interchanges $p_{n-1}(U_{\varepsilon_n}(x_n))$ with its complement.

Notice that every G_n-orbit intersects the interval $U_{\varepsilon_n}(x_n)$. It is not difficult to deduce from this that, if the sequence ε_n tends to zero as n goes to infinity, then the action of $G := \bigcup_n G_n$ is minimal. To ensure the non-conservativity we choose ε_n sufficiently small so that

$$\mathrm{Leb}\big(p_{n-1}^{-1}(p_{n-1}(U_{\varepsilon_n}(x_n)))\big) < \frac{1}{2^{n+1}},$$

and we define a decreasing sequence of sets A_n, each of which is disjoint from its nontrivial G_n-images, by letting

$$A_0 := S^1, \quad A_n := A_{n-1} \setminus p_{n-1}^{-1}\big(p_{n-1}(U_{\varepsilon_n}(x_n))\big).$$

By construction, the intersection $A := \bigcap_n A_n$ is a (measurable) set which is disjoint from all of its images under nontrivial elements in G. Moreover,

$$\mathrm{Leb}(A) \geqslant 1 - \sum_{n \geqslant 1} \mathrm{Leb}\big(p_{n-1}^{-1}(p_{n-1}(U_{\varepsilon_n}(x_0)))\big) \geqslant 1 - \sum_{n \geqslant 1} \frac{1}{2^{n+1}} = 1 - \frac{1}{2} > 0.$$

This shows that the action is non-conservative.

To close this section we would like to point out that, to the best of our knowledge, the only examples of minimal non-ergodic group actions by C^2 circle diffeomorphisms that there exist in the literature are constructed by prescribing a positive measure set which is disjoint from all of its images (i.e., they are actually non-conservative). This motivates the following

Question 4.5. Is every minimal and conservative action of a (non finitely generated) group by C^2 circle diffeomorphisms necessarily ergodic?

Notice that the minimal non-ergodic examples using Quas' construction that we mentioned in the Background (see Section 2.1) are based on a different idea. However, these actions seem to be non C^2 smoothable (in many cases this follows from our Theorem A).

5. Examples

5.1. The smooth, minimal action of the Thompson group *T*. Recall that Thompson's group T is the group of circle homeomorphisms which are piecewise linear in such a way that all the break points, as well as their images, are dyadic rational numbers, and which induce a bijection of the set of dyadic rationals (notice that these properties force the derivatives on the linearity intervals to be integer powers of 2).

As É. Ghys and V. Sergiescu have cleverly noticed in [11], the dynamics of this group is somehow "generated" by a single (non invertible) map, namely $\varphi_0 \colon x \mapsto 2x \mod 1$. Indeed, φ_0 has a (unique) fixed point $x = 0$ whose preorbit is exactly the set of dyadic rationals, and Thompson's group T is the set of homeomorphisms obtained by gluing finitely maps of the form $\varphi_0^{-k} \circ \varphi_0^l$ at dyadic rationals (here, for φ_0^{-k} one can choose any of the corresponding 2^k branches).

The main argument of the construction in [11] consists in replacing φ_0 by another degree-two smooth monotonous map φ fixing the point $x = 0$ and being sufficiently tangent to the identity at this point. One can then define the set of "φ-dyadically rational" points as the φ-preorbit of 0, and one can make correspond, to each element $f \in T$, the map $[f]_\varphi$ which is obtained by gluing (in a coherent way) the branches of $\varphi^{-k} \circ \varphi^l$ instead of $\varphi_0^{-k} \circ \varphi_0^l$ at the corresponding φ-dyadically rational points. The issue here is that, since φ is tangent to the identity at 0, the maps obtained after gluing are smooth (actually, as smooth as the order of the tangency is). Thus, $f \mapsto [f]_\varphi$ is a smooth action of the Thompson group T on the circle.

By choosing appropriately the map φ, the previous action can be made either minimal or having a minimal invariant Cantor set. Here we are going to deal with the first case, which is ensured if φ satisfies $\varphi'(x) > 1$ for all $x \neq 0$.

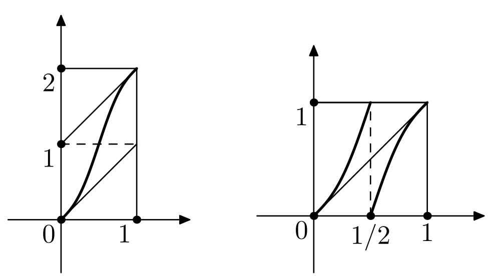

FIGURE 1. The map φ

We can now pass to the

Proof of Theorem B. The first claim of the theorem, namely, the equality NE $=$ $\{0\}$, is rather simple. Indeed, it is quite clear that for every point $x \neq 0$ one can find an element g in the modified Thompson group $[T]_\varphi$ that coincides with φ in a neighborhood of x, and this implies that $g'(x) = \varphi'(x) > 1$. On the other hand, every $g \in [T]_\varphi$ coincides in some right neighborhood of 0 with a map of the form $\varphi^{-k} \circ \varphi^{l}$ for some non negative integers k, l. Therefore

$$g'(0) = (\varphi^{-k} \circ \varphi^{l})'(0) = (\varphi^{-k})'(\varphi^{l}(0)) \cdot (\varphi^{l})'(0) = (\varphi^{-k})'(0) \leqslant 1,$$

where the third equality follows from the fact that 0 is a neutral fixed point of φ, while the last inequality comes from the fact that φ is a non-uniformly expanding map.

Remark 5.1. As it was pointed out to us by É. Ghys, for slightly different maps φ the NE-set may contain finitely many φ-periodic orbits along which the derivative of φ equals 1. For instance, for the map $\varphi\colon x \mapsto 2x - \frac{1}{6\pi}\sin(6\pi x)$, the induced action of T (is minimal and) satisfies $\mathrm{NE}([T]_\varphi) = \{0,\, 1/3,\, 2/3\}$.

To prove the equality $\lambda_{\exp}([T]_\varphi) = 0$, we fix a finite set of elements $\mathcal{F} = \{f_1, \ldots, f_s\}$ which generates $[T]_\varphi$ as a semigroup. Each of these elements f_i coincide locally with maps of the form $\varphi^{-k_{i,j}} \circ \varphi^{l_{i,j}}$. If we let $L := \max_{i,j}\{l_{i,j}\}$, then any composition of the generators having length n writes, near and to the right of a given point x, in the form

$$f_{i_1} \circ \cdots \circ f_{i_n}\big|_{[x,x+\varepsilon]} = \varphi^{-k_{j_1}} \circ \varphi^{l_{j_1}} \circ \cdots \circ \varphi^{-k_{j_n}} \circ \varphi^{l_{j_n}}\big|_{[x,x+\varepsilon]}.$$

Notice that none of the compositions $\varphi^{-1} \circ \varphi$ can be simplified. However, the identity $\varphi \circ \varphi^{-1} = \mathrm{id}$ still holds, and this allows to reduce the above expression to

$$f_{i_1} \circ \cdots \circ f_{i_n}\big|_{[x,x+\varepsilon]} = \varphi^{-k} \circ \varphi^{l}\big|_{[x,x+\varepsilon]},$$

where $l \leqslant Ln$. Thus,

$$(f_{i_1} \circ \cdots \circ f_{i_n})'(x) = (\varphi^{-k} \circ \varphi^{l})'(x) \leqslant (\varphi^{l})'(x) \leqslant (\varphi^{Ln})'(x),$$

where the inequalities follow from the non-uniform expansivity of φ. Hence, to show that the Lyapunov expansion exponent of $[T]_\varphi$ is zero, it suffices to show that the same holds for the map φ. To do this we will use the following result due to T. Inoue [17], which will be discussed at the end of this section since some of the involved ideas will be used latter.

Lemma 5.2 (T. Inoue [17])**.** *For Lebesgue-a.e. point $x \in S^1$, the* time averages *measures*

$$\mu_{n,x} := \frac{1}{n}\sum_{i=0}^{n-1} \delta_{\varphi^{i}(x)}$$

converge to the Dirac measure δ_0 concentrated at the neutral fixed point 0 of φ.

Using Lemma 5.2, classical arguments from Ergodic Theory show that the Lyapunov exponent of the map φ is a.e. equal to zero. Indeed, since for every point $x \in S^1$ and every $n \in \mathbb{N}$ one has

$$\frac{1}{n}\log(\varphi^n)'(x) = \frac{\log\varphi'(x) + \log\varphi'(\varphi(x)) + \cdots + \log\varphi'(\varphi^{n-1}(x))}{n}$$
$$= \int_{S^1} \log\varphi'(s)\, d\mu_{n,x}(s),$$

and since for a.e. $x \in S^1$ one has $\mu_{n,x} \xrightarrow[n\to\infty]{*\text{-weakly}} \delta_0$, one concludes that, for a.e. $x \in S^1$,

$$\frac{1}{n}\log(\varphi^n)'(x) = \int_{S^1} \log\varphi'(s)\, d\mu_{n,x}(s) \xrightarrow[n\to\infty]{} \int_{S^1} \log\varphi'(s)\, d\delta_0 = \log\varphi'(0) = 0.$$

Therefore, the Lyapunov exponent of the map φ is a.e. equal to zero, and this implies that the same holds for the action of $[T]_\varphi$, thus concluding the proof of Theorem B. $\square$

We now give the sketch of the proof of Lemma 5.2 since the ideas will be very useful in the next section. First recall that, for every uniformly expanding, smooth circle map, there exists an absolutely continuous ergodic invariant density is strictly positive and away from zero; moreover, the same holds for maps of the interval having infinitely many branches, provided that there is a uniform bound for the distortion norm and the expansiveness of all of the branches (see for instance [22, Chapter III, Theorem 1.2]). However, the situation which is considered in the lemma is slightly different: although there are only finitely many branches, due to the presence of a parabolic fixed point the map is non-uniformly expanding.

Nevertheless, the neutral fixed point can be somehow "removed" in the following way. For each point c, denote by $[c]_\varphi$ the preimage of c under the (topological) conjugacy between φ and φ_0. Since $[1/3]_\varphi$ is a φ-periodic point of period two, the interval $J := [a, b]$, where $a := [1/3]_\varphi$ and $b := \varphi(a) = [2/3]_\varphi$, is a "fundamental domain" for the expansion both on the left and on the right of the neutral fixed point. Indeed, the restriction of φ to $[0, a]$ (resp. to $[b, 1]$) is one-to-one and onto $[0, b]$ (resp. $[a, 1]$): see Figure 2.

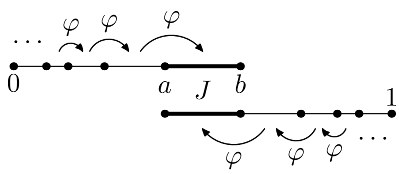

FIGURE 2. A simultaneous fundamental domain

Consider the first-return map $\Phi\colon J \to J$, as well as the return-time function $\tau\colon J \to \mathbb{N}$, which are given by

$$\Phi(x) := \varphi^{\tau(x)}(x), \qquad \tau(x) := \min\{n \geqslant 1\colon \varphi^n(x) \in J\}.$$

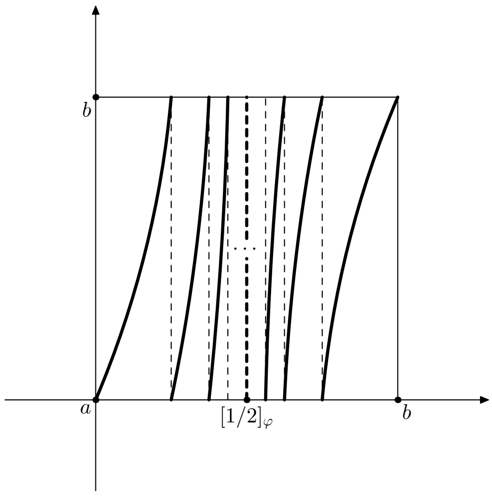

FIGURE 3. The first-return map Φ

The map Φ is in fact an infinite-degree map with infinitely many discontinuity points. However, every maximal interval of continuity I of Φ is mapped onto J, and the distortion and the expansiveness of all of the restrictions Φ_I are uniformly bounded. More precisely, since the images of I under the maps id, φ, $\varphi^2, \ldots,$ $\varphi^{\tau(x)-1}$ are pairwise disjoint, and hence the sum of their lengths does not exceed the total length of the circle, the estimates from Section 4 provide a bound for the distortion norm of Φ_I which is independent of I; moreover, the new map Φ is strictly and uniformly expanding. (Compare Lemma 5.7.) Together with what precedes, this allows to ensure the existence of an absolutely continuous ergodic invariant measure ν for Φ with a strictly positive density.

Consider now the sequence of iterates by the map φ of a Lebesgue generic point $x \in S^1$. Up to a finite number of initial steps, we can suppose that the point x belongs to the interval J, and then its orbit can be divided into segments according to the arrivals to J:

$$x,\, \varphi(x),\, \ldots,\, \varphi^{\tau(x)-1}(x);\;\; \Phi(x),\, \varphi(\Phi(x)),\, \ldots,\, \varphi^{\tau(\Phi(x))-1}(\Phi(x));\;\; \ldots;$$
$$\Phi^m(x),\, \varphi(\Phi^m(x)),\, \ldots,\, \varphi^{\tau(\Phi^m(x))-1}(\Phi^m(x));\, \ldots$$

On the one hand, since the measure ν is absolutely continuous and has positive density, for a Lebesgue generic point x the sequence x, $\Phi(x)$, $\Phi^2(x), \ldots$ is distributed with respect to ν. On the other hand, the return-time function τ has a non locally integrable "singularity" (of type $1/x$) at the point $x = [1/2]_\varphi$. Hence, due to the Birkhoff Ergodic Theorem, for a.e. $x \in J$ one has

$$\frac{\tau(x) + \tau(\Phi(x)) + \cdots + \tau(\Phi^{m-1}(x))}{m} \to +\infty \quad \text{as } m \to \infty,$$

and therefore
$$\frac{m}{\tau(x)+\tau(\Phi(x))+\cdots+\tau(\Phi^{m-1}(x))} \to 0 \quad \text{as } m\to\infty.$$

Now for every fixed $\varepsilon > 0$ the points in $S^1 \setminus U_\varepsilon(0)$ fall into J in a bounded number of iterations. More precisely, there exists a constant $N = N_\varepsilon$ such that for every $x \notin U_\varepsilon(0)$ one has $\varphi^j(x) \in J$ for some $j < N$. Hence, for each $x \in J$, the time spent by a segment of φ-orbit of length n outside $U_\varepsilon(0)$ is comparable to the number of returns to J:
$$\#\{0 \leqslant j \leqslant n-1 \colon \varphi^j(x) \notin U_\varepsilon(0)\} \leqslant N(m(n,\, x)+1),$$
where
$$m(n,\, x) := \max\{m \colon \tau(x)+\cdots+\tau(\Phi^{m-1}(x)) \leqslant n-1\}.$$
This implies that
$$\frac{\#\{0 \leqslant j \leqslant n-1 \colon \varphi^j(x) \notin U_\varepsilon(0)\}}{n} \leqslant \frac{N(m(n,\, x)+1)}{\tau(x)+\tau(\Phi(x))+\cdots+\tau(\Phi^{m(n,x)-1}(x))} \xrightarrow[n\to\infty]{} 0. \tag{4}$$
Thus, the proportion of time spent outside $U_\varepsilon(0)$ tends to 0 for a.e. $x \in J$, and hence for a.e. $x \in S^1$. Since $\varepsilon > 0$ was arbitrary, (up to some technical details) this concludes the proof of Lemma 5.2.

We close this section by giving an explicit construction for Example 1.18. Let us consider the Ghys–Sergiescu's non-minimal action of the Thompson group $[T]_\varphi$, associated to a degree-two smooth circle map φ with the following properties:

- It has exactly two fixed points x_- and x_+.
- It is tangent to the identity at x_- and x_+.
- Outside the invariant interval $[x_-,\, x_+]$, one has $\varphi' > 1$.

The last property guarantees that when we shrink the components of the preimages of $I = (x_-,\, x_+)$ by the powers of φ, the induced map becomes topologically conjugate to φ_0. This implies that the complement Λ of the union of the preimages of I is an exceptional minimal set for $[T]_\varphi$.

For each preimage y of x_+ by a power φ^n of φ, let I_y be the component of $\varphi^{-n}(I)$ containing the point y. By construction, all these intervals are disjoint. By the distortion arguments developped in Section 4, there exists a constant $C > 0$ depending only on φ, such that $(\varphi^n)'(y) \geqslant \frac{C}{|I_y|}$. Thus, the series
$$S = \sum_{\substack{(n,y)\colon \varphi^n(y)=x_+\\ \varphi^{n-1}(y)\neq x_+}} \frac{1}{(\varphi^n)'(y)}$$
converges, and hence, for every $\delta \geqslant 1$, the value of the sum
$$S_\delta := \sum_{\substack{(n,y)\colon \varphi^n(y)=x_+\\ \varphi^{n-1}(y)\neq x_+}} \frac{1}{[(\varphi^n)'(y)]^\delta}$$

is finite. Therefore, the measure

$$\mu_\delta := \frac{1}{S_\delta} \sum_{\substack{n \in \mathbb{N} \\ \varphi^n(y) = x_+ \neq \varphi^{n-1}(y)}} \frac{\mathrm{Dirac}_y}{[(\varphi^n)'(y)]^\delta}$$

is a δ-conformal measure for $[T]_\varphi$ supported on the orbit of the point x_+. (We use here the notation Dirac_x instead of δ_x for the Dirac measure, to avoid the ambiguity with the conformal exponent, also traditionally denoted by δ.)

5.2. The Case of $\mathbf{PSL_2}(\mathbb{Z})$. To deal with the (canonical) action of $\mathrm{PSL}_2(\mathbb{Z})$ on $S^1 = \mathbb{P}(\mathbb{R}^2)$, we pass to an affine chart on $\mathbb{P}(\mathbb{R}^2) = \mathbb{R} \cup \{\infty\}$ using the coordinate $\theta \mapsto \mathrm{ctg}(\theta)$. Then the minimality follows easily from the density of $\mathbb{Q}$ in $\mathbb{R}$: every orbit accumulates to the infinity (i.e., the point $(1:0)$), and $G(\infty) = \mathbb{Q} \cup \{\infty\}$.

To show that the point $(1:0)$ is non-expandable first notice that, in the coordinate above, an element $F = \left[\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\right]$ in $\mathrm{PSL}_2(\mathbb{R})$ is given by

$$x \overset{\tilde{F}}{\longmapsto} \frac{ax+b}{cx+d},$$

and thus its derivative at the point x is

$$\widetilde{F}'(x) = \frac{ad-bc}{(cx+d)^2} = \frac{1}{(cx+d)^2}.$$

For $x = 0$ this gives $\widetilde{F}'(0) = 1/d^2$. Now, coming back to the original coordinate θ, we have $\mathrm{ctg}'(\pi/2) = 1$ and $\mathrm{arcctg}'(x) = 1/(1+x^2)$; therefore, if $d \neq 0$,

$$F'(0) = 1 \cdot \frac{1}{d^2} \cdot \frac{1}{1+(b/d)^2} = \frac{1}{b^2+d^2}. \tag{5}$$

By continuity, the same formula holds when $d = 0$. If F belongs to $\mathrm{PSL}_2(\mathbb{Z})$, then b, d are in $\mathbb{Z}$ and cannot be both equal to 0. Hence, the equality (5) shows that $F'(0) \leqslant 1$. A similar argument shows that the point $(0:1)$ is also non-expandable.

By pursuing slightly the above computations, one easily checks that the derivative of the map F at a point $\theta \in S^1$ is equal to

$$F'(\theta) = \frac{\|(u,\,v)\|^2}{\|F(u,\,v)\|^2}, \tag{6}$$

where $(u,\,v)$ is any nonzero vector in the direction given by the angle θ. This formula will strongly simplify the proof of the nullity of the Lyapunov expansion exponent. For this, instead of working directly with $\mathrm{PSL}_2(\mathbb{Z})$, we will work with the subgroup G_2 which is the kernel of the natural map $\mathrm{PSL}_2(\mathbb{Z}) \to SL_2(\mathbb{Z}/2\mathbb{Z})$. Since G_2 is of finite index in $\mathrm{PSL}_2(\mathbb{Z})$, the corresponding actions have zero or positive Lyapunov expansion exponents simultaneously.

It is well-known that G_2 is a free group, and that a system of generators is given by $f_1 = \left[\left(\begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix}\right)\right]$ and $f_2 = \left[\left(\begin{smallmatrix} 1 & 0 \\ 2 & 1 \end{smallmatrix}\right)\right]$. One way to show this is by applying the Ping-pong Lemma (see e.g. [12]) to the sets

$$A_+ = \{\theta \in [0,\,\pi/4]\}, \qquad B_+ = \{\theta \in [\pi/4,\,\pi/2]\},$$
$$B_- = \{\theta \in [\pi/2,\,3\pi/4]\}, \quad A_- = \{\theta \in [3\pi/4,\,\pi]\}.$$

(Notice that under the identification $S^1 = \mathbb{P}(\mathbb{R}^2)$, the angle θ is measured modulo π, and not modulo 2π, as usually.) Indeed, one has

$$f_1^{-1}(A_+) = A_+ \cup B_- \cup B_+, \quad f_1(A_-) = A_- \cup B_- \cup B_+,$$
$$f_2^{-1}(B_+) = B_+ \cup A_- \cup A_+, \quad f_2(B_-) = B_- \cup A_- \cup A_+.$$

We will denote by $\mathcal{F} = \{f_1, f_1^{-1}, f_2, f_2^{-1}\}$ the finite set of elements generating G_2 as a semigroup. Notice that for the action of (the representatives of) these elements on a vector (u, v), one has the following possibilities:

(1) If $|u| \neq |v|$, $|u| \neq 0$, and $|v| \neq 0$, then there is a unique element in $\mathcal{F}$ which decreases the norm of (u, v), while the other generators strictly increase it.
(2) If $|u| = 0$, then $f_2^{\pm 1}$ preserve the norm of (u, v), while $f_1^{\pm 1}$ increase it.
(2′) If $|v| = 0$, then $f_1^{\pm 1}$ preserve the norm of (u, v), while $f_2^{\pm 1}$ increase it.
(3) If $u = v$, then f_1^{-1} and f_2^{-1} preserve the norm of (u, v), while f_1 and f_2 increase it.
(3′) If $u = -v$, then f_1 and f_2 preserve the norm of (u, v), while f_1^{-1} and f_2^{-1} increase it.

Using (6), one may translate all of this to the original action on the circle, thus showing that for any point θ, one (and only one) of the following two possibilities occurs:

- One of the four maps f_1, f_1^{-1}, f_2, f_2^{-1}, has derivative greater than 1 at θ, while the other three maps have derivative strictly smaller than 1 at this point.
- Two of these maps have derivative equal to 1 at θ, while the other two have derivative smaller than 1 at the same point.

From the first remark above and relation (6) we deduce that every point $(u : v)$ which is different from $(0 : 1)$, $(1 : 0)$, $(1 : 1)$, and $(-1 : 1)$, is expandable by some element of G_2 (and thus of $\mathrm{PSL}_2(\mathbb{Z})$). The latter two points are expanded by elements in $\mathrm{PSL}_2(\mathbb{Z}) \setminus G_2$, for instance, $f = \left[\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\right]$ and $g = \left[\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right]$, respectively. Since we have already seen that the former points are non-expandable, this shows that the NE-set for $\mathrm{PSL}_2(\mathbb{Z})$ is reduced to $\{(0 : 1), (1 : 0)\}$.

Now notice that the remarks above also show that, among the compositions of length smaller than or equal to n, the one that expands the most at a generic[7] point θ can be found by a "greedy" algorithm: apply always the generator which expands at the point obtained after the previous composition.

Lemma 5.3. *Given $N \in \mathbb{N}$ and a generic point θ, let $f_{i_1}, f_{i_2}, \ldots, f_{i_n}$ be a finite sequence of elements in $\mathcal{F}$ such that $n \leqslant N$ and such that the value of the derivative at the point θ of the composition $f_{i_n} \circ \cdots \circ f_{i_2} \circ f_{i_1}$ is maximal among the compositions of length smaller than or equal to N. Then $n = N$, and the composition is obtained by the "greedy" algorithm described above.*

Proof. We may assume that the composition $f_{i_n} \circ \cdots \circ f_{i_2} \circ f_{i_1}$ is irreducible, that is, no generator is applied immediately after its inverse. Let us denote by θ_k the

[7]Here, *generic* just means not contained in the orbit by $\mathrm{PSL}_2(\mathbb{Z})$ of $(1 : 0)$, or equivalently, the set of θ for which $\tan(\theta)$ is irrational.

image of θ under the partial composition $f_{i_k} \circ \cdots \circ f_{i_2} \circ f_{i_1}$. Assume that for some k the generator which is applied at time k is contracting at the corresponding point θ_{k-1} (that is, $f'_{i_k}(\theta_{k-1}) < 1$). Then the inverse of this generator is expanding at the image point θ_k (i.e., $(f_{i_k}^{-1})'(\theta_k) > 1$). Since for each generic point there is only one generator having derivative greater than one, and since $f_{i_{k+1}} \neq f_{i_k}^{-1}$, this implies that $f'_{i_{k+1}}(\theta_k) < 1$. Repeating this argument several times, this allows us to conclude that, for all $j \geqslant k$, one has $f'_{i_j}(\theta_{j-1}) < 1$. This clearly implies that all the "tail" $f_{i_n} \circ \cdots \circ f_{i_{k+1}}$ contracts at θ_k, and hence omitting it increases the derivative at the point θ:

$$(f_{i_n} \circ \cdots \circ f_{i_2} \circ f_{i_1})'(\theta) < (f_{i_{k-1}} \circ \cdots \circ f_{i_2} \circ f_{i_1})'(\theta).$$

However, this is in contradiction with our choice of the sequence $f_{i_1}, f_{i_2}, \ldots, f_{i_n}$.

Therefore, at each time k the generator which is applied is expanding at the point θ_{k-1}. In other words, the sequence coincides with the one provided by the "greedy" algorithm.

Finally, it is clear that $n = N$, as otherwise one could compose with a generator which is expanding at the corresponding point, thus increasing the derivative. $\square$

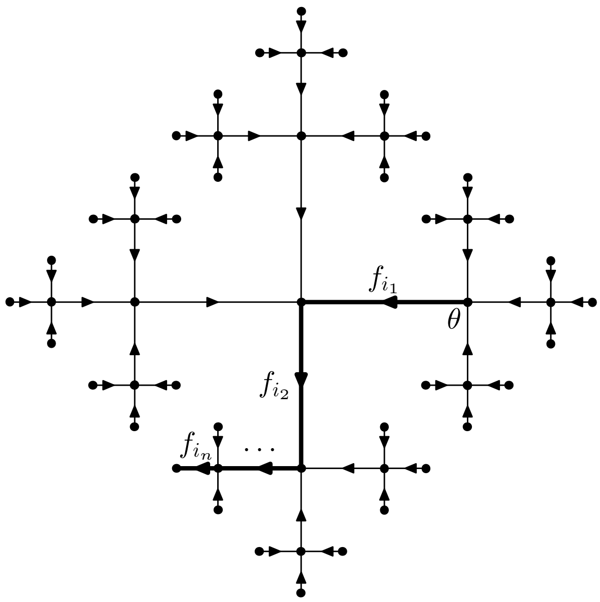

FIGURE 4. The Schreier graph of an orbit with the arrows showing the expanding direction

The "greedy" algorithm reduces the study of the Lyapunov expansion exponent of G_2 to the study of the Lyapunov exponent of a deterministic dynamics, namely the one given by applying the map f_1^{-1} on A_+, the map f_1 on A_-, the map f_2^{-1} on B_+, and the map f_2 on B_-. To deal with this dynamics, let us consider the map

$S\colon S^1 \to [0,1]$ obtained as the "union" of the affine charts on $A_\pm$, $B_\pm$, that is,

$$S(\theta) = \begin{cases} |\tan(\theta)|, & \theta \in A_- \cup A_+ = [-\pi/4,\, \pi/4], \\ |\operatorname{ctg}(\theta)|, & \theta \in B_- \cup B_+ = [\pi/4,\, 3\pi/4]. \end{cases}$$

Since both S and the set $\mathcal{F}$ are invariant under conjugacies by the elements in the finite group $\mathcal{H} = \{\mathrm{id},\, x \mapsto 1/x,\, x \mapsto -x,\, x \mapsto -1/x\}$ (written in the affine chart $(1:x)$), this dynamics descends to the quotient $S^1/\mathcal{H} = [0,1]$. Actually, a straightforward computation shows the following

Proposition 5.4. *Given a generic point $\theta \in S^1$, let f be the "expanding generator" at this point, that is, the element $f \in \mathcal{F}$ such that $f'(\theta) > 1$. Then $S(f(\theta)) = \widetilde{\varphi}(S(\theta))$, where*

$$\widetilde{\varphi}(x) = \begin{cases} \frac{1}{\frac{1}{x}-2} = \frac{x}{1-2x}, & x \in [0,\, 1/3], \\ \frac{1}{x} - 2, & x \in [1/3,\, 1/2], \\ 2 - \frac{1}{x}, & x \in [1/2,\, 1]. \end{cases} \tag{7}$$

In other words, after projecting into the quotient $S^1/\mathcal{H} = [0,1]$, the dynamics of the "greedy" algorithm becomes the dynamics of the non-uniformly expanding map $\widetilde{\varphi}$. This map has two neutral fixed points (namely 0 and 1), and in analogy to Lemma 5.2 one can state the following lemma for which we postpone the proof.

Lemma 5.5. *For Lebesgue-a.e. point $x \in [0,1]$, the time averages concentrate on the set $\{0,1\}$. More precisely, for every $\varepsilon > 0$ we have*

$$\frac{1}{n}\,\#\big\{0 \leqslant j \leqslant n-1 \colon \widetilde{\varphi}^j(x) \in U_\varepsilon(0) \cup U_\varepsilon(1)\big\} \xrightarrow[n\to\infty]{} 1.$$

Since $\widetilde{\varphi}'(0) = \widetilde{\varphi}'(1) = 1$, and since the function $|\log \widetilde{\varphi}'|$ is bounded on $[0,1]$ and continuous near 0 and 1, the lemma above easily implies the following

Corollary 5.6. *For Lebesgue-a.e. $x \in [0,1]$, the Lyapunov exponent of $\widetilde{\varphi}$ at x is equal to zero.*

Now to complete the proof of Theorem C, notice that the derivative of the map S is bounded from above and away from zero. Thus, by Proposition 5.4, the Lyapunov expansion exponent of G_2 (and hence that of $\mathrm{PSL}_2(\mathbb{Z})$) is also equal to zero for Lebesgue-almost every point θ on the circle.

Now, in the same spirit as that of Lemma 5.2, we provide the

Proof of Lemma 5.5. The first step consists in finding a periodic orbit of period two, say $\{a, \widetilde{\varphi}(a)\} = \{a, b\}$ (Figure 5), such that the interval $J := [a,b]$ is at the same time a fundamental domain for the map in a neighborhood of 0 and in a neighborhood of 1. To do this, we consider the function $\widetilde{\varphi}^2$ on the interval $(\widetilde{\varphi}^{-1}(1/2),\, 1/3)$, where the preimage is taken for the branch $\widetilde{\varphi}|_{[0,1/3]}$. Then

$$\widetilde{\varphi}^2(\widetilde{\varphi}^{-1}(1/2)) = \widetilde{\varphi}(1/2) = 0, \qquad \widetilde{\varphi}^2(1/3) = \widetilde{\varphi}(1) = 1,$$

and since the map $\widetilde{\varphi}^2$ in increasing and expanding on $(\widetilde{\varphi}^{-1}(1/2),\, 1/3)$, it has a unique fixed point a therein.

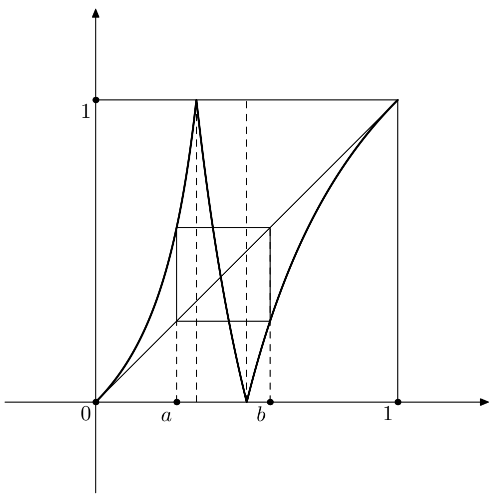

FIGURE 5. The order-two periodic point

Notice that since $a \in (0,\, 1/3)$ and $b = \widetilde{\varphi}(a) \in (1/2,\, 1)$, the interval $(a,\, b)$ is simultaneously a fundamental domain for both $\widetilde{\varphi}|_{(0,1/3)}$ and $\widetilde{\varphi}|_{(1/2,1)}$ (see Figure 6).

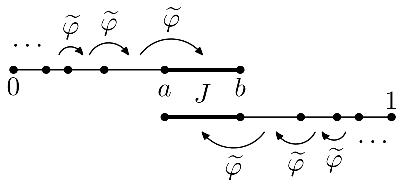

FIGURE 6. A simultaneous fundamental domain

Now consider the first-return map Φ to J, as well as the return-time function τ, given by

$$\Phi(x) := \widetilde{\varphi}^{\tau(x)}(x), \qquad \tau(x) := \min\{n \geqslant 1 \colon \widetilde{\varphi}^{n}(x) \in J\}.$$

The map Φ can be described in the following way (see Figure 7):

- The intervals I_1 and I_2 are mapped onto $[b,\, 1]$, and then they return by the topologically repelling map $\widetilde{\varphi} : [b,\, 1) \to [a,\, 1)$ to the fundamental domain J. They are decomposed into infinitely many continuity intervals whose images by Φ coincide with the whole interval J.
- The interval I_3 is mapped onto J.
- The intervals I_4 and I_5 are mapped onto $[0,\, a]$, and then they return by the topologically repelling map $\widetilde{\varphi} : (0,\, a] \to (0,\, b]$ to the fundamental domain J. They are decomposed into infinitely many continuity intervals whose images by Φ coincide with the whole interval J.

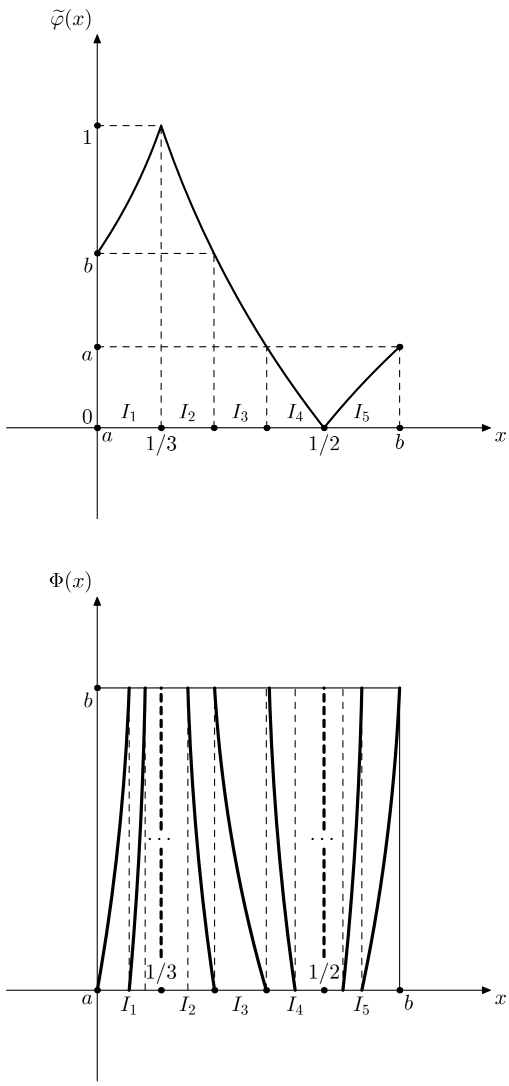

Figure 7. The first-return map Φ

Since $\widetilde{\varphi}$ is non-uniformly expanding on the whole interval [0, 1], and since it is uniformly expanding on J, for all $x \in J$ we have

$$|\Phi'(x)| \geqslant |\widetilde{\varphi}'(x)| \geqslant \inf_{y \in J} |\widetilde{\varphi}'(y)| > 1.$$

Thus, the map Φ is uniformly expanding. The following lemma provides us with a necessary upper bound for the distortion norms of the branches of Φ. We give a more general version (which still holds for non-expanding maps) since the underlying (simple) idea will be relevant in the next section.

Lemma 5.7. *Let x_0 be a fixed point of some C^2 diffeomorphism $f\colon [x_0, a] \to [x_0, b]$ such that $f(x) > x$ for all $x \in (x_0, a]$. Consider the first-entry map $F\colon [x_0, a] \to [a, b]$ into the interval $J = [a, b]$, that is*

$$F(x) := f^{k(x)}(x), \quad k(x) := \min\{k \geqslant 1\colon f^k(x) \in J\}.$$

Let $J_k := f^{-k}(J)$ be the (infinitely many) continuity intervals of F, and denote by f_k the restriction of F to J_k (that is, $f_k := f^k|_{J_k}$). Then the following hold:

1. *There exists a uniform bound for the distortion norms of the maps f_k.*
2. *Starting from some k_0, the maps f_k become uniformly expanding. More precisely, there exists $\lambda > 1$ such that, for all $k \geqslant k_0$ and all $x \in J_k$, one has $f_k'(x) \geqslant \lambda$. Moreover, at the cost of increasing k_0, one can take $\lambda = 2$.*

Proof. First notice that the distortion coefficients of all of the maps f_k are uniformly bounded. Indeed, since the intervals J_k are disjoint,

$$\varkappa(f^k;\, J_k) \leqslant \|\log(f')\|_{C^1} \sum_{i=0}^{k-1} |f^i(J_k)| = \|\log(f')\|_{C^1} \sum_{i=1}^{k} |J_i| \leqslant \|\log(f')\|_{C^1}(b - x_0), \tag{8}$$

and since the right hand expression does not depend on k, this provides the uniform bound for the distortion coefficients. Now letting $C := \exp\big(\|\log f'\|_{C^1}(b - x_0)\big)$, the above estimate gives, for all $k \geqslant 1$ and all $x \in J_k$,

$$(f^k)'(x) \geqslant \frac{1}{C} \cdot \frac{|J|}{|J_k|}.$$

Moreover, since the series $\sum_k |J_k|$ converges, the length $|J_k|$ goes to zero. In particular, there exists $k_0 \in \mathbb{N}$ such that, for all $k \geqslant k_0$, one has $|J_k| \leqslant |J|/2C$. This immediately yields, for all $k \geqslant k_0$ and all $x \in J_k$,

$$(f_k)'(x) \geqslant \frac{|J|}{C|J_k|} \geqslant 2,$$

thus proving the second claim of the lemma. To prove the first claim notice that, according to the control for the distortion coefficients already established, for each interval $J' \subset J$ and each $i \in \{1, \ldots, k\}$ we have

$$|f^{-i}(J')| \leqslant C|J'| \cdot \frac{|J_i|}{|J|}.$$

Hence,

$$\varkappa(f_k;\, f^{-k}(J')) \leqslant \|\log f'\|_{C^1} \sum_{i=1}^{k} C|J'| \cdot \frac{|J_i|}{|J|} \leqslant \frac{C\|\log f'\|_{C^1}(b - x_0)}{|J|} \cdot |J'| = C'|J'|,$$

which gives $\eta(f_k;\, f^{-k}(J')) \leqslant C'$, thus finishing the proof. □

According to [22, Chapter III] (see Theorems 1.1 and 1.2), the preceding lemma guarantees the hypotheses which ensure the existence of an ergodic Φ-invariant measure with a positive continuous density (w.r.t. the Lebesgue measure). Now remark that the return-time function τ is not locally integrable near the points $1/3$ and $1/2$ (this is due to the fact that these points are mapped by $\widetilde{\varphi}$ into the parabolic fixed points 0 and 1 respectively). Hence, using the very same arguments as those of the proof of Lemma 5.2, this allows to finish the proof of Lemma 5.5. □

We close this section with an explicit construction of conformal measures corresponding to Example 1.19.

Notice that, due to formula (6), if a map $F \in \mathrm{PSL}_2(\mathbb{Z})$ sends some point $(m : n)$ into $(a : b)$, where m, n, a, b are integers, and $\gcd(m, n) = \gcd(a, b) = 1$, then

$$F'((m:n)) = \left(\frac{\|(m,\, n)\|}{\|(a,\, b)\|}\right)^2.$$

On the other hand, if $\delta > 1$, then the sum

$$S_\delta := \sum_{\gcd(m,n)=1} \frac{1}{\|(m,\, n)\|^{2\delta}}$$

is finite. One can then easily see that the measure

$$\mu_\delta := \frac{1}{S_\delta} \sum_{\gcd(m,n)=1} \frac{\mathrm{Dirac}_{(m:n)}}{\|(m,\, n)\|^{2\delta}}$$

is δ-conformal.

6. Proofs

6.1. Actions with property $(\star)$

Proof of Theorem A. We begin by dealing with the first claim. For this we notice that the set $\mathrm{NE}(G)$ is closed, since it is a (countable) intersection of closed sets:

$$\mathrm{NE}(G) = \bigcap_{g \in G} \{x \colon g'(x) \leqslant 1\}.$$

Therefore, the finiteness of $\mathrm{NE}(G)$ follows directly from the following

Lemma 6.1. *The set* $\mathrm{NE}(G)$ *is made up of isolated points.*

Proof. For a fixed $y \in \mathrm{NE}(G)$ we will find an interval of the form $(y,\, y + \delta)$ which does not intersect $\mathrm{NE}(G)$. The reader will notice that a similar argument provides an interval of the form $(y - \delta',\, y)$ also disjoint from $\mathrm{NE}(G)$.

By property $(\star)$, there exist $g_+ \in G$ and $\varepsilon > 0$ such that $g_+(y) = y$ and such that g_+ has no other fixed point in $(y,\, y + \varepsilon)$. Replacing g_+ by its inverse if necessary, we may assume y to be a right topologically repelling fixed point of g_+. Let us consider the point $\bar{y} := y + \varepsilon/2 \in (y,\, y + \varepsilon)$, and for each integer $k \geqslant 0$ let $\bar{y}_k := g_+^{-k}(\bar{y})$ and $J_k := (\bar{y}_{k+1},\, \bar{y}_k)$. Taking $a = \bar{y}_1$, $b = \bar{y}$, and applying Lemma 5.7, we see that for some $k_0 \in \mathbb{N}$ one has $(g_+^k)'(x) \geqslant 2$ for all $k \geqslant k_0$ and all $x \in J_k$. Hence, for all $k \geqslant k_0$ we have $\mathrm{NE}(G) \cap J_k = \varnothing$. This clearly implies that $\mathrm{NE}(G) \cap (y,\, \bar{y}_{k_0}) = \varnothing$, thus finishing the proof. □

According to the proof above, for each point $y \in \mathrm{NE}(G)$ one can fix an element $g_+ \in G$ having y as a right topologically repelling fixed point, a positive integer k_0^+, and an interval $I_y^+ := (y,\, \bar{y}_{k_0^+})$ contained in the right repulsion basin of y, such that, if for $x \in I_y^+$ we take the smallest integer $n \geqslant 0$ such that $g_+^n(x) \notin I_y^+$, then $(g_+^{n+k_0^+})'(x) \geqslant 2$. In the same way are defined an element g_- having y as a left topologically repelling fixed point, an interval I_y^-, and a positive integer k_0^-, sharing analogous properties. We then let

$$U_y := I_y^+ \cup I_y^- \cup \{y\}.$$

By definition (and continuity), for every point $y \notin \mathrm{NE}(G)$ there exist $g = g_y \in G$ and a neighborhood V_y of y such that $\inf_{V_y} g' > 1$. The sets $\{U_y : y \in \mathrm{NE}(G)\}$ and $\{V_y : y \notin \mathrm{NE}(G)\}$ form an open cover of the circle, from which we can extract a finite sub-cover

$$\{U_y : y \in \mathrm{NE}(G)\} \cup \{V_{y_1}, \ldots, V_{y_k}\}.$$

Let

$$\lambda := \min\big\{2, \inf_{V_{y_1}} g'_{y_1}, \ldots, \inf_{V_{y_k}} g'_{y_k}\big\}.$$

Since λ is the minimum among finitely many numbers greater than 1, we have $\lambda > 1$.

Now for every $x \in S^1$, either $x \in \mathrm{NE}(G)$, or x lies inside one of the sets $I_y^{\pm}$ or V_{y_j}. In the last case, there exists a map $g \in G$ such that $g'(x) \geqslant \lambda$. Again, either $g(x) \in \mathrm{NE}(G)$, or $g(x)$ belongs to some $I_y^{\pm}$ or V_{y_j}, and in the last case there is $\bar{g} \in G$ such that $\bar{g}'(g(x)) \geqslant \lambda$. Continuing in this way, we see that if we do not fall into a point in $\mathrm{NE}(G)$ by some composition, then we can always continue expanding by a factor at least equal to λ by some element in G. Therefore, for each point not belonging to the orbit of $\mathrm{NE}(G)$, the set of derivatives $\{g'(x) : g \in G\}$ is unbounded. Since for a point x in the orbit of $\mathrm{NE}(G)$ this set is obviously bounded, this proves the second claim of Theorem A.

To complete the proof of the theorem, the only conclusion which is left corresponds to that of the ergodicity of the action. Thus, let $A \subset S^1$ be an invariant measurable set of positive Lebesgue measure, and let a be a density point in A not belonging to the orbit of $\mathrm{NE}(G)$ (notice that, since this orbit is countable, such a point a exists). Then the expansion procedure works by applying the "exit-maps" $g_{\pm}^{n+k_0^{\pm}}$ to points in $I_y^+ \cup I_y^-$ and the map g_y to points in V_y. Now what we need to do is to control the distortion of these compositions in a small neighborhood of a until its image reaches a "macroscopic" length. Although this can be done in terms of distortion coefficients, we prefer working directly with the derivatives of the maps which are involved, since this approach provides another way to deal with the ergodicity conjecture and allows to state later an interesting problem, namely Question 6.6.

To simplify, in what follows we add to our prescribed system of generators the elements of the form g_+ and g_-, as well as their inverses. The main issue below consists in controlling the sum of the derivatives along a sequence of compositions by the derivative of the whole composition.

Lemma 6.2. *There exists a constant $C_1 > 0$ such that, for every $x \notin \mathrm{NE}(G)$, one can find a composition $f_n \circ \cdots \circ f_1$ of elements $f_1, \ldots, f_n$ in $\mathfrak{F}$ such that*

$(f_n \circ \cdots \circ f_1)'(x) \geqslant \lambda$ *and*

$$\frac{\sum_{j=1}^{n}(f_j \circ \cdots \circ f_1)'(x)}{(f_n \circ \cdots \circ f_1)'(x)} \leqslant C_1. \tag{9}$$

Proof. We have already constructed for any $x \notin \mathrm{NE}$ a map $g \in G$ with $g'(x) \geqslant \lambda$; so, we have now only to prove the existence of an upper bound on the left hand side of (9).

If x belongs to a neighborhood of the type V_{y_i}, we can decompose the corresponding $g_{y_i} \in G$ as a composition of generators $f_j \in \mathcal{F}$; the supremum of the left hand side of (9) is then finite for any i, so the maximum (on i) of these supremums provides a uniform upper bound in this case.

Assume now that $x \in I_y^+$ (the case $x \in I_y^-$ is analogous). Recall that the map $g_+ \in G$ was taken to have y as a right-repelling fixed point. Subsequently, the interval I_y^+ was chosen as follows. We take a point $\bar{y}$ within the right basin of repulsion of y, and denote

$$\bar{y}_k := g_+^{-k}(y), \qquad J_0 := [\bar{y}_1, \bar{y}), \qquad J_k := g_+^{-k}(J_0).$$

Then, we know that for some k_0^+ one has $(g_+^n)'(x) \geqslant \lambda$ for all $n \geqslant k_0^+$ and all $x \in J_n$. The interval I_y^+ is defined as $(y, \bar{y}_{k_0^+})$, and it is decomposed as the union of J_n, with $n \geqslant k_0^+$. For a point $x \in J_n$ we put $f_1 = \ldots = f_n := g_+$, thus guaranteeing that $(f_n \circ \cdots \circ f_1)'(x) \geqslant \lambda$. It suffices now to estimate the quotient (9). To do this, we notice that Proposition 4.2 easily implies that

$$\sum_{j=1}^{n}(g_+^j)'(x) \leqslant \exp(C_{\mathcal{F}}) \cdot \frac{\sum_{j=1}^{n}|g_+^j(J_n)|}{|J_n|} \leqslant \frac{\exp(C_{\mathcal{F}})}{|J_n|}$$

and

$$(g_+^n)'(x) \geqslant \exp(-C_{\mathcal{F}}) \cdot \frac{|g^n(J_n)|}{|J_n|} = \exp(-C_{\mathcal{F}}) \cdot \frac{|J_0|}{|J_n|}.$$

Therefore,

$$\frac{\sum_{j=1}^{n}(f_j \circ \cdots \circ f_1)'(x)}{(f_n \circ \cdots \circ f_1)'(x)} = \frac{\sum_{j=1}^{n}(g_+^j)'(x)}{(g_+^n)'(x)} \leqslant \frac{\exp(2C_{\mathcal{F}})}{|J_0|}.$$

□

The previous lemma provides us with a natural "expansion procedure" which yields the following

Lemma 6.3. *There exists a constant C_2 such that, for every point x which does not belong to the orbit of* $\mathrm{NE}(G)$ *and every $M > 1$, one can find $f_1, \ldots, f_n$ in $\mathcal{F}$ such that the composition $f_n \circ \cdots \circ f_1$ has derivative greater than or equal to M at x and*

$$\frac{\sum_{j=1}^{n}(f_j \circ \cdots \circ f_1)'(x)}{(f_n \circ \cdots \circ f_1)'(x)} \leqslant C_2. \tag{10}$$

Proof. Starting with $x_0 = x$ we let

$$x_k := f_{k,n_k} \circ \cdots \circ f_{k,1}(x_{k-1}),$$

where the elements $f_{k,j} \in \mathcal{F}$ (chosen using Lemma 6.2) satisfy

$$\frac{\sum_{j=1}^{n_k} (f_{k,j} \circ \cdots \circ f_{k,1})'(x_{k-1})}{(f_{k,n_k} \circ \cdots \circ f_{k,1})'(x_{k-1})} \leqslant C_1, \qquad (f_{k,n_k} \circ \cdots \circ f_{k,1})'(x_{k-1}) \geqslant \lambda.$$

Let us construct such a sequence for $k = 1, \ldots, K$, where $K \geqslant \log(M)/\log(\lambda)$. Then, for the compositions

$$F_k := (f_{k,n_k} \circ \cdots \circ f_{k,1}) \circ \cdots \circ (f_{1,n_1} \circ \cdots \circ f_{1,1})$$

we obtain

$$F_K'(x) = \prod_{k=1}^{K} (f_{k,n_k} \circ \cdots \circ f_{k,1})'(x_{k-1}) \geqslant \lambda^K \geqslant M.$$

To estimate the quotient in the left hand side expression of (10), we will write it differently. Namely, letting $y := f_n \circ \cdots \circ f_1(x)$, one can easily check that

$$\frac{\sum_{j=1}^{n} (f_j \circ \cdots \circ f_1)'(x)}{(f_n \circ \cdots \circ f_1)'(x)} = \sum_{j=1}^{n} (f_{j+1}^{-1} \circ \ldots f_n^{-1})'(y). \tag{11}$$

In other words, providing a control for the quotient in (10) corresponding to an expansion at x is equivalent to providing a control for the sum of the derivatives for the contraction at y.

Now, to simplify the notation, we will denote by $\widetilde{F}_k$ the composition obtained at each step of the expansion procedure, that is,

$$\widetilde{F}_k := f_{k,n_k} \circ \cdots \circ f_{k,1}.$$

If we denote $y := F_K(x)$, then using (11) we see that the left hand side expression in (10) is equal to

$$\sum_{k=1}^{K} \sum_{j=1}^{n_k} \Big((f_{k,j+1}^{-1} \circ \cdots \circ f_{k,n_k}^{-1}) \circ (\widetilde{F}_{k+1}^{-1} \circ \cdots \circ \widetilde{F}_K^{-1}) \Big)' (y) =$$

$$= \sum_{k=1}^{K} (\widetilde{F}_{k+1}^{-1} \circ \cdots \circ \widetilde{F}_K^{-1})'(y) \cdot \sum_{j=1}^{n_k} (f_{k,j+1}^{-1} \circ \cdots \circ f_{k,n_k}^{-1})'(x_k) \leqslant$$

$$\leqslant \sum_{k=1}^{K} \frac{1}{\lambda^{K-k}} \cdot C_1 \leqslant \frac{C_1}{1 - \lambda^{-1}}. \qquad \square$$

Finally, the bound obtained in the previous lemma provides the desired control of distortion for the compositions on a neighborhood which is expanded up to a macroscopic length. More precisely, the following holds.

Proposition 6.4. *There exists $\varepsilon > 0$ such that, for every point x not belonging to the orbit of* $\mathrm{NE}(G)$, *there exists a sequence V_k of neighbourhoods of x converging to x, and a sequence of elements g_k in G, such that $|g_k(V_k)| = \varepsilon$ and $\varkappa(g_k; V_k) \leqslant$* $\log(2)$.

Proof. We will check the conclusion of the lemma for $\varepsilon = \log(2)/(2C_{\mathcal{F}}C_2)$. Indeed, fix $M > 1$ and consider the composition $f_n \circ \cdots \circ f_1$ associated to x and M provided by the previous lemma. Denoting $\bar{F}_n := f_n \circ \cdots \circ f_1$ and $y := \bar{F}_n(x)$, for the neighborhood $V := \bar{F}_n^{-1}(U_{\varepsilon/2}(y))$ of x we have

$$\varkappa\big(\bar{F}_n;\, \bar{F}_n^{-1}(U_{\varepsilon/2}(y))\big) = \varkappa\big(\bar{F}_n^{-1};\, U_{\varepsilon/2}(y)\big) = \varkappa\big(f_1^{-1} \circ \cdots \circ f_n^{-1},\, U_{\varepsilon/2}(y)\big).$$

By Proposition 4.4, the distortion coefficient of the composition $f_1^{-1} \circ \cdots \circ f_n^{-1}$ is bounded from above by $\log(2)$ in a neighborhood of y of radius

$$r := \frac{\log(2)}{4C_{\mathcal{F}}S},$$

where

$$S := \sum_{j=1}^{n} (f_{j+1}^{-1} \circ \cdots \circ f_n^{-1})'(\bar{F}_n(x)). \tag{12}$$

According to (10) and (11) we have

$$S = \frac{\sum_{j=1}^{n} (f_j \circ \cdots \circ f_1)'(x)}{(f_n \circ \cdots \circ f_1)'(x)} \leqslant C_2.$$

Therefore, the chosen ε is less than or equal to $2r$, which implies the desired estimate for the distortion. Finally, notice that

$$|V| = \big|\bar{F}_n^{-1}(U_{\varepsilon/2}(y))\big| \leqslant \frac{|U_{\varepsilon/2}(y)| \exp\big(\varkappa(\bar{F}_n^{-1};\, U_{\varepsilon/2}(y))\big)}{(f_n \circ \cdots \circ f_1)'(x)} \leqslant \frac{2\varepsilon}{M},$$

and the last expression tends to zero as M goes to infinity. Thus, for the family of neighborhoods $V = V_k$ obtained by the procedure described above for $M = k$ going to infinity, we see that the sequence V_k actually collapses to x, and this concludes the proof of the proposition. $\square$

The preceding proposition provides us with the desired bound for the distortion of the expansion on small neighborhoods of the density point $a \in A$. By the arguments already mentioned in Section 2.1, this implies the ergodicity of the action. Namely, the proportion of points of A in the neighborhoods V_k of a tends to 1, because these neighborhoods collapse to a. Thus, the same holds for their expanded images $I_k := g_k(V_k)$, because the distortion is uniformly bounded, and the set A is invariant. But the length of the intervals I_k equals ε, and by extracting a convergent subsequence of them, at the we find an interval I of positive length on which the measure of A is total. By minimality, this implies that A is a set of full Lebesgue measure.

We have then proved the ergodicity, thus completing the proof of Theorem A. $\square$

Proof of Corollary 1.10. Let G be a finitely generated group of C^2 circle diffeomorphisms for which property $(\star)$ holds with respect to a prescribed Riemannian metric. Given a new Riemannian metric on S^1, let us denote by $c\colon S^1 \to \mathbb{R}$ the function which is the quotient of between the new metric and the old one. If we denote by g' (resp. $g^\bullet$) the derivative of $g \in G$ with respect to the original (resp. the new) metric, then one has $g^\bullet(x) = \frac{c(g(x))}{c(x)} g'(x)$. By Theorem A, a point $x \in S^1$ does not belong to the orbit of any non-expandable point with respect to the initial metric if and only if the set $\{g'(x)\colon g \in G\}$ is unbounded. Now since the value of c is bounded from above and away from zero, this happens if and only if the set $\{g^{\bullet}(x)\colon g \in G\}$ is also unbounded. Therefore, every point x which is non-expandable for the new metric is in the orbit of some point x_0 which is non-expandable for the original one. By property $(\star)$, there exist g_- and g_+ in Γ having x_0 as an isolated fixed point by the left and by the right, respectively. Choosing $h \in G$ such that $x = h(x_0)$, this implies that x is a fixed point which is isolated by the left (resp. by the right) for hg_-h^{-1} (resp. hg_+h^{-1}), and this shows that property $(\star)$ holds with respect to the new metric. □

We would like to close this section with a few comments on the idea of the proof of Theorem A above. For this, let us first introduce some terminology.

Definition 6.5. Given $C > 0$, a point $x \in S^1$ is said to be *C-distortion-expandable* for the action of a finitely generated group G of C^2-circle diffeomorphisms if for each $M > 1$ one can find $f_1, \ldots, f_n$ in $\mathcal{F}$ such that $(f_n \circ \cdots \circ f_1)'(x) \geqslant M$ and

$$\frac{\sum_{j=1}^{n} (f_j \circ \cdots \circ f_1)'(x)}{(f_n \circ \cdots \circ f_1)'(x)} \leqslant C.$$

The arguments of the proof of Proposition 6.4 prove more generally that if for some $C > 0$ the set of C-distortion-expandable points has positive Lebesgue measure and the action is minimal, then the Lebesgue measure of this set equals 1, and the action is ergodic. (In fact, this is the reason why this definition is given only for C^2 actions. Formally speaking, one can consider this condition for C^1-actions as well, but without the C^2 regularity hypothesis it does not imply the control of distortion for the expansion procedure.)

Due to this implication, a positive answer for the following question would also provide a positive answer for the ergodicity conjecture.

Question 6.6. Is it true that, for every finitely generated group of C^2 circle diffeomorphisms whose action is minimal and does not preserve a probability measure, there exists a constant $C > 0$ such that Lebesgue-a.e. point is C-distortion-expandable?

6.2. Conformal measures

Proof of Theorem F. We will use the following fact from basic Measure Theory:

Proposition 6.7. *For any two non-atomic measures μ_1 and μ_2 on the circle, the limit (in $[0, \infty]$)*

$$\rho(x) = \rho_{\mu_1,\mu_2}(x) = \lim_{\varepsilon \to 0} \frac{\mu_1(U_\varepsilon(x))}{\mu_2(U_\varepsilon(x))} \tag{13}$$

exists for $(\mu_1 + \mu_2)$-almost every x. This limit is nonzero for μ_1-almost every x and finite for μ_2-almost every x. The set $A_0 := \{x\colon \rho(x) = 0\}$ (resp., $A_\infty := \{x\colon \rho(x) = \infty\}$) corresponds to the singular part of μ_2 with respect to μ_1 (resp. of μ_1 with respect to μ_2). The restrictions of the measures μ_1 and μ_2 to the set $B := \{0 < \rho(x) < \infty\}$ are equivalent, and the density of μ_1 w.r.t. μ_2 on B equals ρ.

Let us first consider the case of a minimal dynamics (as we will see, the very same arguments can be used in the case of an exceptional minimal set). Let μ be a conformal measure with some exponent δ for an action satisfying the property $(\star)$, and assume that μ does not charge the orbit of $\mathrm{NE}(G)$. Notice that an atom of μ can be placed only at a point x with a bounded set of derivatives $\{g'(x) \colon g \in G\}$. By the second conclusion of Theorem A, we know that such a point must belong to the orbit of a non-expandable one. Therefore, as we assumed that the orbit of $\mathrm{NE}(G)$ is not charged, the measure μ is non-atomic.

Now, let us take any point $x \notin G(\mathrm{NE})$ and let us analyze the behaviour of the limit (13) with the help of Proposition 6.4. This proposition provides us with a sequence of neighborhoods U_k of the point x, as well as expanding compositions $F_k := f_{n_k} \circ \cdots \circ f_1$, so that one has $\varkappa(F_k; U_k) \leqslant \log(2)$. Hence, for every $y \in U_k$

$$\frac{|F_k(U_k)|}{2|U_k|} \leqslant F_k'(y) \leqslant \frac{2|F_k(U_k)|}{|U_k|}. \tag{14}$$

From the definition of δ-conformality it follows that

$$\frac{1}{2^\delta} \cdot \left(\frac{|F_k(U_k)|}{|U_k|}\right)^\delta \mu(U_k) \leqslant \mu(F_k(U_k)) \leqslant 2^\delta \cdot \left(\frac{|F_k(U_k)|}{|U_k|}\right)^\delta \mu(U_k),$$

and hence,

$$\frac{1}{2^\delta} \cdot \left(\frac{|U_k|}{|F_k(U_k)|}\right)^\delta \mu(F_k(U_k)) \leqslant \mu(U_k) \leqslant 2^\delta \cdot \left(\frac{|U_k|}{|F_k(U_k)|}\right)^\delta \mu(F_k(U_k)).$$

Since the length of the image $F_k(U_k)$ equals ε for every k, the measures $\mu(F_k(U_k))$ are bounded from below independently on k. Thus, we have

$$c|U_k|^\delta \leqslant \mu(U_k) \leqslant C|U_k|^\delta \tag{15}$$

for some constants $C > c > 0$.

Now, let us consider the three possible cases for the conformal exponent: $\delta < 1$, $\delta = 1$, and $\delta > 1$. In the first case, we have

$$\lim_{k\to\infty} \frac{\mu(U_k)}{|U_k|} \geqslant \lim_{k\to\infty} \frac{c|U_k|^\delta}{|U_k|} = \infty.$$

Thus, for a subsequence of neighborhoods surrounding x (recall that the neighborhoods U_k provided by Proposition 6.4 are not of arbitrary size, though they collapse to x), the "density" limit (13) is infinite. But due to Proposition 6.7, this limit should be finite for Lebesgue-almost every point x. This contradiction shows that this case is impossible.

On the other hand, if $\delta > 1$,

$$\lim_{k\to\infty} \frac{\mu(U_k)}{|U_k|} \leqslant \lim_{k\to\infty} \frac{C|U_k|^\delta}{|U_k|} = 0.$$

However, according to Proposition 6.7, the limit should be positive for μ-almost every x. Since by assumption the measure μ does not charge the set $G(\mathrm{NE})$, this gives a contradiction which makes this case impossible.

The only case which is left is $\delta = 1$. However, in this case the estimate (15) implies that the measure μ is absolutely continuous with respect to the Lebesgue one, and its density (due to the fact that μ is 1-conformal) is an invariant function. Since the Lebesgue measure is ergodic, this density is constant, and hence the measure μ is proportional to the Lebesgue one, and actually equal to it due to the normalization. This concludes the proof in the case of a minimal action.

Assume now that the group G acts with an exceptional minimal set Λ and satisfies property $(\Lambda\star)$. Once again, we see that if a conformal measure does not charge $G(\mathrm{NE})$, then it must be non-atomic. We still have the same estimates (14) and (15) on the derivative and on the quotient of measures, though the neighborhoods are now considered only for points in Λ. The argument excluding the exponent $\delta > 1$ still works: the density limit $\rho_{\mu,\mathrm{Leb}}$ of the measure μ w.r.t. the Lebesgue one cannot be zero μ-almost everywhere.

Similar arguments to the above ones exclude the case $\delta = 1$: indeed, if δ was equal to 1, this would imply that the measure μ is absolutely continuous with respect to the Lebesgue one, which is impossible, since the set Λ has zero Lebesgue measure.

Finally, the case $\delta < 1$ becomes possible: the density limit of μ with respect to the Lebesgue measure will be infinite only at the points of Λ. However, there can be only one conformal exponent δ and only one conformal measure μ corresponding to this exponent. Indeed, let $\delta_1 \geqslant \delta_2$ be two conformal exponents corresponding to conformal measures μ_1 and μ_2, respectively. Then, by re-applying the same arguments of control of distortion as those above, and noticing that the measures μ_1, μ_2 of an interval $F_k(U_k)$ of length ε are bounded from below, we deduce from (14) that

$$c|U_k|^{\delta_1-\delta_2} \leqslant \frac{\mu_1(U_k)}{\mu_2(U_k)} \leqslant C|U_k|^{\delta_1-\delta_2} \tag{16}$$

for some constants $C > c > 0$.

If $\delta_1 > \delta_2$, the inequalities (16) imply that the density limit (13) for the measures μ_1, μ_2 is zero on a subsequence for every point $x \in \Lambda \setminus G(\mathrm{NE})$. However, this is impossible, since this density limit should be positive for μ_1-almost every point of $\mathrm{supp}(\mu_1) = \Lambda$. Finally, if $\delta_1 = \delta_2$, these conformal measures are equivalent, and the density $\frac{d\mu_1}{d\mu_2}$ is an invariant function. Therefore, once we prove that the measure μ_1 is ergodic, this will imply that $\mu_1 = \mu_2$.

The ergodicity of the measure μ_1 can be deduced in the same way as in Theorem A was deduced the ergodicity of the Lebesgue measure for the minimal case. Namely, if $A \subset \Lambda$ is a measurable invariant set, then μ_1-almost every point in A is a μ_1-density point of A. By expanding arbitrarily small neighborhoods of such a point, using the fact that (due to the minimality of the action on Λ) one has $\mathrm{supp}(\mu_1) = \Lambda$, and choosing a subsequence among the expanded intervals, at the limit we obtain an interval on which the points of A form a subset of full μ_1-measure. Due to the minimality, this implies that A has full μ_1-measure. This concludes the proof of the ergodicity, and thus that of Theorem F. □

6.3. Random dynamics

Proof of Theorem G. Let m be a measure on G having finite first word-moment and such that there is no measure on the circle which is simultaneously invariant by all the maps in $\mathrm{supp}(m)$. By P. Baxendale's theorem (see Section 2.2), there exists an ergodic stationary measure ν such that the corresponding random Lyapunov exponent is strictly negative. We will prove that for ν-almost every point x the Lyapunov expansion exponent at x is positive. More precisely, we will prove that

$$\lambda_{\exp}(\nu;\, G;\, \mathcal{F}) \geqslant \frac{|\lambda_{RD}(\nu;\, m)|}{v_{\mathcal{F}}(m)}, \tag{17}$$

where $\lambda_{\exp}(\nu;\, G;\, \mathcal{F})$ stands for the Lyapunov expansion exponent at ν-almost every point (due to the ergodicity of the measure ν, this exponent is constant ν-almost everywhere), and $v_{\mathcal{F}}(m)$ denotes the *rate of escape* for the convolutions of m:

$$v_{\mathcal{F}}(m) = \lim_{n\to\infty} \frac{1}{n} \int_{G^n} \|g_1 \circ \cdots \circ g_n\|_{\mathcal{F}}\, dm(g_1) \ldots dm(g_n).$$

(The existence of the limit above follows from the subadditivity of the corresponding sequence of integrals.) Notice that a direct consequence of (17) is that

$$\lambda_{\exp}(\nu;\, G;\, \mathcal{F}) \geqslant \frac{|\lambda_{RD}(\nu;\, m)|}{\int_G \|g\|_{\mathcal{F}}\, dm(g)}.$$

To prove the estimate (17), fix $\varepsilon > 0$, and consider the skew-product map

$$F : S^1 \times G^{\mathbb{N}} \to S^1 \times G^{\mathbb{N}}, \quad F(x,\, (g_i)) = (g_1(x),\, (g_{i+1})).$$

Since ν is an ergodic stationary measure, the Random Ergodic Theorem (see, *e.g.*, [7]) asserts that the F-invariant measure $\widetilde{\nu} = \nu \times m^{\mathbb{N}}$ is ergodic.

For each $n \in \mathbb{N}$, consider the sets

$$A_n := \{(x,\, (g_i))\colon\, \log(g_n \circ \cdots \circ g_1)'(x) < -n(|\lambda_{RD}(m;\, \nu)| - \varepsilon)\},$$

and

$$B_n := \{(x,\, (g_i))\colon\, \|g_n \circ \cdots \circ g_1\|_{\mathcal{F}} < n(v_{\mathcal{F}}(m) + \varepsilon)\}.$$

Notice that the measures of both sets A_n and B_n tend to 1 as n tends to infinity (this follows immediately from the definitions of the random Lyapunov exponent and of the rate of escape). Clearly, the same holds for the measures of the sets $F^n(A_n \cap B_n)$ (as F preserves the measure $\widetilde{\nu}$), as well as for the ν-measures of the projections of these sets on the circle. Now, a point y belongs to the projection $C_n := \pi_{S^1}(F^n(A_n \cap B_n))$ if and only if there exist $x,\, g_1,\, \ldots,\, g_n$ such that

$$y = (g_n \circ \cdots \circ g_1)(x), \quad \log(g_n \circ \cdots \circ g_1)'(x) < -n(|\lambda_{RD}(m;\, \nu)| - \varepsilon),$$
$$\text{and} \quad \|g_n \circ \cdots \circ g_1\|_{\mathcal{F}} < n(v_{\mathcal{F}}(m) + \varepsilon).$$

This implies that for the composition $g_1^{-1} \circ \cdots \circ g_n^{-1}$ one has

$$\frac{\log(g_1^{-1} \circ \cdots \circ g_n^{-1})'(y)}{\|g_1^{-1} \circ \cdots \circ g_n^{-1}\|_{\mathcal{F}}} \geqslant \frac{|\lambda_{RD}(m;\, \nu)| - \varepsilon}{v_{\mathcal{F}}(m) + \varepsilon}.$$

Since $\nu(C_n) \to 1$, the set of points y belonging to an infinite number of sets C_n is of full ν-measure. Hence,

$$\lambda_{\exp}(\nu;\, G;\, \mathcal{F}) \geqslant \frac{|\lambda_{RD}(m;\, \nu)| - \varepsilon}{v_{\mathcal{F}}(m) + \varepsilon},$$

and since $\varepsilon > 0$ was arbitrary,

$$\lambda_{\exp}(\nu; G; \mathcal{F}) \geqslant \frac{|\lambda_{RD}(m; \nu)|}{v_{\mathcal{F}}(m)},$$

which concludes the proof of the theorem. □

As we already noticed in Remark 1.24, assuming more restrictive assumptions on the moments, one can prove that the "exponentially expanding" compositions can be chosen of any length. To do this, due to the Borel–Cantelli Lemma, it suffices to check that the series $\sum_n (1 - \nu(C_n))$ converges. And indeed, by establishing some control for the "large deviations", one can show (under certain additional assumptions) that the measures of the sets A_n and B_n tend to 1 exponentially, which immediately implies this convergence.

Acknowledgments

The authors would like to express their gratitude to Étienne Ghys and Yulij Ilyashenko for many fruitful discussions and their interest in the problem, as well as the anonymous referee for his valuable remarks on the text. The second author would also like to thank Dominique Cerveau, Yves Guivarc'h, Pierre de la Harpe, Vadim Kaimanovich, Frank Loray, and Nicolas Monod, for valuable remarks and suggestions. All the authors would like to thank the Universities of Geneva and Santiago de Chile for their hospitality.

References

[1] V. A. Antonov, *Modeling of processes of cyclic evolution type. Synchronization by a random signal*, Vestnik Leningrad. Univ. Mat. Mekh. Astronom. (1984), no. 2, 67–76 (Russian). MR 756386

[2] P. H. Baxendale, *Lyapunov exponents and relative entropy for a stochastic flow of diffeomorphisms*, Probab. Theory Related Fields **81** (1989), no. 4, 521–554. MR 995809

[3] R. Bowen, *A horseshoe with positive measure*, Invent. Math. **29** (1975), no. 3, 203–204. MR 0380890

[4] J. Cantwell and L. Conlon, *Foliations and subshifts*, Tohoku Math. J. (2) **40** (1988), no. 2, 165–187. MR 943817

[5] B. Deroin and V. Kleptsyn, *Random conformal dynamical systems*, Geom. Funct. Anal. **17** (2007), no. 4, 1043–1105. MR 2373011

[6] B. Deroin, V. Kleptsyn, and A. Navas, *Sur la dynamique unidimensionnelle en régularité intermédiaire*, Acta Math. **199** (2007), no. 2, 199–262. MR 2358052

[7] A. Furman, *Random walks on groups and random transformations*, Handbook of dynamical systems, Vol. 1A, North-Holland, Amsterdam, 2002, pp. 931–1014. MR 1928529

[8] H. Furstenberg, *Strict ergodicity and transformation of the torus*, Amer. J. Math. **83** (1961), 573–601. MR 0133429

[9] H. Furstenberg, *Noncommuting random products*, Trans. Amer. Math. Soc. **108** (1963), 377–428. MR 0163345

[10] H. Furstenberg, *Random walks and discrete subgroups of Lie groups*, Advances in Probability and Related Topics, Vol. 1, Dekker, New York, 1971, pp. 1–63. MR 0284569

[11] É. Ghys and V. Sergiescu, *Sur un groupe remarquable de difféomorphismes du cercle*, Comment. Math. Helv. **62** (1987), no. 2, 185–239. MR 896095

[12] É. Ghys, *Groups acting on the circle*, Enseign. Math. (2) **47** (2001), no. 3–4, 329–407. MR 1876932

[13] Y. Guivarc'h and Y. Le Jan, *Asymptotic winding of the geodesic flow on modular surfaces and continued fractions*, Ann. Sci. École Norm. Sup. (4) **26** (1993), no. 1, 23–50. MR 1209912
[14] G. Hector, *Actions de groupes de difféomorphismes de* {0, 1}, Géométrie différentielle (Colloq., Univ. Santiago de Compostela, Santiago de Compostela, 1972), Springer, Berlin, 1974, pp. 14–22. Lecture Notes in Math., Vol. 392. MR 0362338
[15] M.-R. Herman, *Sur la conjugaison différentiable des difféomorphismes du cercle à des rotations*, Inst. Hautes Études Sci. Publ. Math. (1979), no. 49, 5–233. MR 538680
[16] S. Hurder, *Exceptional minimal sets and the Godbillon–Vey class*, To appear in Ann. Inst. Fourier (Grenoble).
[17] T. Inoue, *Ratio ergodic theorems for maps with indifferent fixed points*, Ergodic Theory Dynam. Systems **17** (1997), no. 3, 625–642. MR 1452184
[18] V. Kaimanovich and V. Le Prince, *Matrix random products with singular harmonic measure*, Preprint, 2008.
[19] A. Katok and B. Hasselblatt, *Introduction to the modern theory of dynamical systems*, Encyclopedia of Mathematics and its Applications, vol. 54, Cambridge University Press, Cambridge, 1995. MR 1326374.
[20] V. A. Kleptsyn and M. B. Nalskii, *Convergence of orbits in random dynamical systems on a circle*, Funktsional. Anal. i Prilozhen. **38** (2004), no. 4, 36–54, 95–96 (Russian). MR 2117507; *Letter to the editors: "Convergence of orbits in random dynamical systems on a circle"*, **39** (2005), no. 2, 95 (Russian). MR 2161524. English translation: Funct. Anal. Appl. **38** (2004), no. 4, 267–282; **39** (2005), no. 2, 163.
[21] I. Liousse, Unpublished text, 2007.
[22] R. Mañé, *Introdução à teoria ergódica*, Projeto Euclides, vol. 14, Instituto de Matemática Pura e Aplicada (IMPA), Rio de Janeiro, 1983. MR 800092. English translation: *Ergodic theory and differentiable dynamics*, Springer-Verlag, Berlin, 1987. MR 889254
[23] A. Navas, *Sur les groupes de difféomorphismes du cercle engendrés par des éléments proches des rotations*, Enseign. Math. (2) **50** (2004), no. 1–2, 29–68. MR 2084334
[24] A. Navas, *Grupos de difeomorfismos del círculo*, Ensaios Matemáticos, vol. 13, Sociedade Brasileira de Matemática, Rio de Janeiro, 2007. MR 2394157. English version (Groups of circle diffeomorphisms) is to appear.
[25] F. Oliveira and L. F. C. da Rocha, *Minimal non-ergodic C^1-diffeomorphisms of the circle*, Ergodic Theory Dynam. Systems **21** (2001), no. 6, 1843–1854. MR 1869073
[26] A. N. Quas, *Non-ergodicity for C^1 expanding maps and g-measures*, Ergodic Theory Dynam. Systems **16** (1996), no. 3, 531–543. MR 1395051
[27] J. C. Rebelo, *A theorem of measurable rigidity in* $\mathrm{Diff}^\omega(\mathbb{S}^1)$, Ergodic Theory Dynam. Systems **21** (2001), no. 5, 1525–1561. MR 1855846
[28] D. Sullivan, *The density at infinity of a discrete group of hyperbolic motions*, Inst. Hautes Études Sci. Publ. Math. (1979), no. 50, 171–202. MR 556586
[29] D. Sullivan, *Conformal dynamical systems*, Geometric dynamics (Rio de Janeiro, 1981), Lecture Notes in Math., vol. 1007, Springer, Berlin, 1983, pp. 725–752. MR 730296

Université Paris-Sud, Lab. de Mathématiques, Bât 425, 91405 Orsay Cedex, France
E-mail address: Bertrand.Deroin@math.u-psud.fr

Institut de Recherches Mathématiques de Rennes (UMR 6625 CNRS), Campus Beaulieu, 35042 Rennes, France
E-mail address: Victor.Kleptsyn@univ-rennes1.fr

Universidad de Santiago de Chile, Alameda 3363, Santiago, Chile
E-mail address: anavas@usach.cl