

## A DENJOY TYPE THEOREM FOR COMMUTING CIRCLE DIFFEOMORPHISMS WITH DERIVATIVES HAVING DIFFERENT HÖLDER DIFFERENTIABILITY CLASSES

VICTOR KLEPTSYN AND ANDRÉS NAVAS

ABSTRACT. Let  $d \geq 2$  be an integer number, and let  $f_k, k \in \{1, \dots, d\}$ , be  $C^{1+\tau_k}$  commuting circle diffeomorphisms, with  $\tau_k \in ]0, 1[$  and  $\tau_1 + \dots + \tau_d > 1$ . We prove that if the rotation numbers of the  $f_k$ 's are independent over the rationals (that is, if the corresponding action of  $\mathbb{Z}^n$  on the circle is free), then they are simultaneously (topologically) conjugate to rotations.

2000 MATH. SUBJ. CLASS. 22F05, 37C15, 37C80, 37E10.

KEY WORDS AND PHRASES. Denjoy theorem, centralizers, Hölder class of the derivative.

### INTRODUCTION

Starting from the seminal works by Poincaré [13], [14] and Denjoy [3], a deep theory for the dynamics of circle diffeomorphisms has been developed by many authors [1], [7], [8], [18], and most of the fundamental related problems have been already solved. Quite surprisingly, the case of several commuting diffeomorphisms is rather special, as it was pointed out for the first time by Moser [9] in relation to the problem of the smoothness for the simultaneous conjugacy to rotations. Roughly speaking, in this case it should be enough to assume a joint Diophantine condition on the rotation numbers which does not imply a Diophantine condition for any of them (see the recent work [5] for the solution of the  $C^\infty$  case of Moser's problem).

A similar phenomenon concerns the classical Denjoy theorem. Indeed, in [4] it was proved that if  $d \geq 2$  is an integer number and  $\tau > 1/d$ , then the elements  $f_1, \dots, f_d$  of any family of  $C^{1+\tau}$  commuting circle diffeomorphisms are simultaneously (topologically) conjugate to rotations provided that their rotation numbers are independent over the rationals (that is, no nontrivial linear combination of them with rational coefficients equals a rational number). In other words, the classical (and nearly optimal)  $C^2$  hypothesis for Denjoy theorem can be weakened in the case of several commuting diffeomorphisms. The first and main result of this work is a generalization of this fact to the case of different regularities.

---

Received August 14, 2007.

The first named author was funded by the Swiss National Science Foundation and by the RFBR grants 7-01-00017-a and CNRS-L\_a 05-01-02801. The second named author was funded by the Conicyt grants 1060829 and 7060237 and by the Research Network on Low Dimensional Dynamics.

**Theorem A.** *Let  $d \geq 2$  be an integer number and  $\tau_1, \dots, \tau_d$  be real numbers in  $]0, 1[$  such that  $\tau_1 + \dots + \tau_d > 1$ . If  $f_k, k \in \{1, \dots, d\}$ , are respectively  $C^{1+\tau_k}$  circle diffeomorphisms which have rotation numbers independent over the rationals and which do commute, then they are simultaneously (topologically) conjugate to rotations.*

It is maybe possible to modify the probabilistic arguments of [4] in order to deal with the present case. However, the methods that we introduce here are different. Indeed, for the proof of the result above we use a key new argument which is somehow more deterministic.

Theorem A is (almost) optimal (in the Hölder scale), in the sense that if one decreases slightly the regularity assumptions then it is no longer true. The following result relies on classical constructions by Bohl [2], Denjoy [3], Herman [7], and Pixton [12], and its proof consists on an easy extension of the construction given by Tsuboi in [17].

**Theorem B.** *Let  $d \geq 2$  be an integer number and  $\tau_1, \dots, \tau_d$  be real numbers in  $]0, 1[$  such that  $\tau_1 + \dots + \tau_d < 1$ . If  $\rho_1, \dots, \rho_d$  are elements in  $\mathbb{R}/\mathbb{Z}$  which are independent over the rationals, then there exist  $C^{1+\tau_k}$  circle diffeomorphisms  $f_k, k \in \{1, \dots, d\}$ , having rotation numbers  $\rho_k$ , which do commute, and such that none of them is topologically conjugate to a rotation.*

It is well known that the techniques developed for Denjoy theory can be applied to the study of group actions on the interval. In this direction we should point out that the methods of this paper also allow to extend (in a straightforward way) the so called “generalized Kopell lemma” and the “Denjoy–Szekerés type theorem” (Theorems B and C of [4] respectively) for Abelian groups of interval diffeomorphisms under analogous hypothesis of different regularities. Furthermore, the construction of counter-examples for both of them when these hypothesis do not hold can be also extended to this context. We leave the verification of all of this to the reader.

**Acknowledgments.** It is a pleasure to thank Bassam Fayad and Sergey Voronin for their encouragements, as well as the Independent University of Moscow for the hospitality during the conference “Laminations and Group Actions in Dynamics” held in February 2007. The first author would like to thank the hospitality of the University of Geneva, where he had a post-doctoral position at the time this work was written.

## 1. A GENERAL PRINCIPLE REVISITED

As it is well known since the classical works by Denjoy, Schwartz and Sacksteder [3], [15], [16], if  $I$  is a wandering interval<sup>1</sup> for the dynamics of a finitely generated semigroup  $\Gamma$  of  $C^{1+lip}$  diffeomorphisms of the closed interval or the circle (on which we will always consider the normalized length), one can control the distortion of the elements of  $\Gamma$  over (a slightly larger interval than)  $I$  in terms of the sum of the lengths of the images of  $I$  along the corresponding sequence of compositions and a

<sup>1</sup>We say that an interval is *wandering* if its images by different elements of the underlying semigroup are disjoint.

uniform Lipschitz constant for the derivatives of the (finitely many) generators of  $\Gamma$ . If  $\tau$  belongs to  $]0, 1[$  and  $\Gamma$  consists of  $C^{1+\tau}$  diffeomorphisms, the same is true provided that the sum of the  $\tau$ -powers of the lengths of the corresponding images of  $I$  is finite (this last condition does not follow from the disjointness of these intervals!): see for instance [4, Lemma 2.2]. It is not difficult to prove a similar statement for the case of different regularities, and this is precisely the content of the following lemma. However, to the difference of [4], here we will deal with *finite* sequences of compositions by a technical reason which will be clear at the end of the next section.

**Lemma 1.1.** *Let  $\Gamma$  be a semigroup of (orientation preserving) diffeomorphisms of the circle or the closed interval which is generated by finitely many elements  $g_k, k \in \{1, \dots, l\}$ , which are respectively of class  $C^{1+\tau_k}$ , where  $\tau_k \in ]0, 1[$ . Let  $C_k$  denote the  $\tau_k$ -Hölder constant of the function  $\log(g'_k)$ , and let  $C = \max\{C_1, \dots, C_l\}$  and  $\tau = \max\{\tau_1, \dots, \tau_l\}$ . Given  $n_0 \in \mathbb{N}$ , for each  $n \leq n_0$  let us chose  $k_n \in \{1, \dots, l\}$ , and for a fixed interval  $I$  let  $S > 0$  be a constant such that*

$$S \geq \sum_{n=0}^{n_0-1} |g_{k_n} \cdots g_{k_1}(I)|^{\tau_{k_{n+1}}}. \tag{1}$$

If  $n \leq n_0$  is such that  $g_{k_n} \cdots g_{k_1}(I)$  does not intersect  $I$  but is contained in the  $L$ -neighborhood of  $I$ , where  $L := \frac{|I|}{2 \exp(2^\tau CS)}$ , then  $g_{k_n} \cdots g_{k_1}$  has a hyperbolic fixed point.

*Proof.* Let  $J = [a, b]$  be the (closed)  $2L$ -neighborhood of  $I$ , and let  $I'$  (resp.  $I''$ ) the connected component of  $J \setminus I$  to the right (resp. to the left) of  $I$ . We will prove by induction on  $j \in \{0, \dots, n_0\}$  that the following two conditions are satisfied:

- (i)<sub>j</sub>  $|g_{k_j} \cdots g_{k_1}(I')| \leq |g_{k_j} \cdots g_{k_1}(I)|$ ,
- (ii)<sub>j</sub>  $\sup_{\{x,y\} \subset I \cup I'} \frac{(g_{k_j} \cdots g_{k_1})'(x)}{(g_{k_j} \cdots g_{k_1})'(y)} \leq \exp(2^\tau CS)$ .

Condition (ii)<sub>0</sub> is trivially satisfied, whereas condition (i)<sub>0</sub> is satisfied since  $|I'| = 2L \leq |I|$ . Assume that (i)<sub>i</sub> and (ii)<sub>i</sub> hold for each  $i \in \{0, \dots, j-1\}$ . Then for every  $x, y$  in  $I \cup I'$  we have

$$\begin{aligned} \left| \log \left( \frac{(g_{k_j} \cdots g_{k_1})'(x)}{(g_{k_j} \cdots g_{k_1})'(y)} \right) \right| &\leq \sum_{i=0}^{j-1} \left| \log(g'_{k_{i+1}}(g_{k_i} \cdots g_{k_1}(x))) - \log(g'_{k_{i+1}}(g_{k_i} \cdots g_{k_1}(y))) \right| \\ &\leq \sum_{i=0}^{j-1} C_{k_{i+1}} |g_{k_i} \cdots g_{k_1}(x) - g_{k_i} \cdots g_{k_1}(y)|^{\tau_{k_{i+1}}} \\ &\leq C \sum_{i=0}^{j-1} (|g_{k_i} \cdots g_{k_1}(I)| + |g_{k_i} \cdots g_{k_1}(I')|)^{\tau_{k_{i+1}}} \\ &\leq C 2^\tau \sum_{i=0}^{j-1} |g_{k_i} \cdots g_{k_1}(I)|^{\tau_{k_{i+1}}} \\ &\leq C 2^\tau S. \end{aligned}$$

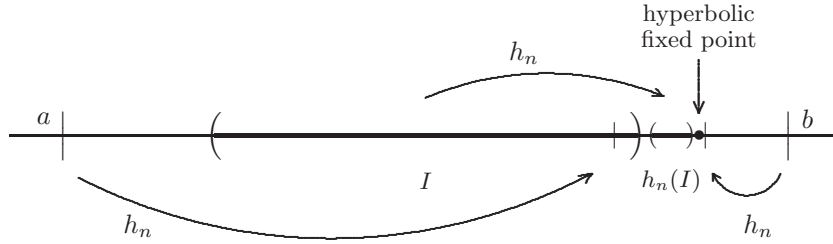


FIGURE 1.

This shows (ii)<sub>j</sub>. To verify (i)<sub>j</sub> first note that there must exist  $x \in I$  and  $y \in I'$  such that

$$|g_{k_j} \cdots g_{k_1}(I)| = |I| \cdot (g_{k_j} \cdots g_{k_1})'(x) \quad \text{and} \quad |g_{k_j} \cdots g_{k_1}(I')| = |I'| \cdot (g_{k_j} \cdots g_{k_1})'(y).$$

Therefore, by (ii)<sub>j</sub>,

$$\frac{|g_{k_j} \cdots g_{k_1}(I')|}{|g_{k_j} \cdots g_{k_1}(I)|} = \frac{(g_{k_j} \cdots g_{k_1})'(x)}{(g_{k_j} \cdots g_{k_1})'(y)} \cdot \frac{|I'|}{|I|} \leq \exp(2^\tau CS) \frac{|I'|}{|I|} = 1,$$

which proves (i)<sub>j</sub>. Obviously, similar arguments show that (i)<sub>j</sub> and (ii)<sub>j</sub> also hold for every  $j \in \{0, \dots, n_0\}$  when replacing  $I'$  by  $I''$ .

Now for simplicity let us denote  $h_j = g_{k_j} \cdots g_{k_1}$ . Assume that  $h_n(I)$  is contained in the  $L$ -neighborhood of the interval  $I$  and does not intersect  $I$  (see Figure 1). Then property (i)<sub>n</sub> gives  $h_n(J) \subset J$ , and this already implies that  $h_n$  has a fixed point  $x$  in  $J$ . (The reader will see that the existence of this fixed point together with the fact that  $h_n \neq id$  is the only information that we will retain for the proof of Theorem A.)

To conclude we would like to show that the fixed point  $x$  is hyperbolic. To do this just notice that there exists  $y \in I$  such that

$$h'_n(y) = \frac{|h_n(I)|}{|I|} \leq \frac{L}{|I|}.$$

Therefore, by (ii)<sub>n</sub>,

$$h'_n(x) \leq h'_n(y) \exp(2^\tau CS) \leq \frac{L \exp(2^\tau CS)}{|I|} = \frac{1}{2},$$

and this finishes the proof. □

## 2. PROOF OF THEOREM A

Recall the following well known argument (see for instance [6, Proposition 6.17] or [11, Lemma 4.14]). If  $f_1, \dots, f_d$  are commuting circle homeomorphisms, then there is a common invariant probability measure  $\mu$  on  $S^1$ . Moreover, if the rotation number of at least one of them is irrational, then there is no finite orbit for the group action, and the measure  $\mu$  has no atom. Therefore, the distribution function

$$F_\mu: S^1 \rightarrow \mathbb{R}/\mathbb{Z}, \quad F_\mu(x) := \mu([0, x]),$$

gives a (simultaneous) semiconjugacy between the maps  $f_1, \dots, f_d$  and the rotations corresponding to their rotation numbers. Thus, for the proof of Theorem A we have to show that this semiconjugacy is in fact a conjugacy, and our strategy for proving this (under the hypothesis of the theorem) is the classical one and goes back to Schwartz [16]. Indeed, in the contrary case the support of  $\mu$  would be a (minimal) invariant Cantor set, and the connected components of its complement would correspond to the maximal wandering open intervals. Fixing one of these intervals, say  $I$ , we will search for a sequence of compositions  $h_n = f_{k_n} \cdots f_{k_1}$  satisfying the hypothesis of Lemma 1.1. This will allow us to conclude that some  $h_n$  has a (hyperbolic) fixed point, thus implying that its rotation number is equal to zero. However, this is in contradiction to the fact that the rotation numbers of the  $f_k$ 's are independent over the rationals (it is easy to verify that the rotation number restricted to any group of circle homeomorphisms which preserves a probability measure on  $S^1$  is a group homomorphism: see again [6] or [11]).

In order to ensure the existence of the sequence  $(h_n)$ , the main idea of [4] was to endow the space of all (infinite) sequences of compositions with a natural probability measure, and then to prove that the "generic ones" satisfy many nice properties as for instance the convergence of the sum (1) as  $n_0$  goes to infinity. It seems difficult to apply such a probabilistic argument to the case of different regularities, and we will need to introduce a new argument which is somehow more deterministic, since it gives partial information on the sequence that we find. For simplicity we will first deal with the case  $d = 2$ .

**2.1. The case  $d = 2$ .** Although not explicitly stated in [4], the main probabilistic argument for the proof of the generalized Denjoy theorem therein is not a dynamical issue, but it is just a statement concerning the finiteness of the sum of the  $\tau$ -powers of some positive real numbers. To be more concrete (at least in the case  $d = 2$  and when  $\tau > 1/2$ ), if  $(\ell_{i,j})$  is a double-indexed sequence of positive numbers with finite total sum (where  $i$  and  $j$  are nonnegative integers), then with respect to some natural probability distribution on the space of infinite paths  $(i(n), j(n))_{n \geq 0}$  satisfying  $i(0) = j(0) = 0$ ,  $i(n+1) \geq i(n)$ ,  $j(n+1) \geq j(n)$  and  $i(n+1) + j(n+1) = 1 + i(n) + j(n)$ , one has almost everywhere the convergence of the sum

$$\sum_{n \geq 0} \ell_{i(n), j(n)}^\tau.$$

The first goal of this section is to prove the existence of paths sharing a similar property in the case of different exponents  $\tau_1, \tau_2$  in  $]0, 1[$  (with  $\tau_1 + \tau_2 > 1$ ). A substantial difference here is that we will construct our sequence by concatenating infinitely many finite paths, and each one of these paths will be chosen among finitely many ones. To do this we begin with the following elementary lemma.

**Lemma 2.1.** *Let  $\ell_{i,j}$  be positive real numbers, where  $i \in \{1, \dots, m\}$  and  $j \in \{1, \dots, n\}$ . Assume that the total sum of the  $\ell_{i,j}$ 's is less than or equal to 1. If  $\tau$  belongs to  $]0, 1[$ , then there exists  $k \in \{1, \dots, n\}$  such that*

$$\sum_{i=1}^m \ell_{i,k}^\tau \leq \frac{m^{1-\tau}}{n^\tau}.$$

*Proof.* We will show that the mean value of the function  $k \mapsto \sum_{i=1}^m \ell_{i,k}^\tau$  is less than or equal to  $m^{1-\tau}/n^\tau$ , from where the claim of the lemma follows immediately. To do this first notice that, by Hölder’s inequality, for each fixed  $k \in \{1, \dots, n\}$  one has

$$\sum_{i=1}^m \ell_{i,k}^\tau = \langle (\ell_{i,k}^\tau)_{i=1}^m, (1)_{i=1}^m \rangle \leq \|(\ell_{i,k}^\tau)_{i=1}^m\|_{1/\tau} \cdot \|(1)_{i=1}^m\|_{1/(1-\tau)} = \left( \sum_{i=1}^m \ell_{i,k} \right)^\tau m^{1-\tau}.$$

Thus, by using Hölder’s inequality again one obtains

$$\begin{aligned} \frac{1}{n} \sum_{k=1}^n \left( \sum_{i=1}^m \ell_{i,k}^\tau \right) &= \frac{m^{1-\tau}}{n} \left\langle \left( \left( \sum_{k=1}^n \ell_{i,k} \right)^\tau \right)_{k=1}^n, (1)_{k=1}^n \right\rangle \\ &\leq \frac{m^{1-\tau}}{n} \left\| \left( \left( \sum_{k=1}^n \ell_{i,k} \right)^\tau \right)_{k=1}^n \right\|_{1/\tau} \cdot \|(1)_{k=1}^n\|_{1/(1-\tau)} \\ &= \frac{m^{1-\tau}}{n} \left( \sum_{k=1}^n \sum_{i=1}^m \ell_{i,k} \right)^\tau n^{1-\tau} \\ &\leq \frac{m^{1-\tau}}{n^\tau}, \end{aligned}$$

which finishes the proof. □

Now we explain the main idea of our construction. Let us assume that the total sum of the double-indexed sequence of positive numbers  $\ell_{i,j}$  is  $\leq 1$ , and suppose that the numbers  $\tau_1 \in ]0, 1[$  and  $\tau_2 \in ]0, 1[$  such that  $\tau_1 + \tau_2 > 1$  are fixed. Denoting by  $[[a, b]]$  the set of integers between  $a$  and  $b$  (with  $a$  and/or  $b$  included when they are in  $\mathbb{Z}$ ), let us consider any sequence of rectangles  $R_m \subset \mathbb{N}_0 \times \mathbb{N}_0$  such that  $R_0 = \{(0, 0)\}$ ,  $R_{2m+1} = [[i_m, i_{m+1}] \times [[j_m, j_{m+2}]$  and  $R_{2m+2} = [[i_m, i_{m+2}] \times [[j_{m+1}, j_{m+2}]$ , where  $(i_m)_{m \geq 1}$  and  $(j_m)_{m \geq 1}$  are strictly increasing sequences of nonnegative integers numbers satisfying  $i_0 = i_1 = 0$  and  $j_0 = j_1 = 0$  (see Figure 2). Denoting by  $X_m$  and  $Y_m$  respectively the number of points on the horizontal and vertical sides of each  $R_m$ , a direct application of Lemma 2.1 gives us, for  $\varepsilon := 1 - \tau_1 - \tau_2 > 0$  and each  $m \geq 0$ :

- an integer  $r(2m + 1) \in [[i_m, i_{m+1}]]$  such that

$$\sum_{j=j_m}^{j_{m+2}} \ell_{r(2m+1),j}^{\tau_2} \leq \frac{Y_{2m+1}^{1-\tau_2}}{X_{2m+1}^{\tau_2}} = \frac{Y_{2m+1}^{\tau_1}}{X_{2m+1}^{\tau_2}} \cdot Y_{2m+1}^{-\varepsilon},$$

- an integer  $r(2m + 2) \in [[j_{m+1}, j_{m+2}]]$  such that

$$\sum_{i=i_m}^{i_{m+2}} \ell_{i,r(2m+2)}^{\tau_1} \leq \frac{X_{2m+2}^{1-\tau_1}}{Y_{2m+2}^{\tau_1}} = \frac{X_{2m+2}^{\tau_2}}{Y_{2m+2}^{\tau_1}} \cdot X_{2m+2}^{-\varepsilon}.$$

Starting from the origin and following the corresponding horizontal and vertical lines, we find an infinite path  $(i(n), j(n))_{n \geq 0}$  satisfying

$$\begin{aligned} i(0) = j(0) = 0, \quad i(n + 1) \geq i(n), \quad j(n + 1) \geq j(n), \\ i(n + 1) + j(n + 1) = 1 + i(n) + j(n), \end{aligned}$$

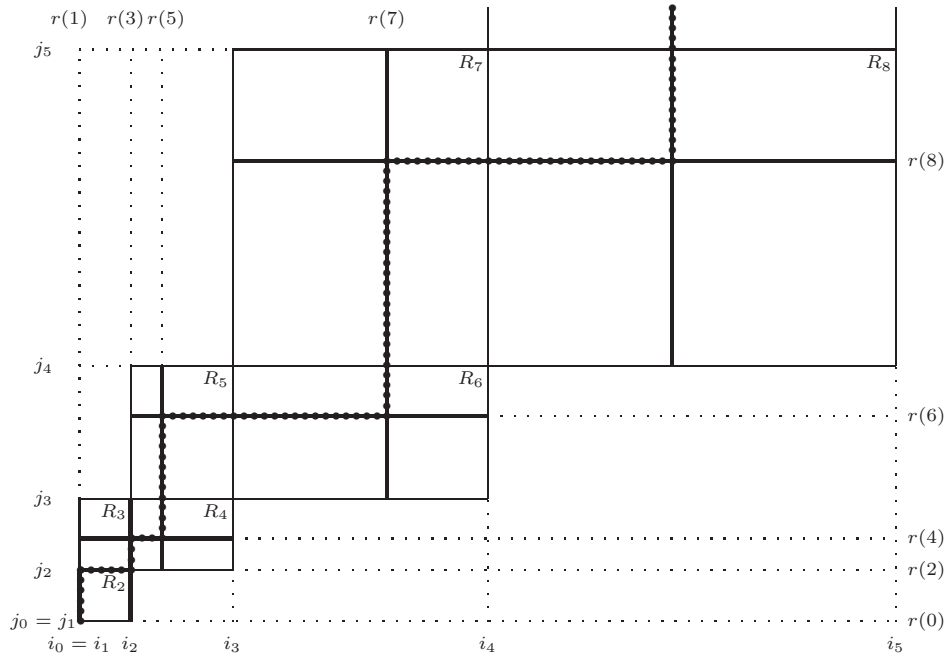


FIGURE 2.

and such that the sum

$$\sum_{n \geq 0} \ell_{i(n),j(n)}^{\tau_{\alpha(n)}} \tag{2}$$

is bounded by

$$\sum_{m \geq 0} \left[ \frac{Y_{2m+1}^{\tau_1}}{X_{2m+1}^{\tau_2}} \cdot Y_{2m+1}^{-\varepsilon} + \frac{X_{2m+2}^{\tau_2}}{Y_{2m+2}^{\tau_1}} \cdot X_{2m+2}^{-\varepsilon} \right], \tag{3}$$

where  $\alpha(n) := 1$  if  $|i(n+1) - i(n)| = 1$  and  $\alpha(n) := 2$  if  $|j(n+1) - j(n)| = 1$ .

Now let us consider any choice such that  $i_m = \lfloor 4^{m\tau_1} \rfloor$  and  $j_m = \lfloor 4^{m\tau_2} \rfloor$  for  $m$  large enough. Writing  $a_m \simeq b_m$  when  $(a_m)$  and  $(b_m)$  are sequences of positive numbers such that  $(a_m/b_m)$  remains bounded and away from zero, for such a choice we have  $X_m \simeq 2^{m\tau_1}$  and  $Y_m \simeq 2^{m\tau_2}$ . Thus,

$$\frac{X_m^{\tau_2}}{Y_m^{\tau_1}} \simeq \frac{(2^{m\tau_1})^{\tau_2}}{(2^{m\tau_2})^{\tau_1}} = 1,$$

and therefore there exists  $C' > 0$  such that, for each  $m \geq 0$ ,

$$\frac{1}{C'} \leq \frac{X_m^{\tau_2}}{Y_m^{\tau_1}} \leq C'.$$

This implies that, for  $\bar{C} = 1 + \max\{(4^{2\tau_1} - 1)^{-\varepsilon}, (4^{2\tau_2} - 1)^{-\varepsilon}\}$ , the sum in (3) is bounded by

$$S := C'\bar{C} \left( \sum_{m \geq 0} \left[ \left(\frac{1}{4^{m\tau_2}}\right)^\varepsilon + \left(\frac{1}{4^{m\tau_1}}\right)^\varepsilon \right] \right) = C'\bar{C} \left( \frac{4^{\tau_2\varepsilon}}{4^{\tau_2\varepsilon} - 1} + \frac{4^{\tau_1\varepsilon}}{4^{\tau_1\varepsilon} - 1} \right), \quad (4)$$

and so the value of the sum (2) is finite (and also bounded by  $S$ ).

We can now proceed to the proof of Theorem A in the case  $d = 2$ . Assume by contradiction that  $f_k, k \in \{1, 2\}$ , are respectively  $C^{1+\tau_k}$  commuting circle diffeomorphisms which are not simultaneously conjugate to rotations and which have rotation numbers independent over the rationals. Let  $I$  be a connected component of the complement of the invariant minimal Cantor set for the group action, and let  $\ell_{i,j} = |f_1^i f_2^j(I)|$ . We obviously have  $\sum_{i,j} \ell_{i,j} \leq 1$ , and so we can apply all our previous discussion to this sequence. In particular, there exists an infinite path  $(i(n), j(n))$  starting at the origin and such that the sum

$$\sum_{n \geq 0} \ell_{i(n),j(n)}^{\tau_{\alpha(n)}}$$

is bounded by the number  $S > 0$  defined by (4). If for  $n \geq 1$  we let  $k_n = \alpha(n-1) \in \{1, 2\}$ , then we obtain a sequence of compositions  $h_n = f_{k_n} \dots f_{k_1}$  such that the preceding sum coincides term by term with

$$\sum_{n \geq 0} |f_{k_n} \dots f_{k_1}(I)|^{\tau_{k_{n+1}}}.$$

Thus, in order to apply Lemma 1.1 to get a contradiction, we just need to verify that, for some  $n \geq 1$ , the hypothesis that  $h_n(I) = f_{k_n} \dots f_{k_1}(I)$  is contained in the  $L$ -neighborhood of  $I$  is satisfied (where  $L := \frac{|I|}{2 \exp(2^\tau C S)}$ ,  $\tau := \max\{\tau_1, \tau_2\}$ , and  $C := \max\{C_1, C_2\}$ , with  $C_k$  being the  $\tau_k$ -Hölder constant for the function  $\log(f'_k)$ ).

To do this first notice that, if we collapse all the connected components of the complement of the minimal invariant Cantor set, then we obtain a topological circle  $\hat{S}^1$  on which the original diffeomorphisms induce naturally minimal homeomorphisms  $\hat{f}_1$  and  $\hat{f}_2$  which are simultaneously conjugate to rotations. Moreover, the  $L$ -neighborhood of  $I$  becomes a nondegenerate interval  $\hat{U}$ ; thus, there exists  $N \in \mathbb{N}$  such that the intervals  $\hat{f}_1^{-1}(\hat{U}), \dots, \hat{f}_1^{-N}(\hat{U})$ , as well as  $\hat{f}_2^{-1}(\hat{U}), \dots, \hat{f}_2^{-N}(\hat{U})$ , cover the circle  $\hat{S}^1$ . This easily implies that for any image  $I_0$  of  $I$  by some element of the semigroup generated by  $f_1$  and  $f_2$  there exists  $k$  and  $k'$  in  $\{1, \dots, N\}$  such that  $f_1^k(I_0)$  and  $f_2^{k'}(I_0)$  are contained in the  $L$ -neighborhood of  $I$ . Now it is easy to see that, for the sequence of compositions that we found, for every  $\bar{N} \in \mathbb{N}$  there exists some integer  $r \in \mathbb{N}$  such that  $k_r = k_{r+1} = \dots = k_{r+\bar{N}}$ . For  $\bar{N} = N$  this obviously implies that at least one of the intervals  $h_{r+1}(I), \dots, h_{r+N}(I)$  is contained in the  $L$ -neighborhood of  $I$ , thus finishing the proof.

We would like to close this section by giving a different type of choice for the sequence of rectangles which is simpler to describe and for which the preceding arguments are also valid. (For simplicity, we will use a similar construction to deal with the case  $d > 2$ , although the preceding one still applies). This sequence



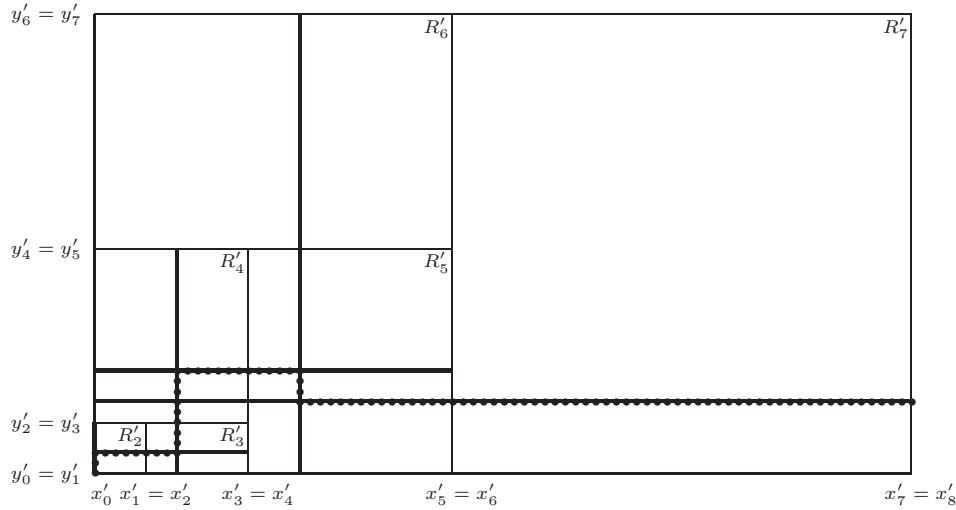


FIGURE 3.

$(R'_m)_{m \geq 0}$  is of the form  $[[0, x'_m]] \times [[0, y'_m]]$ , where  $(x'_m)$  and  $(y'_m)$  are nondecreasing sequences of positive integer numbers such that  $x'_0 = y'_0 = 0$ ,  $x'_m > x'_{m-1}$  and  $y'_m = y'_{m-1}$  if  $m$  is odd, and  $x'_m = x'_{m-1}$  and  $y'_m > y'_{m-1}$  if  $m$  is even. If  $(\ell_{i,j})$  is a double-indexed sequence of positive real numbers with total sum  $\leq 1$ , we chose these integer numbers in such a way that  $x'_{2m+1} = x'_{2m+2} = [4^{m\tau_1}]$  and  $y'_{2m} = y'_{2m+1} = [4^{m\tau_2}]$  for  $m$  large enough. As before, inside the rectangle  $R_m$  there is a “good” vertical (resp. horizontal) segment of line  $L_m$  for  $m$  even (resp. odd). Therefore, for each  $M_0 \in \mathbb{N}$  we can concatenate these segments between  $L_{m-1} \cap L_m$  and  $L_m \cap L_{m+1}$  at the  $m^{\text{th}}$  step for  $m < M_0$ , and between  $L_{M_0-1} \cap L_{M_0}$  and the point of  $L_{M_0}$  on the boundary of  $R_{M_0}$  at the last step (see Figure 3). In this way we obtain a path (starting at the origin) of finite length  $n(M_0) - 1$  for which the sum

$$\sum_{n=0}^{n(M_0)-1} \ell_{i(n),j(n)}^{\tau_{\alpha(n)}}$$

is bounded by some number  $S > 0$  which is independent of  $M_0$ .

Now let  $f_k$ ,  $k \in \{1, 2\}$ , be two commuting circle diffeomorphisms of class  $C^{1+\tau_k}$  which are not simultaneously conjugate to rotations. Fix again one of the maximal wandering open intervals  $I$  for the dynamics, and let  $\ell_{i,j} = |f_1^i f_2^j(I)|$ . (Notice that  $\sum_{i,j} \ell_{i,j} \leq 1$ .) The method above gives us a family of finite paths, and each of these paths determines uniquely a sequence of compositions. Remark however that there is a little difference here, since we allow the use of the inverses of the  $f_k$ 's. Therefore, in order to apply Lemma 1.1, we must consider  $\{f_1, f_1^{-1}, f_2, f_2^{-1}\}$  as being our system of generators, and therefore we put  $\tau = \max\{\tau_1, \tau_2\}$  and  $C = \max\{C_1, C_2, C'_1, C'_2\}$ , where  $C_i$  (resp.  $C'_i$ ) is a  $\tau_i$ -Hölder constant for the function  $\log(f'_i)$  (resp.  $\log((f_i^{-1})')$ ). As in the previous proof, we need to verify that, for some  $M_0 \in \mathbb{N}$ , there exists a nontrivial element in the sequence of compositions  $(h_n)$

associated to its corresponding finite path which sends  $I$  inside the  $L$ -neighborhood of itself, where  $L := \frac{|I|}{2^{\exp(2^\tau CS)}}$ . As before, for proving this it suffices to show that for every  $N$  there exists  $r \in \mathbb{N}$  such that one has  $h_{r+i+1} = f_1 h_{r+i}$  for each  $i \in \{0, \dots, N-1\}$ , or  $h_{r+i+1} = f_2 h_{r+i}$  for each  $i \in \{0, \dots, N-1\}$ . However, this last property is always satisfied if  $M_0$  is big enough so that the number of points with integer coordinates in the line segment  $L_{M_0}$  contained in  $R_{M_0} \setminus R_{M_0-1}$  is greater than  $N$ . Notice that it is in this last argument where we use the fact that we keep only finite sequences of compositions, although our method combined with a diagonal type argument easily shows the existence of an infinite sequence for which the sum (2) converges.

**2.2. The general case.** In the case  $d = 2$ , the “good” paths leading to the sequence of compositions which allows to apply Lemma 1.1 were obtained by concatenating horizontal and vertical lines. When  $d > 2$  we will need to concatenate lines in several (namely,  $d$ ) directions, and the geometrical difficulty for doing this is evident: in dimension bigger than 2, two lines in different directions do not necessarily intersect. To overcome this difficulty we will use the fact that, at each step (*i.e.* inside each rectangle), there is not only one finite path which is good, but this is the case for a “large proportion” of finite paths. We first reformulate Lemma 2.1 in this direction.

**Lemma 2.2.** *Let  $\ell_{i,j}$  be positive real numbers, where  $i \in \{1, \dots, m\}$  and  $j \in \{1, \dots, n\}$ . Assume that the total sum of the  $\ell_{i,j}$ ’s is less than or equal to 1. If  $\tau$  belongs to  $]0, 1[$  and  $A > 1$ , then for a proportion of indexes  $k \in \{1, \dots, n\}$  greater than or equal to  $(1 - 1/A)$  we have*

$$\sum_{i=1}^m \ell_{i,k}^\tau \leq A \cdot \frac{m^{1-\tau}}{n^\tau}.$$

*Proof.* As in the proof of Lemma 2.1, the mean value of the function

$$k \mapsto \sum_{i=1}^m \ell_{i,k}^\tau \tag{5}$$

is less than or equal to  $m^{1-\tau}/n^\tau$ . The claim of the lemma then follows as a direct application of Chebychev’s inequality: the proportion of points for which the value of (5) is greater than this mean value times  $A$  cannot exceed  $1/A$ .  $\square$

Now let  $(\ell_{i_1, \dots, i_d})$  be a multi-indexed sequence of positive real numbers having total sum  $\leq 1$ , and let  $\tau_1, \dots, \tau_d$  be real numbers in  $]0, 1[$ . Starting with  $R_0 = [[0, 0]]^d$ , let us consider a sequence  $(R_m)_{m \geq 0}$  of rectangles of the form  $R_m = [[0, x_{1,m}]] \times \dots \times [[0, x_{d,m}]]$  satisfying  $x_{k,m} \geq x_{k,m-1}$  for each  $k \in \{1, \dots, d\}$ , with strict inequality if and only if  $k \equiv m \pmod{d}$ . For each  $m \geq 1$  denote by  $s(m) \in \{1, \dots, d\}$  the residue class (mod  $d$ ) of  $m$ , and denote by  $F_m$  the face

$$[[0, x_{1,m}]] \times \dots \times [[0, x_{s(m)-1,m}]] \times \{0\} \times [[0, x_{s(m)+1,m}]] \times \dots \times [[0, x_{d,m}]]$$

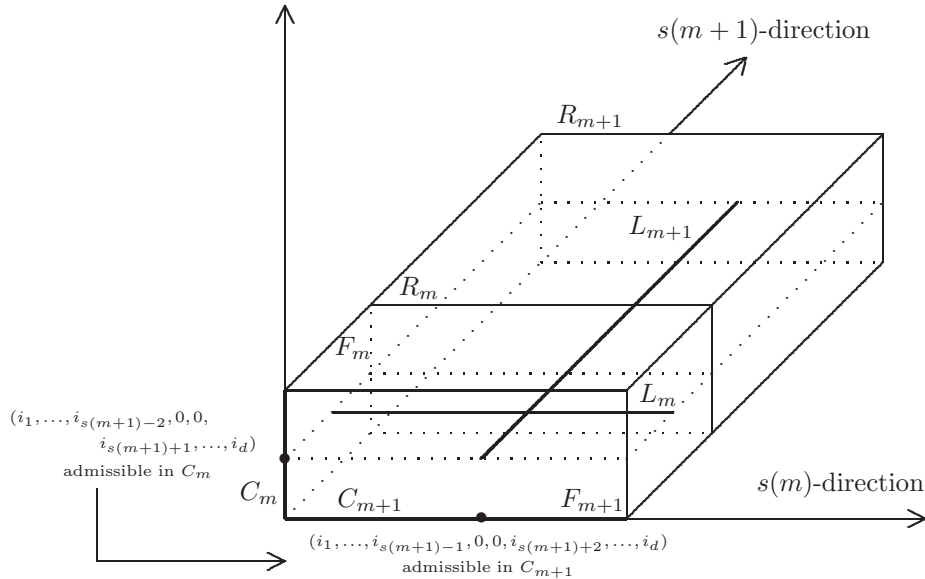


FIGURE 4.

of  $R_m$ . For each  $(i_1, \dots, i_{s(m)-1}, 0, i_{s(m)+1}, \dots, i_d)$  belonging to this face  $F_m$  we consider the sum

$$\sum_{j=0}^{x_{s(m),m}} \ell_{i_1, \dots, i_{s(m)-1}, j, i_{s(m)+1}, \dots, i_d}^{\tau_{s(m)}}$$

By Lemma 2.2, if  $A_m > 1$  then the proportion of points in  $F_m$  for which this sum is bounded by

$$A_m \cdot \frac{(1 + x_{s(m),m})^{1-\tau_{s(m)}}}{\prod_{j \neq s(m)} (1 + x_{j,m})^{\tau_{s(m)}}} = A_m \cdot \frac{X_{s(m),m}^{1-\tau_{s(m)}}}{\prod_{j \neq s(m)} X_{j,m}^{\tau_{s(m)}}$$

is at least equal to  $(1 - 1/A_m)$ , where  $X_{j,m} := 1 + x_{j,m}$ . In order to concatenate the corresponding lines we will use the following elementary lemma.

**Lemma 2.3.** *Let us choose inside each rectangle  $(R_m)_{m \geq 1}$  a set  $\mathcal{L}(m)$  of (complete) lines in the corresponding  $s(m)$ -direction whose proportion (with respect to all the lines in that direction inside  $(R_m)$ ) is at least  $(1 - 1/A_m)$ . If  $M_0 \in \mathbb{N}$  is such that  $\sum_{m=1}^{M_0} 1/A_m < 1$ , then there exists a sequence of lines  $L_m \in \mathcal{L}(m)$ ,  $m \in \{0, \dots, M_0\}$ , such that  $L_{m+1}$  intersects  $L_m$  for every  $m < M_0$ .*

*Proof.* Let us denote by  $C_m$  the  $(d - 2)$ -dimensional face of  $R_m$  given by

$$[[0, x_{1,m}]] \times \dots \times [[0, x_{s(m)-1,m}]] \times \{0\} \times \{0\} \times [[0, x_{s(m)+2,m}]] \times \dots \times [[0, x_{d,m}]].$$

Call a point  $(i_1, \dots, i_{s(m)-1}, 0, 0, i_{s(m)+2}, \dots, i_d) \in C_m$  *admissible* if there exists a sequence of lines  $L_i \in \mathcal{L}(i)$ ,  $i \in \{0, \dots, m\}$ , such that  $L_i$  intersects  $L_{i+1}$  for every  $i \in \{0, \dots, m - 1\}$ , and such that  $L_m$  projects in the  $s(m)$ -direction into a point  $(i_1, \dots, i_{s(m)-1}, 0, i_{s(m)+1}, i_{s(m)+2}, \dots, i_d) \in F_m$  for some  $i_{s(m)+1} \in$

$[[0, x_{s(m)+1, m+1}]]$ . We will show that the proportion of admissible points in  $C_{M_0}$  is greater than or equal to

$$P := 1 - \sum_{m=1}^{M_0} A_m > 0.$$

To do this, for each  $m \geq 0$  let us denote by  $P_m$  the proportion of admissible points in  $C_m$ . Since  $R_0$  reduces to the origin, it suffices to show that, for all  $m \geq 0$ ,

$$P_{m+1} \geq P_m - \frac{1}{A_{m+1}}.$$

To prove this inequality first notice that each line  $L_{m+1} \in \mathcal{L}(m+1)$  determines uniquely a point  $(i_1, \dots, i_{s(m+1)-1}, 0, i_{s(m+1)+1}, \dots, i_d) \in F_{m+1}$ . The projection into  $C_m$  of this line then corresponds to the point

$$(i_1, \dots, i_{s(m+1)-2}, 0, 0, i_{s(m+1)+1}, \dots, i_d).$$

If this is an admissible point of  $C_m$  then we can concatenate the line  $L_{m+1}$  to the sequence of lines corresponding to it (see Figure 4). Now the proportion of lines in  $\mathcal{L}(m+1)$  being at least  $1 - 1/A_{m+1}$ , the proportion of those lines which project on  $C_m$  into an admissible point is at least equal to

$$1 - \frac{1}{A_{m+1}} - (1 - P_m) = P_m - \frac{1}{A_{m+1}}.$$

By projecting in the  $(s(m+1) + 1)$ -direction, this obviously implies that the proportion of admissible points in  $C_{m+1}$  is also greater than or equal to  $P_m - 1/A_{m+1}$ , thus finishing the proof.  $\square$

Observe that a sequence of lines  $L_m$  as above determines a finite path (starting at the origin) of points  $(x_1(n), \dots, x_d(n))$  having nonnegative integer coordinates such that the distance between two consecutive ones is equal to 1. Moreover, if we denote by  $n(M_0)$  the length of this path plus 1, the corresponding sum

$$\sum_{n=0}^{n(M_0)-1} \ell_{x_1(n), \dots, x_d(n)}^{\tau_{\alpha(n)}} \tag{6}$$

is bounded by

$$\sum_{m=0}^{M_0} A_m \cdot \frac{(1 + x_{s(m), m})^{1-\tau_{s(m)}}}{\prod_{i \neq s(m)} (1 + x_{i, m})^{\tau_{s(m)}}} = \sum_{m=0}^{M_0} A_m \cdot \frac{X_{s(m), m}^{1-\tau_{s(m)}}}{\prod_{j \neq s(m)} X_{j, m}^{\tau_{s(m)}}}, \tag{7}$$

where  $\alpha(n)$  equals the unique index in  $\{1, \dots, d\}$  s. t.  $|x_{\alpha(n)}(n+1) - x_{\alpha(n)}(n)| = 1$ .

Now let us define  $A_m = 2^{\varepsilon m \tau_{s(m)}/2} A$ , where  $A$  is a large enough constant so that  $\sum_{m \geq 0} 1/A_m < 1$ , and let us consider any choice of the  $x_{k, m}$ 's so that  $X_{k, m} \simeq 2^{m \tau_k}$ . For such a choice we have

$$\frac{X_{k, m}^{1-\tau_k}}{\prod_{j \neq k} X_{j, m}^{\tau_k}} = X_{k, m}^{-\varepsilon} \cdot \prod_{j \neq k} \frac{X_{k, m}^{\tau_j}}{X_{j, m}^{\tau_k}} \simeq 2^{-\varepsilon m \tau_k} \cdot \prod_{j \neq k} \frac{(2^{m \tau_k})^{\tau_j}}{(2^{m \tau_j})^{\tau_k}} = 2^{-\varepsilon m \tau_k}, \tag{8}$$

where  $\varepsilon := 1 - \tau_1 - \dots - \tau_d > 0$ . Therefore, for each  $M_0 \in \mathbb{N}$  the preceding lemma provides us a sequence of lines  $L_m$ ,  $m \in \{0, \dots, M_0\}$ , such that  $L_{m+1}$  intersects

$L_m$  for each  $m < M_0$ , and such that the corresponding expression (7) is bounded from above by

$$\sum_{m=0}^{M_0} 2^{\varepsilon m \tau_s(m)/2} A \cdot \frac{X_{k,m}^{1-\tau_k}}{\prod_{j \neq k} X_{j,m}^{\tau_k}} \leq AC' \sum_{m \geq 0} 2^{-\varepsilon m \tau_s(m)/2} \leq AC' \sum_{m \geq 0} 2^{-\varepsilon m \tau'/2} =: S < \infty, \quad (9)$$

where  $\tau' := \min\{\tau_1, \dots, \tau_d\}$  and  $C'$  is a constant (independent of  $M_0$ ) giving an upper bound for the quotient between the left and the right hand expressions in (8).

With all this information in mind we may proceed to the proof of Theorem A in the case  $d > 2$  in the very same way as in the (second proof for the) case  $d = 2$ . Indeed, assume that  $f_k, k \in \{1, \dots, d\}$ , are circle diffeomorphisms as in the statement of the theorem which are not conjugate to rotations, and let  $I$  be a maximal open wandering interval for the dynamics (*i.e.* a connected component of the complement of the minimal invariant Cantor set). Obviously, we may apply all our previous discussion to the multi-indexed sequence  $(\ell_{i_1, \dots, i_d})$  defined by  $\ell_{i_1, \dots, i_d} = |f_1^{i_1} \dots f_d^{i_d}(I)|$ . In particular, for each  $M_0 \in \mathbb{N}$  we can find a finite path so that the sum (6) is bounded by the number  $S > 0$  defined by (9) (which is independent of  $M_0$ ). Each such a path induces canonically a finite sequence of compositions by the  $f_k$ 's and their inverses. Therefore, in order to apply Lemma 1.1 to get a contradiction, we need to verify that some of such sequences contains a (nontrivial) element  $h_n$  which sends  $I$  into its  $L$ -neighborhood for  $L := \frac{|I|}{2^{\exp(2^r CS)}}$ , where  $\tau := \max\{\tau_1, \dots, \tau_d\}$  and  $C := \max\{C_1, \dots, C_d, C'_1, \dots, C'_d\}$ , with  $C_k$  (resp.  $C'_k$ ) being the  $\tau_k$ -Hölder constant of the function  $\log(f'_k)$  (resp.  $\log((f_k^{-1})')$ ). To ensure this last property let  $U$  be the  $L$ -neighborhood of  $I$ , and let  $N \in \mathbb{N}$  be such that, given any wandering interval, among the first  $N$  iterates of  $f_1$ , as well as for  $f_2, \dots, f_d$ , at least one of them sends this interval inside  $U$ . If we take  $M_0$  large enough so that the number of points with integer coordinates in  $L_{M_0}$  which are contained in  $R_{M_0} \setminus R_{M_0-1}$  exceeds  $N$ , then one can easily see that the associated sequence of compositions contains the desired element  $h_n$ . This finishes the proof of Theorem A.

### 3. PROOF OF THEOREM B

The strategy for the proof of Theorem B is well known. We prescribe the rotation numbers  $\rho_1, \dots, \rho_d$  (which are supposed to be independent over the rationals), we fix a point  $p \in S^1$ , and for each  $(i_1, \dots, i_d) \in \mathbb{Z}^d$  we replace the point  $R_{\rho_1}^{i_1} \dots R_{\rho_d}^{i_d}(p)$  by an interval  $I_{i_1, \dots, i_d}$  of length  $\ell_{i_1, \dots, i_d}$  in such a way that the total sum of the  $\ell_{i_1, \dots, i_d}$ 's is finite. Doing this we obtain a new circle on which the rotations  $R_{\rho_k}$  induce nice homeomorphisms if we extend them appropriately to the intervals  $I_{i_1, \dots, i_d}$  (outside these intervals the induced homeomorphisms are canonically defined). More precisely, as it is well explained in [4], [7], [10], [17], if there exists a constant  $C' > 0$  so that for all  $(i_1, \dots, i_d) \in \mathbb{Z}^d$  and all  $k \in \{1, \dots, d\}$  one

has

$$\left| \frac{\ell_{i_1, \dots, 1+i_k, \dots, i_d}}{\ell_{i_1, \dots, i_k, \dots, i_d}} - 1 \right| \frac{1}{\ell_{i_1, \dots, i_k, \dots, i_d}^{\tau_k}} \leq C', \tag{10}$$

then one can perform the extension to the intervals  $I_{i_1, \dots, i_d}$  in such a way the resulting maps  $f_k$ ,  $k \in \{1, \dots, d\}$ , are respectively  $C^{1+\tau_k}$  diffeomorphisms and commute, and moreover their derivatives are identically equal to 1 on the invariant minimal Cantor set.<sup>2</sup> Indeed, one possible extension is given by  $f_k(x) = (\varphi_{I_{i_1, \dots, i_k, \dots, i_d}})^{-1} \circ \varphi_{I_{i_1, \dots, 1+i_k, \dots, i_d}}(x)$ , where  $x$  belongs to the interior of the interval  $I_{i_1, \dots, i_k, \dots, i_d}$ . Here,  $\varphi_I: ]a, b[ \rightarrow \mathbb{R}$  denotes the map

$$\varphi_I(x) = \frac{-1}{b-a} \operatorname{ctg}\left(\pi \frac{x-a}{b-a}\right).$$

It turns out that a good choice for the lengths is

$$\ell_{i_1, \dots, i_d} = \frac{1}{1 + |i_1|^{1/\tau_1} + \dots + |i_d|^{1/\tau_d}}.$$

Indeed, on the one hand, if we decompose the sum of the  $\ell_{i_1, \dots, i_d}$ 's according to the biggest  $|i_j|^{1/\tau_j}$  we obtain

$$\sum_{(i_1, \dots, i_d) \in \mathbb{Z}^d} \ell_{i_1, \dots, i_d} \leq 1 + \sum_{k=1}^d \sum_{\substack{|i_j|^{1/\tau_j} \leq |i_k|^{1/\tau_k} \\ \text{for all } j \in \{1, \dots, d\} \\ |i_k| \geq 1}} \frac{1}{1 + |i_1|^{1/\tau_1} + \dots + |i_d|^{1/\tau_d}},$$

and therefore, for some constant  $C > 0$ , this sum is bounded by

$$\begin{aligned} & 1 + \sum_{k=1}^d \sum_{n \geq 0} \frac{\operatorname{card}\{(i_1, \dots, i_d) : |i_j|^{1/\tau_j} \leq n^{1/\tau_k} \text{ for all } j \in \{1, \dots, d\}, i_k = n\}}{1 + n^{1/\tau_k}} \\ & \leq 1 + C \sum_{k=1}^d \sum_{n \geq 1} \frac{1}{n^{1/\tau_k}} \prod_{j \neq k} n^{\tau_j/\tau_k} = 1 + C \sum_{k=1}^d \sum_{n \geq 1} \frac{n^{(\sum_{j \neq k} \tau_j)/\tau_k}}{n^{1/\tau_k}} \\ & = 1 + C \sum_{k=1}^d \sum_{n \geq 1} \frac{n^{(1-\tau_k-\varepsilon)/\tau_k}}{n^{1/\tau_k}} = 1 + C \sum_{k=1}^d \sum_{n \geq 1} \frac{1}{n^{1+\varepsilon/\tau_k}}, \end{aligned}$$

where  $\varepsilon := 1 - (\tau_1 + \dots + \tau_d)$ . (Remark that, since  $\varepsilon > 0$ , the last infinite sum converges.)

On the other hand, the left hand expression in (10) is equal to

$$\begin{aligned} F(i_1, \dots, i_d) := & \left| \frac{|1 + i_k|^{1/\tau_k} - |i_k|^{1/\tau_k}}{1 + |i_1|^{1/\tau_1} + \dots + |1 + i_k|^{1/\tau_k} + \dots + |i_d|^{1/\tau_d}} \right| \times \\ & \times (1 + |i_1|^{1/\tau_1} + \dots + |i_k|^{1/\tau_k} + \dots + |i_d|^{1/\tau_d})^{\tau_k}. \end{aligned}$$

---

<sup>2</sup>Condition (10) is also necessary under these requirements. Indeed, there must exist a point in  $I_{i_1, \dots, i_k, \dots, i_d}$  for which the derivative of the corresponding map  $f_k$  equals  $\ell_{i_1, \dots, 1+i_k, \dots, i_d} / \ell_{i_1, \dots, i_k, \dots, i_d}$ . Since the derivative of  $f_k$  at the end points of  $I_{i_1, \dots, i_k, \dots, i_d}$  is assumed to be equal to 1, condition (10) holds for  $C'$  being the  $\tau_k$ -Hölder constant of the derivative of  $f_k$ .

In order to obtain an upper bound for this expression first notice that, if  $i_k \geq 0$ , then

$$F(i_1, \dots, i_k, \dots, i_d) \leq F(i_1, \dots, -1 - i_k, \dots, i_d).$$

Therefore, we can restrict to the case where  $i_k < 0$ . For this case, denoting  $B = 1 + \sum_{j \neq k} |i_j|^{1/\tau_j}$  and  $a = |i_k|$  we have

$$\begin{aligned} F(i_1, \dots, i_d) &= \frac{a^{1/\tau_k} - (a-1)^{1/\tau_k}}{B + (a-1)^{1/\tau_k}} \cdot (B + a^{1/\tau_k})^{\tau_k} = \\ &= \frac{a^{1/\tau_k} - (a-1)^{1/\tau_k}}{(B + (a-1)^{1/\tau_k})^{1-\tau_k}} \cdot \left( \frac{B + a^{1/\tau_k}}{B + (a-1)^{1/\tau_k}} \right)^{\tau_k}. \end{aligned}$$

Both factors in the last expression are decreasing in  $B$ . Thus, since  $B \geq 1$ ,

$$F(i_1, \dots, i_d) \leq \frac{a^{1/\tau_k} - (a-1)^{1/\tau_k}}{(1 + (a-1)^{1/\tau_k})^{1-\tau_k}} \cdot \left( \frac{1 + a^{1/\tau_k}}{1 + (a-1)^{1/\tau_k}} \right)^{\tau_k}.$$

Now notice that  $a \geq 1$ . For  $a = 1$  the right hand expression above equals  $2^{\tau_k}$ . If  $a > 1$  then the mean value theorem gives the estimate  $a^{1/\tau_k} - (a-1)^{1/\tau_k} \leq a^{\frac{1}{\tau_k}-1}/\tau_k$ , and therefore the preceding expression is bounded from above by

$$\begin{aligned} \frac{1}{\tau_k} \frac{a^{\frac{1}{\tau_k}-1}}{((a-1)^{1/\tau_k})^{1-\tau_k}} \cdot \left( \frac{a^{1/\tau_k}}{(a-1)^{1/\tau_k}} \right)^{\tau_k} &= \frac{1}{\tau_k} \left( \frac{a}{a-1} \right)^{\frac{1}{\tau_k}-1} \cdot \left( \frac{a}{a-1} \right) \leq \\ &\leq \frac{1}{\tau_k} \cdot 2^{\frac{1}{\tau_k}-1} \cdot 2 = \frac{2^{1/\tau_k}}{\tau_k}. \end{aligned}$$

We have then shown that for any  $(i_1, \dots, i_d) \in \mathbb{Z}^d$  one has

$$F(i_1, \dots, i_d) \leq \frac{1}{\tau_k} 2^{1/\tau_k}.$$

In other words, if  $\tau' = \min\{\tau_1, \dots, \tau_d\}$  then inequality (10) with the constant  $C' = 2^{1/\tau'}/\tau'$  holds for each  $(i_1, \dots, i_d) \in \mathbb{Z}^d$  and every  $k \in \{1, \dots, d\}$ , and this finishes the proof of Theorem B.

## REFERENCES

- [1] V. I. Arnold, *Small denominators. I. Mapping the circle onto itself*, Izv. Akad. Nauk SSSR Ser. Mat. **25** (1961), 21–86 (Russian). MR [0140699](#).
- [2] P. Bohl, *Über die Hinsichtlich der Unabhängigen und Abhängigen Variablen Periodische Differentialgleichung Erster Ordnung*, Acta Math. **40** (1916), no. 1, 321–336. MR [1555141](#)
- [3] A. Denjoy, *Sur les courbes définies par des équations différentielles à la surface du tore*, J. Math. Pures et Appl. **11** (1932), 333–375.
- [4] B. Deroin, V. Kleptsyn, and A. Navas, *Sur la dynamique unidimensionnelle en régularité intermédiaire*, Acta Math. **199** (2007), no. 2, 199–262. MR [2358052](#)
- [5] B. Fayad and K. Khanin, *Smooth linearization of commuting circle diffeomorphisms*, Ann. Math., to appear. Preprint version: [arXiv:math/0605214](#) [math.DS].
- [6] É. Ghys, *Groups acting on the circle*, Enseign. Math. (2) **47** (2001), no. 3–4, 329–407. MR [1876932](#)
- [7] M.-R. Herman, *Sur la conjugaison différentiable des difféomorphismes du cercle à des rotations*, Inst. Hautes Études Sci. Publ. Math. (1979), no. 49, 5–233. MR [538680](#)
- [8] Y. Katznelson and D. Ornstein, *The differentiability of the conjugation of certain diffeomorphisms of the circle*, Ergodic Theory Dynam. Systems **9** (1989), no. 4, 643–680. MR [1036902](#)

- [9] J. Moser, *On commuting circle mappings and simultaneous Diophantine approximations*, Math. Z. **205** (1990), no. 1, 105–121. MR [1069487](#)
- [10] A. Navas, *Growth of groups and diffeomorphisms of the interval*, preprint [arXiv:math/0508353](#) [math.DS], to appear in Geom. and Funct. Anal.
- [11] A. Navas, *Grupos de difeomorfismos del círculo*, Ensaios Matemáticos, 13, Sociedade Brasileira de Matemática, Rio de Janeiro, 2007. MR [2394157](#)
- [12] D. Pixton, *Nonsmoothable, unstable group actions*, Trans. Amer. Math. Soc. **229** (1977), 259–268. MR [0438397](#)
- [13] H. Poincaré, *Mémoire sur les courbes définies par une équation différentielle*, Journal de Mathématiques **7** (1881), 375–422.
- [14] H. Poincaré, *Mémoire sur les courbes définies par une équation différentielle*, Journal de Mathématiques **8** (1882), 251–296.
- [15] R. Sacksteder, *Foliations and pseudogroups*, Amer. J. Math. **87** (1965), 79–102. MR [0174061](#)
- [16] A. J. Schwartz, *A generalization of a Poincaré–Bendixson theorem to closed two-dimensional manifolds*, Amer. J. Math. **85** (1963), 453–458; errata, *ibid* **85** (1963), 753. MR [0155061](#)
- [17] T. Tsuboi, *Homological and dynamical study on certain groups of Lipschitz homeomorphisms of the circle*, J. Math. Soc. Japan **47** (1995), no. 1, 1–30. MR [1304186](#)
- [18] J.-C. Yoccoz, *Centralisateurs et conjugaison différentiable des difféomorphismes du cercle. Petits diviseurs en dimension 1*, Astérisque (1995), no. 231, 89–242. MR [1367354](#)

INSTITUT DE RECHERCHES MATHÉMATIQUES DE RENNES (UMR 6625 CNRS), CAMPUS BEAULIEU, 35042 RENNES, FRANCE

*E-mail address:* [Victor.Kleptsyn@univ-rennes1.fr](mailto:Victor.Kleptsyn@univ-rennes1.fr)

UNIVERSIDAD DE SANTIAGO DE CHILE, ALAMEDA 3363, SANTIAGO, CHILE

*E-mail address:* [anavas@usach.cl](mailto:anavas@usach.cl)